FRACTIONAL CALCULUS OF THE GENERALIZED WRIGHT FUNCTION

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Abstract

The paper is devoted to the study of the fractional calculus of the generalized Wright function $p\Psi_q(z)$ defined for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$ and real $\alpha_i, \beta_j \in \mathbb{R}$ ($i = 1, 2, \cdots p; j = 1, 2, \cdots q$) by the series

$$p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \cdot z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) \cdot k!}.$$ 

It is proved that the Riemann-Liouville fractional integrals and derivative of the Wright function are also the Wright functions but of greater order. Special cases are considered.

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1. Introduction

The paper deals with the generalized Wright function defined for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$ and real $\alpha_i, \beta_j \in \mathbb{R} = (-\infty, \infty)$ ($\alpha_i, \beta_j \neq 0; \ i = 1, 2, \cdots, p; \ j = 1, 2, \cdots, q$) by the series

$$p \Psi_q(z) \equiv p \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \left( \begin{array}{c} z \\ \infty \end{array} \right) = \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \quad (1)$$

Here $\Gamma(z)$ is the Euler gamma-function [3, Section 1.1]. The function in (1) was introduced by Wright [21] and is called the generalized Wright function, see [3, Section 4.1]. Conditions for the existence of the generalized Wright function (1) together with its representation in terms of the Mellin-Barnes integral and of the $H$-function were established in [6].

The special case of the function (1) in the form

$$\phi(\beta, b; z) \equiv 0 \Psi_1 \left[ \begin{array}{c} (b, \beta) \\ (b, \beta)_{1,q} \end{array} \right] \left( \begin{array}{c} z \\ \infty \end{array} \right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta k + b)} \frac{z^k}{k!}, \quad (2)$$

with complex $z, b \in \mathbb{C}$ and real $\beta \in \mathbb{R}$, known as the Wright function [4, Section 18.1], was introduced by Wright in [19]. When $\beta = \delta, b = \nu + 1$ and $z$ is replaced by $-z$, the function $\phi(\delta, \nu + 1; -z)$ is denoted by $J_{\nu}^{\delta}(z)$:

$$J_{\nu}^{\delta}(z) \equiv \phi(\delta, \nu + 1; -z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!}, \quad (3)$$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function, see [7, p. 352] and [14, (8.3)]. Some other particular cases of the generalized Wright function (1), generalizing the classical Mittag-Leffler function, were presented in [6, Section 6].

Wright in [20], [24] investigated the asymptotic expansions of the function $\phi(\beta, b; z)$ for large values of $z$ in the cases $\beta > 0$ and $-1 < \beta < 0$, respectively, making use of the "steepest descent" method. In [20] he gives an application of the obtained results to the asymptotic theory of partitions. In [21]-[23] Wright extended the last results to the generalized Wright function (1) and proved several theorems on the asymptotic expansion of $p \Psi_q(z)$ for all values of the argument $z$ under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \quad (4)$$
The properties of the Wright function (2) were studied in a series of papers. Some of them can be found in [4, Section 18.1]. We also mention that some fractional integral relations for the function (2) were presented in [2], asymptotic relations for zeros of the Wright function \( \phi(\beta, b; z) \) were established in [8], and distributions of these zeros were investigated in [9]. Applications of the Wright function (2) to the operational calculus were given in [15], to integral transforms of Hankel type - in [5] and [18], to partial differential equations of fractional order - in [1] and [10]-[13], see also [16, Section 4.1.2]. We also note [2], where solution in closed form of the integral equation of the first with the Wright function as a kernel was obtained.

