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ON BILATERAL BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

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Abstract: The present research article deals with the derivation of continued fraction involving bilateral basic hypergeometric series by making use of known three term relations and other known results of R.P. Agarwal.

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1. Introduction

The role of continued fractions in mathematical analysis and number theory has been recognized since the times of Euler and Gauss. It is significant to quote that ordinary basic hypergeometric series and their representation in terms of continued fractions had provided a platform for further developments of mathematical literatures related with the q-series. During the beginning of the 20^{th} century a fresh impetus to this field of mathematics rejuvenated by the works of Ramanujan. Ramanujan's contribution to continued fraction associated with analytic functions is remarkable and his Notebooks [11] contain a large number of beautiful results associated with hypergeometric functions (basic and ordinary) and continued fractions. Chapters 1, 2 and 3 of Agarwal [6] deals with a number of interesting results of Ramanujan's on continued fractions. Most

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of the continued fractions for the quotient of two basic hypergeometric series namely ${}_2\phi_1$ extended for the quotient of two basic bilateral series of ${}_2\psi_2$ type and a number of author's namely Denis [9,10], Singh [12,13,14], Agarwal [7,8], Bhargava and Adiga [1], Srivastava [2,3], Srivastava and Mishra [4], Srivastava et al. [5] and many more established important application for basic bilateral series and continued fractions. In the present paper we have made an attempt to establish certain results involving bilateral basic hypergeometric series and continued fractions.

2. Notations and Definitions

For a real or complex q ($|q| < 1$), we define the $q-$shifted factorial by

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \tag{2.1}$$

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu;q)_\infty}. \tag{2.2}$$

For arbitrary parameters a and ν, so that

$$(a;q)_n = \begin{cases} 1; & n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}); & n=1,2,3\dots \end{cases} \tag{2.3}$$

$$[a;q]_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n[q/a;q]_n}, \tag{2.4}$$

$$(a_1,a_2,...a_k;q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n, \tag{2.5}$$

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n. \tag{2.6}$$

The generalized bilateral basic hypergeometric series is defined as:

$${}_r\psi_r\left[\begin{array}{l} a_1,a_2,\dots,a_r;q;z \\ b_1,b_2,\dots,b_r \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{[a_1,a_2,\dots,a_r;q]_n z^n}{[b_1,b_2,\dots,b_r;q]_n}, \tag{2.7}$$

where $|b_1,b_2,...b_r/a_1,a_2,a_3,...a_r| < |z| < 1$ for convergence.

An expression of the form

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_2+}\cdots\frac{a_n}{b_n} \tag{2.8}$$

is said to be a terminating continued fraction and as $n \to \infty$, it is said to be an infinite continued fraction. Other notations and definitions appearing in this paper have their usual meaning.

We shall make use of the following three term relations in establishing our result:

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{array}{l}\alpha q^i, \beta q^i; q, x\\ \delta, \gamma q^{2i}\end{array}\right]\\
&\qquad = A_i\,{}_2\psi_2\left[\begin{array}{l}\alpha q^i, \beta q^{i+1}; q, x\\ \delta, \gamma q^{2i+1}\end{array}\right] + xB_i\,{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1}, \beta q^{i+1}; q, x\\ \delta, \gamma q^{2i+2}\end{array}\right]
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{array}{l}\alpha q^i, \beta q^{i+1}; q, x\\ \delta, \gamma q^{2i+1}\end{array}\right]\\
&\qquad = C_i\,{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1}, \beta q^{i+1}; q, x\\ \delta, \gamma q^{2i+2}\end{array}\right] + xD_i\,{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1}, \beta q^{i+2}; q, x\\ \delta, \gamma q^{2i+3}\end{array}\right],
\end{aligned}
\tag{2.10}
$$

where

$$
A_i = \frac{(1-\beta q^i)(\gamma q^{2i+1}-\delta)}{(1-\gamma q^{2i})(\beta q^{i+1}-\delta)}, \quad B_i = q^{i+1}\frac{(1-\alpha q^i)(1-\beta q^i)(\beta-\gamma q^i)}{(1-\gamma q^{2i})(1-\gamma q^{2i+1})(\beta q^{i+1}-\delta)},
$$

$$
C_i = \frac{(1-\alpha q^i)(\gamma q^{2i+2}-\delta)}{(1-\gamma q^{2i+1})(\alpha q^{i+1}-\delta)}, \quad D_i = q^{i+1}\frac{(1-\beta q^{i+1})(1-\alpha q^i)(\alpha-\gamma q^{i+1})}{(1-\gamma q^{2i+1})(1-\gamma q^{2i+2})(\alpha q^{i+1}-\delta)}
$$

valid for $|\alpha|, |\gamma| < |\delta| < 1, 4|\beta x\delta| < |\delta-\gamma|^2, (\delta\gamma/\alpha\beta) < |x| < 1$, see Agarwal [6], (3.3), (3.4) on p. 183.

