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ON THE INCOMPLETE GAMMA FUNCTION

Mongkolsery Lin[1 §], Brian Fisher[2], Somsak Orankitjaroen[3]

[1,3]Department of Mathematics
Faculty of Science
Mahidol University
Bangkok 10400, THAILAND

[2]Department of Mathematics
University of Leicester
Leicester, LE1 7RH, UK

Abstract: The lower incomplete Gamma function $\gamma(\lambda, x)$ is defined for $\lambda > 0$ by

$$\gamma(\lambda, x) = \int_0^x t^{\lambda-1} e^{-t}\, dt$$

and more generally, the function $\gamma_s(\lambda, x)$ is defined for $\lambda > 0$ by

$$\gamma_s(\lambda, x) = \int_0^x t^{\lambda-1} e^{-t} \ln^s t\, dt$$

for $s = 0, 1, 2, \ldots$. If $\lambda \leq 0$, the function $\gamma_s(\lambda, x)$ is defined by

$$\gamma_s(\lambda, x) = \mathrm{N}\!-\!\lim_{\epsilon \to 0} \int_\epsilon^x t^{\lambda-1} e^{-t} \ln^s t\, dt$$

for $s = 0, 1, 2, \ldots$. It is proved that

$$\gamma_s(0, x) = \frac{e^{-x} \ln^{s+1} x}{s+1} + \frac{\gamma_{s+1}(1, x)}{s+1}$$

for $s = 0, 1, 2, \ldots$ and

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§Correspondence author

$$\gamma_1(-r,x) = \sum_{i=1}^{r} \frac{(-1)^{r+i}(i-1)!}{r!}[\gamma_0(-i,x) - x^{-i}e^{-x}\ln x] + \frac{(-1)^r}{r!}\gamma_1(0,x)$$

for $r = 1, 2, \ldots$. Further results on the incomplete gamma function are also proved.

AMS Subject Classification: 33B10
Key Words: gamma function, neutrix, neutrix limit

1. Introduction

The Gamma function, see for example Sneddon [4], is usually defined for $\lambda > 0$ by the integral

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}\,dt.$$

If $\lambda > 0$, it follows that

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda).$$

This equation is then used to define $\Gamma(\lambda)$ for $\lambda < 0$ and $\lambda \neq -1, -2, \ldots$.

It follows by induction that if $-n < \lambda < -n+1$ then

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}\Big[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\Big]\,dt.$$

It was then proved in [3] that

$$\Gamma^{(s)}(\lambda) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{\lambda-1}\ln^r t\, e^{-t}\,dt \tag{1}$$

for $s = 0, 1, 2, \ldots$ and $\lambda \neq 0, -1, -2, \ldots$. where N is the neutrix, see van der Corput [1], having domain $N' = \{\epsilon : \ 0 < \epsilon < \infty\}$ with negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{s-1}\epsilon, \quad \ln^s\epsilon : \quad \lambda < 0, \quad s = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as ϵ tends to zero. It was proved that the neutrix limit in equation (1) existed for $r, s = 0, 1, 2, \ldots$.

It was also proved in [3] that the neutrix limit in equation (1) also existed for $\lambda = 0, -1, -2, \ldots$, and this suggested that $\Gamma^{(s)}(-r)$ could be defined by

$$\Gamma^{(s)}(-r) = \mathrm{N}\text{–}\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} t^{-r-1} \ln^s t \, e^{-t} \, dt$$

for $r, s = 0, 1, 2, \ldots$.

The upper incomplete Gamma function $\Gamma(\lambda, x)$ is defined for $\lambda > 0$ by

$$\Gamma(\lambda, x) = \int_{x}^{\infty} t^{\lambda-1} e^{-t} \, dt$$

and more generally, the function $\Gamma_s(\lambda, x)$ is defined for $\lambda > 0$ by

$$\Gamma_s(\lambda, x) = \int_{x}^{\infty} t^{\lambda-1} e^{-t} \ln^s t \, dt$$

for $s = 0, 1, 2, \ldots$.