The present paper is devoted to the study of the Riemann-Liouville fractional integration and differentiation of the Wright function (1). For \( \alpha \in \mathbb{C} \ (\text{Re}(\alpha) > 0) \), such a left- and right-hand sided fractional integration operators are defined by

\[
(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (x > 0);
\]

and

\[
(I_{-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad (x > 0),
\]

respectively [17, Section 5.1], and the corresponding fractional differentiation operators have the forms

\[
(D_{0+}^\alpha f)(x) = \left( \frac{d}{dx} \right)^{\lfloor \text{Re}(\alpha) \rfloor + 1} (I_{0+}^{1-\alpha+\lfloor \text{Re}(\alpha) \rfloor} f)(x)
\]

\[
= \left( \frac{d}{dx} \right)^{\lfloor \text{Re}(\alpha) \rfloor + 1} \frac{1}{\Gamma(1 - \alpha + \lfloor \text{Re}(\alpha) \rfloor)} \int_0^x \frac{f(t)}{(x-t)^{\alpha-\lfloor \text{Re}(\alpha) \rfloor}} dt \quad (x > 0)
\]

and

\[
(D_{-}^\alpha f)(x) = \left( -\frac{d}{dx} \right)^{\lfloor \text{Re}(\alpha) \rfloor + 1} (I_{-}^{1-\alpha+\lfloor \text{Re}(\alpha) \rfloor} f)(x)
\]

\[
= \left( -\frac{d}{dx} \right)^{\lfloor \text{Re}(\alpha) \rfloor + 1} \frac{1}{\Gamma(1 - \alpha + \lfloor \text{Re}(\alpha) \rfloor)} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-\lfloor \text{Re}(\alpha) \rfloor}} dt \quad (x > 0),
\]

respectively, where \( \lfloor \text{Re}(\alpha) \rfloor \) is the integral part of \( \text{Re}(\alpha) \).

The paper is organized as follows. Some known results are presented in Section 2. The fractional integration and differentiation of the generalized
Wright function (1) is established in Sections 3 and 4, respectively. The corresponding results for the Wright function (2) and the Bessel-Maitland function (3) are presented in Section 5.

2. Preliminaries

In this section we present the conditions for the existence of the generalized Wright function \( p \Psi_q (z) \) in (1) proved in [6], and the known formulas for the fractional integration (5) and (6) of a power function [17]. To formulate the first result we use the following notation:

\[
\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \\
\delta = \prod_{i=1}^{p} |\alpha_i|^{-\alpha_i} \prod_{j=1}^{q} |\beta_j|^{\beta_j}, \\
\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.
\]

**Theorem 1.** Let \( a_i, b_j \in \mathbb{C} \) and \( \alpha_i, \beta_j \in \mathbb{R} \) \((i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)\).

(a) If \( \Delta > -1 \), then the series in (1) is absolutely convergent for all \( z \in \mathbb{C} \).

(b) If \( \Delta = -1 \), then the series in (1) is absolutely convergent for all values of \( |z| < \delta \) and of \( |z| = \delta, \Re(\mu) > 1/2 \).

**Corollary 1.1.** Let \( a_i, b_j \in \mathbb{C} \) and \( \alpha_i, \beta_j \in \mathbb{R} \) \((i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)\) be such that the condition in (4) is satisfied. Then the generalized Wright function \( p \Psi_q (z) \) is an entire function of \( z \).

**Corollary 1.2.** Let \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{C} \).

(a) If \( \alpha > -1 \), then the series in (2) is absolutely convergent for all \( z \in \mathbb{C} \).

(b) If \( \alpha = -1 \), then the series in (2) is absolutely convergent for all values of \( |z| < 1 \) and of \( |z| = 1, \Re(\beta) > 1 \).

**Corollary 1.3.** If \( \alpha > -1 \) and \( \beta \in \mathbb{C} \), then the Wright function \( \phi(\alpha, \beta; z) \) is an entire function of \( z \).
Corollary 1.4. If $\delta > -1$ and $\nu \in \mathbb{C}$, then the Bessel-Maitland function $J_{\nu}^\delta(z)$ is an entire function of $z$.

The next assertion is well known, see [17, (2.44) and Table 9.3, formula 1].

Lemma 1. Let $\alpha \in \mathbb{C}$ ($\text{Re}(\alpha) > 0$) and $\gamma \in \mathbb{C}$.

(a) If $\text{Re}(\gamma) > 0$, then

$$
(I_{0+}^\alpha t^{\gamma-1}) (x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} x^{\alpha + \gamma - 1}.
$$

(b) If $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$, then

$$
(I_{-t}^\alpha -\gamma) (x) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} x^{\alpha - \gamma}.
$$

3. Fractional integration of the generalized Wright function

In this section we establish a formula for the fractional integration of the generalized Wright function (1). We begin with the left-hand sided fractional integral (5).