We shall also use the following results:

$$
\begin{aligned}
&cq^i(1-aq^i)\,{}_2\psi_2\left[\begin{array}{l}aq^{i+1}, bq^i; q, x\\ cq^{i+1}, d\end{array}\right]\\
&\qquad = (1-cq^i)aq^i\,{}_2\psi_2\left[\begin{array}{l}aq^i, bq^i; q, x\\ cq^i, d\end{array}\right] - (a-c)q^i\,{}_2\psi_2\left[\begin{array}{l}aq^i, bq^i; q, x\\ cq^{i+1}, d\end{array}\right]
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&\frac{-bxq^i(1-aq^i)}{(1-cq^i)}\,{}_2\psi_2\left[\begin{array}{l}aq^{i+1}, bq^{i+1}; q, x\\ cq^{i+1}, d\end{array}\right]\\
&\qquad = \frac{(\frac{d}{q}-bq^i)}{1-bq^i}\,{}_2\psi_2\left[\begin{array}{l}aq^i, bq^i; q, x\\ cq^i, d\end{array}\right] - \frac{d}{q}\,{}_2\psi_2\left[\begin{array}{l}aq^i, bq^{i+1}; q, x\\ cq^i, d\end{array}\right],
\end{aligned}
\tag{2.12}
$$

see Agarwal [6], pp. 187, 188.

3. Main Results

In this section we shall establish the following continued fraction for ${}_2\psi_2$ series.

$$\frac{{}_2\psi_2\left[\begin{array}{l}\alpha q, \beta q; q, x \\ \delta, \gamma q^2\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}\alpha q^2, \beta q^2; q, x \\ \delta, \gamma q^4\end{array}\right]} = \alpha_0 + \frac{\beta_0}{\frac{1}{\gamma_0}-}\,\frac{\delta_0}{\alpha_1+}\,\frac{\beta_1}{\frac{1}{\gamma_1}-}\,\frac{\delta_1}{\alpha_2+}\ldots, \tag{3.1}$$

where for $i \geq 0$:

$$\alpha_i = \frac{(1-\alpha q^{i+1})(\gamma q^{2i+3}-\delta)(1-\beta q^{i+1})(\gamma q^{2i+4}-\delta)}{(1-\gamma q^{2i+2})(\beta q^{i+2}-\delta)(1-\gamma q^{2i+3})(\alpha q^{i+2}-\delta)}$$

$$+xq^{i+2}\frac{(1-\alpha q^{i+1})(1-\beta q^{i+1})(\beta-\gamma q^{i+1})}{(1-\gamma q^{2i+3})(1-\gamma q^{2i+2})(\beta q^{i+2}-\delta)},$$

$$\beta_i = xq^{i+2}\frac{(1-\alpha q^{i+1})(1-\beta q^{i+1})(1-\beta q^{i+2})(\alpha-\gamma q^{i+2})(\gamma q^{2i+3}-\delta)}{(1-\gamma q^{2i+2})(1-\gamma q^{2i+3})(1-\gamma q^{2i+4})(\alpha q^{i+2}-\delta)(\beta q^{i+2}-\delta)} \quad ,$$

$$\gamma_i = \frac{(1-\gamma q^{2i+4})(\beta q^{i+3}-\delta)}{(1-\beta q^{i+2})(\gamma q^{2i+5}-\delta)},$$

and

$$\delta_i = xq^{i+3}\frac{(1-\alpha q^{i+2})(1-\beta q^{i+2})(1-\gamma q^{2i+4})(\beta-\gamma q^{i+2})(\beta q^{i+3}-\delta)}{(1-\gamma q^{2i+4})(1-\gamma q^{2i+5})(1-\beta q^{i+2})(\beta q^{i+3}-\delta)(\gamma q^{2i+5}-\delta)},$$

$$\frac{{}_2\psi_2\left[\begin{array}{l}a, b; q, x \\ c, d\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}a, b; q, x \\ cq, d\end{array}\right]} = A_0 + \frac{B_0}{C_0-}\,\frac{D_0}{E_0+}\,\frac{F_0}{\frac{1}{A_1}+}\,\frac{B_1}{C_1-}\,\frac{D_1}{E_1+}\,\frac{F_1}{\frac{1}{A_2}+\ldots}, \tag{3.2}$$

where

$$A_i = \frac{(a-c)}{a(1-cq^i)}, \quad B_i = \frac{c(1-aq^i)}{a(1-cq^i)}, \quad C_i = \frac{d(1-aq^i)}{(d-aq^{i+1})},$$