The lower incomplete Gamma function $\gamma(\lambda, x)$ is defined for $\lambda > 0$ by

$$\gamma(\lambda, x) = \int_{0}^{x} t^{\lambda-1} e^{-t} \, dt$$

and more generally, the function $\gamma_s(\lambda, x)$ is defined for $\lambda > 0$ by

$$\gamma_s(\lambda, x) = \int_{0}^{x} t^{\lambda-1} e^{-t} \ln^s t \, dt$$

for $s = 0, 1, 2, \ldots$. In particular, we have

$$\gamma_0(\lambda, x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{\lambda+i}}{i!(\lambda+i)}.$$

If $\lambda \leq 0$, we define the function $\gamma_s(\lambda, x)$ by

$$\gamma_s(\lambda, x) = \mathrm{N}\text{–}\lim_{\epsilon \to 0} \int_{\epsilon}^{x} t^{\lambda-1} e^{-t} \ln^s t \, dt \tag{2}$$

for $s = 0, 1, 2, \ldots$. The existence of the neutrix limit in equation (2) follows from the existence of the neutrix limit in equation (1).

It follows that

$$\Gamma^{(s)}(\lambda) = \gamma_s(\lambda, x) + \Gamma_s(\lambda, x) \tag{3}$$

for all λ and $s = 0, 1, 2, \ldots$.

2. Main Results

In this section we will present our results.

Theorem 1.

$$\gamma_s(0,x) = \frac{e^{-x}\ln^{s+1} x}{s+1} + \frac{\gamma_{s+1}(1,x)}{s+1}, \tag{4}$$

for $s = 0, 1, 2, \ldots$.

In particular, we have

$$\gamma_0(0,x) = \ln x + \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{i i!}, \tag{5}$$

$$\gamma_1(0,x) = \frac{1}{2} e^{-x} \ln^2 x + \frac{1}{2}\gamma_2(1,x). \tag{6}$$

Proof. We have

$$\begin{aligned} \int_\epsilon^x t^{-1} \ln^s t e^{-t}\, dt &= \frac{1}{s+1}\int_\epsilon^x e^{-t}\, d\ln^{s+1} t \\ &= \frac{e^{-x}\ln^{s+1} x}{s+1} - \frac{e^{-\epsilon}\ln^{s+1}\epsilon}{s+1} \\ &\quad + \frac{1}{s+1}\int_\epsilon^x \ln^{s+1} t e^{-t}\, dt \end{aligned}$$

and so

$$\begin{aligned} \gamma_s(0,x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-1}\ln^s t e^{-t}\, dt \\ &= \frac{e^{-x}\ln^{s+1} x}{s+1} + \frac{1}{s+1}\int_0^x \ln^{s+1} t e^{-t}\, dt \\ &= \frac{e^{-x}\ln^{s+1} x}{s+1} + \frac{\gamma_{s+1}(1,x)}{s+1}, \end{aligned}$$

proving equation (4) for $s = 0, 1, 2, \ldots$, the function $\gamma_{s+1}(1,x)$ being defined in the normal sense for $s = 0, 1, 2, \ldots$.

When $s = 0$, we get

$$\begin{aligned} \int_\epsilon^x t^{-1} e^{-t}\, dt &= \int_\epsilon^x e^{-t}\, d\ln t \\ &= e^{-x}\ln x - e^{-\epsilon}\ln\epsilon + \int_\epsilon^x \ln t e^{-t}\, dt \end{aligned}$$

and so

$$\begin{aligned}\gamma_0(0,x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-1}e^{-t}\,dt \\ &= e^{-x}\ln x+\int_0^x \ln t e^{-t}\,dt \\ &= e^{-x}\ln x+\gamma_1(1,x), \end{aligned} \tag{7}$$

where

$$\begin{aligned}\gamma_1(1,x) &= \int_0^x \ln t e^{-t}\,dt \\ &= \int_0^x \ln t\,d(1-e^{-t}) \\ &= (1-e^{-x})\ln x-\int_0^x t^{-1}(1-e^{-t})\,dt \\ &= (1-e^{-x})\ln x+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}. \end{aligned} \tag{8}$$

Equation (5) now follows from equations (7) and (8).