Theorem 2. Let $\alpha, \gamma \in \mathbb{C}$ be complex numbers such that $\text{Re}(\alpha) > 0$ and $\text{Re}(\gamma) > 0$, and let $a \in \mathbb{C}$, $\mu > 0$. If the condition (4) is satisfied, then the fractional integration $I_{0+}^\alpha$ of the generalized Wright function (1) is given for $x > 0$ by

$$
(I_{0+}^\alpha \left( t^{\gamma-1} \frac{\Psi_{q}}{p} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \mid at^{\mu} \\ (b_j, \beta_j)_{1,q} \mid a x^{\mu} \end{array} \right] \right) ) (x)
$$

$$
= x^{\gamma + \alpha - 1} \frac{\Psi_{q+1}}{p+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p}, (\gamma, \mu) \mid (b_j, \beta_j)_{1,q}, (\gamma + \alpha, \mu) \mid a x^{\mu} \end{array} \right].
$$

Proof. According to (4) and Corollary 1.1, the generalized Wright functions in both sides of (11) exist for $x > 0$. By (5) and (1) we have

$$
(I_{0+}^\alpha \left( t^{\gamma-1} \frac{\Psi_{q}}{p} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \mid at^{\mu} \\ (b_j, \beta_j)_{1,q} \mid \end{array} \right] \right) ) (x)
$$
\[
\left( I_{0+}^\alpha \left[ t^{\gamma-1} \sum_{k=0}^\infty \prod_{i=1}^p \Gamma(\alpha_i + \alpha_i k) \frac{(at^\mu)^k}{k!} \right] \right)(x).
\] (12)

According to [17, Lemma 15.1] a term-by-term integration of a series in the right-hand side of (12) is possible. Carrying out such an integration and using (9) we obtain

\[
\left( I_{0+}^\alpha \left( t^{\gamma-1} p \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) \right)(x)
\]

\[
= \sum_{k=0}^\infty \prod_{i=1}^p \Gamma(\alpha_i + \alpha_i k) \prod_{j=1}^q \Gamma(\beta_j + \beta_j k) \frac{(at^\mu)^k}{k!}.
\]

According to (1) from here we deduce (11), which completes the proof of theorem.

The following result yields the right-hand sided fractional integration (6) of the generalized Wright function (1).

**Theorem 3.** Let \( \alpha, \gamma \in \mathbb{C} \) be complex numbers such that \( \text{Re}(\gamma) > \text{Re}(\alpha) > 0 \), and let \( a \in \mathbb{C}, \mu > 0 \). If the condition (4) is satisfied, then the fractional integration \( I_{-}^\alpha \) of the generalized Wright function (1) is given by

\[
\left( I_{-}^\alpha \left( t^{-\gamma} p \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) \right)(x)
\]

\[
= x^{\alpha - \gamma - 1} \sum_{k=0}^\infty \prod_{i=1}^p \Gamma(\gamma - \alpha, \mu) \prod_{j=1}^q \Gamma(\gamma, \mu) \frac{(ax^\mu)^k}{k!}.
\] (13)

**Proof.** According to (4) and Corollary 1.1, the generalized Wright functions in both sides of (13) exist for \( x > 0 \). The fractional integrals (5) and (6) are connected by the relation

\[
\left( I_{0+}^\alpha f \left[ \frac{1}{t} \right] \right)(x) = x^{\alpha - 1} \left( I_{0+}^\alpha [t^{-\alpha - 1} f(t)] \right) \left( \frac{1}{x} \right).
\]
Using this formula and taking into account (11) with $\gamma$ replaced by $\gamma - \alpha$, we have

$$
\left( \int_{t^{-\gamma}}^{\infty} p^{-1} \left( \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid at^\mu \right) \right)(x)
= x^{\alpha-1} \left( \int_{0+}^{\infty} p^{-1} \left( \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid at^\mu \right) \right) \left( \frac{1}{x} \right)
= x^{\alpha-\gamma} p_1 \psi_{q+1} \left[ \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid a x^{-\mu} \right],
$$

and (13) is proved.