$$D_i = axq^{i+1}\frac{(1-aq^i)(1-bq^i)}{d-aq^{i+1}}, \quad E_i = bxq^{i+1}\frac{(a-c)(1-bq^i)}{c(1-cq^{i+1})(d-bq^{i+1})},$$

$$F_i = \frac{(1-bq^i)(cd-bxaq^{i+1})}{c(d-bq^{i+1})}.$$

Proof of the result (3.1). In (2.9) and (2.10) replacing i by $i+1$, and then multiplying (2.10) by A_{i+1} and adding with (2.9), we get

$$ {}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1},\beta q^{i+1};q,x\\ \delta,\gamma q^{2i+2}\end{array}\right] = (A_{i+1}C_{i+1}+xB_{i+1}){}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+2};q,x\\ \delta,\gamma q^{2i+4}\end{array}\right] + xA_{i+1}D_{i+1}{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+3};q,x\\ \delta,\gamma q^{2i+5}\end{array}\right], \quad (3.3) $$

which gives

$$ \frac{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1},\beta q^{i+1};q,x\\ \delta,\gamma q^{2i+2}\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+2};q,x\\ \delta,\gamma q^{2i+4}\end{array}\right]} = \alpha_i + \frac{\beta_i}{\dfrac{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+1},\beta q^{i+1};q,x\\ \delta,\gamma q^{2i+2}\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+2};q,x\\ \delta,\gamma q^{2i+4}\end{array}\right]}}, \quad (3.4) $$

where α_i and β_i are given in (3.1). In (2.9) replacing i by $i+2$ and simplifying, we get

$$ \frac{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+3};q,x\\ \delta,\gamma q^{2i+5}\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+2};q,x\\ \delta,\gamma q^{2i+4}\end{array}\right]} = \gamma_i - \frac{\delta_i}{\dfrac{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+2},\beta q^{i+2};q,x\\ \delta,\gamma q^{2i+4}\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}\alpha q^{i+3},\beta q^{i+3};q,x\\ \delta,\gamma q^{2i+6}\end{array}\right]}}, \quad (3.5) $$

where γ_i and δ_i are given in (3.1).

Now using (3.4) and (3.5) repeatedly and setting $i=0$ we get required result (3.1).

Proof of the result (3.2). From (2.11) we have

$$ \frac{{}_2\psi_2\left[\begin{array}{l}aq^{i},bq^{i};q,x\\ cq^{i},d\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}aq^{i},bq^{i};q,x\\ cq^{i+1},d\end{array}\right]} = A_i + \frac{B_i}{\dfrac{{}_2\psi_2\left[\begin{array}{l}aq^{i},bq^{i};q,x\\ cq^{i+1},d\end{array}\right]}{{}_2\psi_2\left[\begin{array}{l}aq^{i+1},bq^{i};q,x\\ cq^{i+1},d\end{array}\right]}}, \quad (3.6) $$

where A_i and B_i are given in (3.2).

In (2.12) replacing i by $i+1$ and interchanging a and b, further simplifying we get,

$$\frac{{}_2\psi_2\left[\begin{matrix} aq^i, bq^i; q, x \\ cq^{i+1}, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^i; q, x \\ cq^{i+1}, d \end{matrix}\right]} = C_i - \frac{D_i}{\dfrac{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^i; q, x \\ cq^{i+1}, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+2}, d \end{matrix}\right]}}, \tag{3.7}$$

where C_i and D_i are given in (3.2).

In (2.11) replacing i by $i+1$ and in (2.12) replacing a by aq, c by cq and simplifying, we get

$$\frac{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^i; q, x \\ cq^{i+1}, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+2}, d \end{matrix}\right]} = E_i + \frac{F_i}{\dfrac{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+2}, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+1}, d \end{matrix}\right]}}, \tag{3.8}$$

where E_i and F_i are given in (3.2).

Now using (3.6), (3.7) and (3.8) and further simplifying we have,

$$\frac{{}_2\psi_2\left[\begin{matrix} aq^i, bq^i; q, x \\ cq^i, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^i, bq^i; q, x \\ cq^{i+1}, d \end{matrix}\right]} = A_i + \frac{B_i}{C_i -}\,\frac{D_i}{E_i +}\,\frac{F_i}{\dfrac{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+2}, d \end{matrix}\right]}{{}_2\psi_2\left[\begin{matrix} aq^{i+1}, bq^{i+1}; q, x \\ cq^{i+1}, d \end{matrix}\right]}}. \tag{3.9}$$

Now, repeating (3.9) again and again by putting $i = 0, 1, 2, ...$ we get the required result (3.2).

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