When $r=1$, we get

$$\begin{aligned}\int_\epsilon^x t^{-1}\ln t\,e^{-t}\,dt &= \frac{1}{2}\int_\epsilon^x e^{-t}\,d\ln^2 t \\ &= \frac{1}{2}e^{-x}\ln^2 x-\frac{1}{2}e^{-\epsilon}\ln^2\epsilon+\frac{1}{2}\int_\epsilon^x e^{-t}\ln^2 t\,dt \end{aligned}$$

and so

$$\begin{aligned}\gamma_1(0,x) &= \frac{1}{2}e^{-x}\ln^2 x-\frac{1}{2}\,\mathrm{N}\!-\!\lim_{\epsilon\to 0}\Big[e^{-\epsilon}\ln^2\epsilon-\int_\epsilon^x e^{-t}\ln^2 t\,dt\Big] \\ &= \frac{1}{2}e^{-x}\ln^2 x+\frac{1}{2}\gamma_2(1,x), \end{aligned}$$

the function $\gamma_2(1,x)$ being defined in the normal sense. This proves equation (6). □

Corollary 2.

$$\Gamma_0(0,x) = -\gamma-\ln x-\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}, \tag{9}$$

$$\Gamma_1(0,x) = \frac{1}{2}[\Gamma''(1)-e^{-x}\ln^2 x-\gamma_2(1,x)], \tag{10}$$

where γ denotes the Euler's constant.

Proof. We have from (3) that

$$\Gamma_0(0,x) = \Gamma(0) - \gamma_0(0,x). \tag{11}$$

It was proved in [2] that

$$\Gamma^{(s)}(0) = \frac{\Gamma^{(s+1)}(1)}{s+1} \tag{12}$$

for $s = 0, 1, 2, \ldots$. In particular

$$\Gamma(0) = \Gamma'(1) = -\gamma \tag{13}$$

and so equation (9) follows from equations (3), (5), (11) and (13).

Next, using equations (3), (6) and (12), we would have

$$\begin{aligned} \Gamma_1(0,x) &= \Gamma'(0) - \gamma_1(0,x) \\ &= \frac{1}{2}[\Gamma''(1) - e^{-x}\ln^2 x - \gamma_2(1,x)], \end{aligned}$$

proving equation (10). □

Theorem 3.

$$\gamma_0(-r,x) = -\frac{x^{-r}e^{-x}}{r} + \frac{(-1)^r}{rr!} - \frac{1}{r}\gamma_0(-r+1,x), \tag{14}$$

for $r = 1, 2, \ldots$.

In particular,

$$\gamma_0(-1,x) = -x^{-1}e^{-x} - 1 - \ln x - \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{ii!}. \tag{15}$$

Proof. We have

$$\begin{aligned} \int_\epsilon^x t^{-r-1}e^{-t}\,dt &= -\frac{1}{r}\int_\epsilon^x e^{-t}\,dt^{-r} \\ &= -\frac{x^{-r}e^{-x}}{r} + \frac{\epsilon^{-r}e^{-\epsilon}}{r} - \frac{1}{r}\int_\epsilon^x t^{-r}e^{-t}\,dt \end{aligned}$$

and so

$$\begin{aligned} \gamma_0(-r,x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-r-1}e^{-t}\,dt \\ &= -\frac{x^{-r}e^{-x}}{r} + \frac{(-1)^r}{rr!} - \frac{1}{r}\int_0^\infty t^{-r}e^{-t}\,dt \\ &= -\frac{x^{-r}e^{-x}}{r} + \frac{(-1)^r}{rr!} - \frac{1}{r}\gamma_0(-r+1,x), \end{aligned}$$

proving equation (14) for $r = 1, 2, \ldots$.