**4. Fractional differentiation of the generalized Wright function**

In this section we establish a formula for the fractional differentiation of the generalized Wright function (1). As in Section 3, we begin with the left-hand sided fractional differentiation (7).

**Theorem 4.** Let $\alpha, \gamma \in \mathbb{C}$ and Re$(\alpha) > 0$ and Re$(\gamma) > 0$, and let $\alpha \in \mathbb{C}, \mu > 0$. If condition (4) is satisfied, then the fractional differentiation $D^\alpha_{0+}$ of the generalized Wright function (1) is given for $x > 0$ by

$$
\left( D^\alpha_{0+} \left( t^{\gamma-1} p^{-1} \psi_{q} \left[ \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid at^\mu \right] \right) \right)(x)
= x^{\gamma-\alpha-1} p_1 \psi_{q+1} \left[ \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid a x^\mu \right]. \quad (14)
$$

**Proof.** According to (1) and Corollary 1.1, the generalized Wright functions on both sides of (14) exist for $x > 0$. Let $n = \lfloor \text{Re}(\alpha) \rfloor + 1$, where $\lfloor \text{Re}(\alpha) \rfloor$ is an integer part of Re$(\alpha)$. Using (7) and (1) and taking into account (11), with $\alpha$ replaced by $\alpha - n$, we have

$$
\left( D^\alpha_{0+} \left( t^{\gamma-1} p^{-1} \psi_{q} \left[ \frac{a_i, \alpha_i}{(b_j, \beta_j)_{1,q}} \mid at^\mu \right] \right) \right)(x)
$$
\[
\left( D_{0+}^{\alpha} \right) \left( t^{\gamma-1} p^\gamma \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^\mu \right) (x)
\]

\[
= \frac{d}{dx} \left( \int_0^x \left( t^{\gamma-1} p^\gamma \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^\mu \right) dt \right) (x)
\]

\[
= \frac{d}{dx} \left( \int_0^x \left( t^{\gamma+n-\alpha-1} p^{\gamma+n-\alpha} \Psi_{q+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p,\gamma,\mu} \\ (b_j, \beta_j)_{1,q,\gamma+n-\alpha,\mu} \end{array} \right] a x^\mu \right) dt \right) (x)
\]

\[
= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \Gamma(\gamma + \mu k) a^k x^{\gamma+n-\alpha+\mu k-1} \right).
\]

According to [17, Lemma 15.1], a term-by-term differentiation of the series on the right-hand side of (15) is possible. Therefore

\[
\left( D_{0+}^{\alpha} \right) \left( t^{\gamma-1} p^\gamma \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^\mu \right) (x)
\]

\[
= \frac{d}{dx} \left( \int_0^x \left( t^{\gamma+n-\alpha-1} p^{\gamma+n-\alpha} \Psi_{q+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p,\gamma,\mu} \\ (b_j, \beta_j)_{1,q,\gamma+n-\alpha,\mu} \end{array} \right] a x^\mu \right) dt \right) (x)
\]

\[
= \sum_{k=0}^{\infty} \prod_{i=1}^{p} \frac{\Gamma(a_i + \alpha_i k) \Gamma(\gamma + \mu k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{a^k x^{\gamma+n-\alpha+\mu k-1}}{\Gamma(\gamma - \alpha + \mu k) k!}.
\]

Thus, in accordance with (1), (14) is proved. \hfill \blacksquare

The next result yields the right-hand sided fractional differentiation (8) of the generalized Wright function (1).

**Theorem 5.** Let \( \alpha, \gamma \in \mathbb{C} \) be complex numbers such that \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\gamma) > [\text{Re}(\alpha)] + 1 - \text{Re}(\alpha) \), and let \( a \in \mathbb{C}, \mu > 0 \). If condition (4) is satisfied, then the fractional differentiation \( D_{a+}^\alpha \) of the generalized Wright function (1) is given by

\[
\left( D_{a+}^\alpha \right) \left( t^{\gamma-1} p^\gamma \Psi_q \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^{-\mu} \right) (x)
\]

\[
= x^{-\alpha-\gamma} p^{\gamma+n+1} \Psi_{q+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p,\gamma+\alpha,\mu} \\ (b_j, \beta_j)_{1,q,\gamma,\mu} \end{array} \right] a x^{-\mu}.
\]
Proof. By (4) and Corollary 1.1, the generalized Wright functions in both sides of (16) exist for \( x > 0 \). Let \( n = \lceil \text{Re}(\alpha) \rceil + 1 \). Using (8) and (1) and taking into account (13) with \( \alpha \) replaced by \( n - \alpha \), similarly to the proof of Theorem 4, we obtain

\[
\left( D^{\alpha} \left( t^{-\gamma} T_1^{\alpha} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^{-\mu} \right) \right) (x)
= \left( -\frac{d}{dx} \right)^n \left( I^{n-\alpha} \left( t^{-\gamma} T_1^{\alpha} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] at^{-\mu} \right) \right) (x)
= \left( -\frac{d}{dx} \right)^n \left( x^{n-\alpha-\gamma} T_1^{\alpha+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p}, (\gamma - n + \alpha, \mu) \\ (b_j, \beta_j)_{1,q}, (\gamma, \mu) \end{array} \right] ax^{-\mu} \right)
= \left( -\frac{d}{dx} \right)^n \sum_{k=0}^{\infty} \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \frac{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)}{\Gamma(\gamma + \mu k)} \frac{(\gamma - n + \alpha + \mu k) a^k}{k!} x^{n-\alpha-\gamma-\mu k}.
\]

By the reflection formula for the gamma-function, see for example, [17, (1.60)],

\[
\frac{1}{\Gamma(1 - \gamma - \alpha - \mu k)} = \frac{\Gamma(\gamma + \alpha + \mu k)}{\Gamma(\gamma + \alpha + \mu k) \Gamma(1 - \gamma - \alpha - \mu k)}
= \frac{\Gamma(\gamma + \alpha + \mu k) \sin[(\gamma + \alpha + \mu k)\pi]}{\pi}
\]

and

\[
\Gamma(\gamma - n + \alpha + \mu k) \Gamma(1 + n - \alpha - \gamma - \mu k) = \frac{\pi}{\sin[(\gamma - n + \alpha + \mu k)\pi]}
= \frac{\pi}{\sin[(\gamma + \alpha + \mu k)\pi] \cos(n\pi)} = \frac{(-1)^n \pi}{\sin[(\gamma + \alpha + \mu k)\pi]}.
\]
Substituting these relations into (17) we obtain

\[
D^{\alpha}_{\gamma} \left( t^{-\gamma} \Psi_p \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \
(b_j, \beta_j)_{1,q} \\
\end{array} \right] \right) (x) = x^{-\alpha - \gamma} \sum_{k=0}^{\infty} \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \prod_{j=1}^{q} \Gamma(b_j + \beta_j k) (-1)^n \frac{\Gamma(\gamma + \alpha + \mu k)}{\Gamma(\gamma + \mu k)} \frac{(ax^{-\mu})^k}{k!},
\]

which, in accordance with (1), yields (16).

\[
D^{\alpha}_{\gamma} \left( t^{-\gamma} \Psi_p \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \
(b_j, \beta_j)_{1,q} \\
\end{array} \right] \right) (x) = x^{-\alpha - \gamma} \sum_{k=0}^{\infty} \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \prod_{j=1}^{q} \Gamma(b_j + \beta_j k) (-1)^n \frac{\Gamma(\gamma + \alpha + \mu k)}{\Gamma(\gamma + \mu k)} \frac{(ax^{-\mu})^k}{k!},
\]

5. Fractional calculus of the Wright and the Bessel-Maitland functions

In this section we establish fractional integration and differentiation of the Wright function \( \phi(\beta, b; z) \) and Bessel-Maitland function \( J_{\nu}^\delta (z) \). Using (2), from Theorems 2-3 and Theorems 4-5 we deduce formulas for the fractional integration and differentiation of \( \phi(\beta, b; z) \).

Theorem 6. Let \( \alpha, \gamma, b, a \in \mathbb{C} \) and \( \mu > 0 \) and \( \beta > -1 \).