When $r = 1$, equation (14) gives

$$\gamma_0(-1, x) = -x^{-1}e^{-x} - 1 - \gamma_0(0, x)$$

and equation (15) follows on using equation (5). □

Corollary 4.

$$\Gamma_0(-1, x) = \gamma + x^{-1}e^{-x} + \ln x + \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{ii!}. \tag{16}$$

Proof. Using equations (3) and (14), we have

$$\begin{aligned} \Gamma_0(-1, x) &= \Gamma(-1) - \gamma_0(-1, x) \\ &= -1 + \gamma + x^{-1}e^{-x} + 1 + \ln x + \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{ii!}, \end{aligned}$$

since it was proved in [2] that

$$\Gamma(-r) = \frac{(-1)^r}{r!}[\phi(r) - \gamma] \tag{17}$$

for $r = 1, 2, \ldots$, where $\phi(r) = \sum_{i=1}^{r} 1/i$. Equation (16) follows. □

Theorem 5.

$$\begin{aligned} \gamma_0(-r, x) &= \sum_{i=1}^{r} \left[\frac{(-1)^r}{ir!} - \frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!} \right] \\ &\qquad + \frac{(-1)^r}{r!}\gamma_0(0, x) \qquad (18) \\ &= \sum_{i=1}^{r} \left[\frac{(-1)^r}{ir!} - \frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!} \right] \\ &\qquad + \frac{(-1)^r}{r!} \left[\ln x + \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{ii!} \right], \qquad (19) \end{aligned}$$

for $r = 1, 2, \ldots$.

In particular,

$$\gamma_0(-1, x) = -x^{-1}e^{-x} - 1 - \ln x - \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{ii!}. \tag{20}$$

Proof. When $r = 1$, equation (18) reduces to

$$\gamma_0(-1, x) = -1 - x^{-1}e^{-x} - \gamma_0(0, x)$$

and so equation (18) holds by equation (14).

Now suppose that equation (18) satisfies for some r. Then on using equation (14) and the assumption, we have

$$\begin{aligned}
\gamma_0(-r-1, x) &= -\frac{x^{-r-1}e^{-x}}{r+1} + \frac{(-1)^{r+1}}{(r+1)(r+1)!} - \frac{1}{r+1}\gamma_0(-r, x) \\
&= -\frac{x^{-r-1}e^{-x}}{r+1} + \frac{(-1)^{r+1}}{(r+1)(r+1)!} \\
&\quad + \sum_{i=1}^{r}\left[\frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{(r+1)!} - \frac{(-1)^r}{i(r+1)!}\right] \\
&\quad + \frac{(-1)^{r+1}}{(r+1)!}\gamma_0(0, x) \\
&= \sum_{i=1}^{r+1}\left[\frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{(r+1)!} - \frac{(-1)^r}{i(r+1)!}\right] \\
&\quad + \frac{(-1)^{r+1}}{(r+1)!}\gamma_0(0, x)
\end{aligned}$$

proving equation (18).

Equation (19) follows from equation (18) on using equation (5).

Equation (20) follows from equation (19) on putting $r = 1$. □

Corollary 6.

$$\begin{aligned}
\Gamma_0(-r, x) &= \sum_{i=1}^{r}\frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!} \\
&\quad - \frac{(-1)^r}{r!}\left[\gamma + \ln x + \sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}\right], \qquad (21)
\end{aligned}$$

for $r = 1, 2, \ldots$.

Proof. It follows from equations (3) and (19) that

$$\begin{aligned}\Gamma_0(-r,x) &= \Gamma(-r)-\gamma_0(-r,x)\\ &= \frac{(-1)^r}{r!}\phi(r)-\frac{(-1)^r}{r!}\gamma\\ &\quad -\sum_{i=1}^{r}\left[\frac{(-1)^r}{ir!}-\frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!}\right]\\ &\quad -\frac{(-1)^r}{r!}\left[\ln x+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}\right]\\ &= \sum_{i=1}^{r}\frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!}\\ &\quad -\frac{(-1)^r}{r!}\left[\gamma+\ln x+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}\right],\end{aligned}$$

on using equation (17). This completes the proof of the corollary. □

Theorem 7.