(a) If \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\gamma) > 0 \), then the fractional integration \( I_{0+}^\alpha \) of the Wright function (2) is given for \( x > 0 \) by

\[
(I_{0+}^\alpha [t^{\gamma-1} \phi (\beta, b; at^\mu)]) (x) = x^{\gamma + \alpha - 1} \Psi_2 \left[ \begin{array}{c} (\gamma, \mu) \\
(b, \beta), (\gamma + \alpha, \mu) \\
\end{array} \right] \frac{ax^\mu}{\Gamma(\alpha \mu)}. \tag{18}
\]

(b) If \( \text{Re}(\gamma) > \text{Re}(\alpha) > 0 \), then the fractional integration \( I_{-}^\alpha \) of the Wright function (2) is given for \( x > 0 \) by

\[
(I_{-}^\alpha [t^{\gamma-1} \phi (\beta, b; at^\mu)]) (x) = x^{\alpha - \gamma} \Psi_2 \left[ \begin{array}{c} (\gamma - \alpha, \mu) \\
(b, \beta), (\gamma, \mu) \\
\end{array} \right] \frac{ax^{-\mu}}{\Gamma(\alpha \mu)}. \tag{19}
\]

Corollary 6.1. Let \( \alpha, \gamma, a \in \mathbb{C} \) and \( \mu > 0 \).

(a) If \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\gamma) > 0 \), then

\[
(I_{0+}^\alpha [t^{\gamma-1} \phi (\mu, \gamma; at^\mu)]) (x) = x^{\gamma + \alpha - 1} \phi (\mu, \gamma + \alpha; ax^\mu). \tag{20}
\]

(b) If \( \text{Re}(\gamma) > \text{Re}(\alpha) > 0 \), then

\[
(I_{-}^\alpha [t^{\gamma} \phi (\mu, \gamma - \alpha; at^{-\mu})]) (x) = x^{\alpha - \gamma} \phi (\mu, \gamma; ax^{-\mu}). \tag{21}
\]
Theorem 7. Let $\alpha, \gamma, b, a \in \mathbb{C}$ and $\mu > 0$ and $\beta > -1$.

(a) If $\text{Re}(\alpha) > 0$ and $\text{Re}(\gamma) > 0$, then the fractional differentiation $D_{0+}^\alpha$ of the Wright function (2) is given for $x > 0$ by

$$
(D_{0+}^\alpha \left[t^{\gamma-1} \phi (\beta, b; at^\mu)\right]) (x) = x^{\gamma-\alpha-1} 1_{\Psi_2} \left[ \begin{array}{c} (\gamma, \mu) \\ (b, \beta, (\gamma-\alpha, \mu) \end{array} \right] a x^\mu.
$$

(b) If $\text{Re}(\gamma) > [\text{Re}(\alpha)] + 1 - \text{Re}(\alpha)$, then the fractional differentiation $D_{a}^\alpha$ of the Wright function (2) is given for $x > 0$ by

$$
(D_{a}^\alpha \left[t^{-\gamma} \phi (\beta, b; at^{-\mu})\right]) (x) = x^{-\alpha-\gamma} 1_{\Psi_2} \left[ \begin{array}{c} (\gamma + \alpha, \mu) \\ (b, \beta, (\gamma-\alpha, \mu) \end{array} \right] a x^{-\mu}.
$$

Corollary 7.1. Let $\alpha, \gamma, a \in \mathbb{C}$ and $\mu > 0$.

(a) If $\text{Re}(\alpha) > 0$ and $\text{Re}(\gamma) > 0$, then

$$
(D_{0+}^\alpha \left[t^{\gamma-1} \phi (\mu, \gamma; at^\mu)\right]) (x) = x^{\gamma-\alpha-1} \phi (\mu, \gamma-\alpha; ax^\mu).
$$

(b) If $\text{Re}(\gamma) > [\text{Re}(\alpha)] + 1 - \text{Re}(\alpha)$, then

$$
(D_{a}^\alpha \left[t^{-\gamma} \phi (\mu, \gamma+\alpha; at^{-\mu})\right]) (x) = x^{\alpha-\gamma} \phi (\mu, \gamma; ax^{-\mu}.
$$

Similarly, in accordance with (3), from Theorems 2-3 and Theorems 4-5 we obtain the fractional integration and differentiation of $J_{\nu}^\delta (z)$.

Theorem 8. Let $\alpha, \gamma, \nu, a \in \mathbb{C}$ and $\mu > 0$ and $\delta > -1$.

(a) If $\text{Re}(\alpha) > 0$ and $\text{Re}(\gamma) > 0$, then the fractional integration $I_{0+}^\alpha$ of the Bessel-Maitland function (3) is given for $x > 0$ by

$$
(I_{0+}^\alpha \left[t^{\gamma-1} J_{\nu}^\delta (at^\mu)\right]) (x) = x^{\gamma+\alpha-1} 1_{\Psi_2} \left[ \begin{array}{c} (\gamma, \mu) \\ (\nu + 1, \delta, (\gamma + \alpha, \mu) \end{array} \right] a x^\mu.
$$

(b) If $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$, then the fractional integration $I_{a}^\alpha$ of the Bessel-Maitland function (3) is given for $x > 0$ by

$$
(I_{a}^\alpha \left[t^{-\gamma} J_{\nu}^\delta (at^{-\mu})\right]) (x) = x^{\alpha-\gamma} 1_{\Psi_2} \left[ \begin{array}{c} (\gamma - \alpha, \mu) \\ (\nu + 1, \delta, (\gamma, \mu) \end{array} \right] a x^{-\mu}.
$$
Corollary 8.1. Let $\alpha, \nu, a \in \mathbb{C}$ be complex numbers such that $\Re(\alpha) > 0$ and $\Re(\nu) > -1$, and let $\mu > 0$. Then there hold the relations

$$\tag{28}
(I_0^+ [t^\nu J_\nu^\mu (at^\mu)]) (x) = x^{\nu+\alpha} J_{\nu+1+\alpha}^\mu (ax^\mu).
$$

and

$$\tag{29}
(I_- [t^{-\alpha-\nu-1} J_{\nu+1+\alpha}^\mu (at^{-\mu})]) (x) = x^{-\nu-1} J_{\nu+1+\alpha}^\mu (ax^{-\mu}).
$$

Theorem 9. Let $\alpha, \gamma, b, \nu \in \mathbb{C}$ and $\mu > 0$ and $\delta > -1$.

(a) If $\Re(\alpha) > 0$ and $\Re(\gamma) > 0$, then the fractional differentiation $D_0^\alpha$ of the Bessel-Maitland function (3) is given for $x > 0$ by

$$\tag{30}
(D_0^\alpha [t^{-\gamma} J_\nu^\delta (at^\mu)]) (x) = x^{\gamma-\alpha-1} \Psi_2 \begin{bmatrix}
(\gamma, \mu) \\
(\nu + 1, \delta), (\gamma - \alpha, \mu) \\
\end{bmatrix}
ax^\mu.
$$

(b) If $\Re(\gamma) > |\Re(\alpha)| + 1 - \Re(\alpha)$, then the fractional differentiation $D_\alpha^\gamma$ of the Bessel-Maitland function (3) is given for $x > 0$ by

$$\tag{31}
(D_\alpha^\gamma [t^\nu J_\nu^\delta (at^{-\mu})]) (x) = x^{-\alpha-\gamma} \Psi_2 \begin{bmatrix}
(\gamma + \alpha, \mu) \\
(\nu + 1, \delta), (\gamma, \mu) \\
\end{bmatrix}
ax^{-\mu}.
$$

Corollary 9.1. Let $\alpha, \nu, a \in \mathbb{C}$ and $\mu > 0$.

(a) If $\Re(\alpha) > 0$ and $\Re(\nu) > -1$, then

$$\tag{32}
(D_0^\alpha [t^\nu J_\nu^\mu (at^\mu)]) (x) = x^{\nu-\alpha} J_{\nu+1-\alpha}^\mu (ax^\mu).
$$

(b) If $\Re(\alpha) > 0$ and $\Re(\nu) > |\Re(\alpha)|$, then

$$\tag{33}
(D_\alpha^\nu [t^{\alpha-\nu-1} J_\nu^\mu (at^{-\mu})]) (x) = x^{-\nu-1} J_{\nu+1-\alpha}^\mu (ax^{-\mu}).
$$

References


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