$$\begin{aligned}\gamma_1(-r,x) &= \sum_{i=1}^{r}\frac{(-1)^{r+i}(i-1)!}{r!}[\gamma_0(-i,x)-x^{-i}e^{-x}\ln x]\\ &\quad +\frac{(-1)^r}{r!}\gamma_1(0,x),\end{aligned}\tag{22}$$

for $r=1,2,\ldots.$

In particular,

$$\begin{aligned}\gamma_1(-1,x) &= -x^{-1}e^{-x}\ln x-\frac{1}{2}e^{-x}\ln^2 x-\frac{1}{2}\gamma_2(1,x)\\ &\quad -x^{-1}e^{-x}-1-\ln x-\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}.\end{aligned}\tag{23}$$

Proof. We have

$$\begin{aligned}\int_{\epsilon}^{x}t^{-r-1}e^{-t}\ln t\,dt &= -\frac{1}{r}\int_{\epsilon}^{x}e^{-t}\ln t\,dt^{-r}\\ &= -\frac{1}{r}x^{-r}e^{-x}\ln x+\frac{1}{r}\epsilon^{-r}e^{-\epsilon}\ln\epsilon\\ &\quad +\frac{1}{r}\int_{\epsilon}^{x}\left(t^{-r-1}e^{-t}-t^{-r}e^{-t}\ln t\right)dt\end{aligned}$$

and it follows that

$$\begin{aligned}\gamma_1(-r,x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-r-1}e^{-t}\ln t\,dt \\ &= -\frac{1}{r}x^{-r}e^{-x}\ln x+\frac{1}{r}\gamma_0(-r,x)-\frac{1}{r}\gamma_1(-r+1,x). \qquad (24)\end{aligned}$$

When $r=1$, equation (22) reduces to

$$\gamma_1(-1,x)=-x^{-1}e^{-x}\ln x+\gamma_0(-1,x)-\gamma_1(0,x)$$

and so equation (22) holds by equation (24).

Now suppose that equation (22) satisfies for some r. Then using equations (24) and our assumption, we have

$$\begin{aligned}\gamma_1(-r-1,x) &= -\frac{1}{r+1}x^{-r-1}e^{-x}\ln x+\frac{1}{r+1}\gamma_0(-r-1,x) \\ &\quad -\frac{1}{r+1}\gamma_1(-r,x) \\ &= -\frac{1}{r+1}x^{-r-1}e^{-x}\ln x+\frac{1}{r+1}\gamma_0(-r-1,x) \\ &\quad -\sum_{i=1}^{r}\frac{(-1)^{r+i}(i-1)!}{(r+1)!}[\gamma_0(-i,x)-x^{-i}e^{-x}\ln x] \\ &\quad -\frac{(-1)^r}{(r+1)!}\gamma_1(0,x) \\ &= \sum_{i=1}^{r+1}\frac{(-1)^{r+i+1}(i-1)!}{(r+1)!}[\gamma_0(-i,x)-x^{-i}e^{-x}\ln x] \\ &\quad +\frac{(-1)^{r+1}}{(r+1)!}\gamma_1(0,x)\end{aligned}$$

and so equation (22) holds for $r+1$. Equation (22) now follows by induction.

Equation (23) follows on taking $r=1$ in equation (22) and using equations (6) and (15). □

Corollary 8.

$$\begin{aligned}\Gamma_1(-1,x) &= \frac{\Gamma''(1)}{2}+\gamma+x^{-1}e^{-x}\ln x+\frac{1}{2}e^{-x}\ln^2 x+\frac{1}{2}\gamma_2(1,x) \\ &\quad +x^{-1}e^{-x}+\ln x+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!}. \qquad (25)\end{aligned}$$

Proof. Using equations (3) and (15), we have

$$\begin{aligned}\Gamma_1(-1,x) &= \Gamma'(-1)-\gamma_1(-1,x)\\ &= \frac{\Gamma''(1)}{2}+\gamma-1+x^{-1}e^{-x}\ln x+\frac{1}{2}e^{-x}\ln^2 x\\ &\quad +\frac{1}{2}\gamma_2(1,x)+x^{-1}e^{-x}+1+\ln x+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{ii!},\end{aligned}$$

since it was proved in [2] that

$$\Gamma^{(s)}(-1)=\sum_{i=1}^{s}\frac{s!}{(i+1)!}\Gamma^{(i+1)}(1)+s!(\gamma-1)$$

for $s=1,2,\ldots$. In particular

$$\Gamma'(-1)=\frac{\Gamma''(1)}{2}+\gamma-1,$$

proving equation (25). □

Theorem 9.

$$\gamma_s(-r,x)=-\frac{1}{r}x^{-r}e^{-x}\ln^s x-\frac{1}{r}\gamma_s(-r+1,x)+\frac{s}{r}\gamma_{s-1}(-r,x), \qquad (26)$$

for $r,s=1,2,\ldots$.

Proof. We have

$$\begin{aligned}\int_\epsilon^x t^{-r-1}e^{-t}\ln^s t\,dt &= -\frac{1}{r}\int_\epsilon^x e^{-t}\ln^s t\,dt^{-r}\\ &= -\frac{1}{r}x^{-r}e^{-x}\ln^s x+\frac{1}{r}\epsilon^{-r}e^{-\epsilon}\ln^s\epsilon\\ &\quad +\frac{1}{r}\int_\epsilon^x\left(st^{-r-1}e^{-t}\ln^{s-1}t-t^{-r}e^{-t}\ln^s t\right)dt\end{aligned}$$

and it follows that

$$\begin{aligned}\gamma_s(-r,x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-r-1}e^{-t}\ln^s t\,dt\\ &= -\frac{1}{r}x^{-r}e^{-x}\ln^s x+\frac{s}{r}\gamma_{s-1}(-r,x)-\frac{1}{r}\gamma_s(-r+1,x)\end{aligned}$$

proving equation (26). □

Theorem 10.

$$\gamma_s(-r,x) = -\sum_{i=0}^{s} \frac{s!}{i!r^{s+1-i}} \left[x^{-r}e^{-x}\ln^i x + \gamma_i(-r+1,x)\right] + \frac{(-1)^r s!}{r^{s+1}r!}, \qquad (27)$$

for $r, s = 1, 2, \ldots$.

In particular,

$$\gamma_s(-1,x) = -\sum_{i=0}^{s} \frac{s!}{i!}\left[x^{-1}e^{-x}\ln^i x + \frac{e^{-x}\ln^{i+1} x}{i+1} + \frac{\gamma_{i+1}(1,x)}{i+1}\right] - s!, \qquad (28)$$

for $s = 1, 2, \ldots$.

Proof. When $s = 1$, equation (27) reduces to

$$\begin{aligned}\gamma_1(-r,x) &= -\frac{1}{r^2}x^{-r}e^{-x} - \frac{1}{r}x^{-r}e^{-x}\ln x - \frac{1}{r^2}\gamma_0(-r+1,x)\\ &\quad -\frac{1}{r}\gamma_1(-r+1,x) + \frac{(-1)^r}{r^2 r!}\end{aligned}$$

and so equation (27) holds by equations (14) and (26) when $s = 1$ for $r = 1, 2, \ldots$.

Now suppose that equation (27) satisfies for some s and $r = 1, 2, \ldots$. Then using equation (26) and our assumption, we have

$$\begin{aligned}\gamma_{s+1}(-r,x) &= -\frac{1}{r}x^{-r}e^{-x}\ln^{s+1} x - \frac{1}{r}\gamma_{s+1}(-r+1,x)\\ &\quad +\frac{s+1}{r}\gamma_s(-r,x)\\ &= -\frac{1}{r}x^{-r}e^{-x}\ln^{s+1} x - \frac{1}{r}\gamma_{s+1}(-r+1,x)\\ &\quad -\sum_{i=0}^{s}\frac{(s+1)!}{i!r^{s+2-i}}\left[x^{-r}e^{-x}\ln^i x + \gamma_i(-r+1,x)\right]\\ &\quad +\frac{(-1)^r(s+1)!}{r^{s+2}r!}\\ &= -\sum_{i=0}^{s+1}\frac{(s+1)!}{i!r^{s+2-i}}\left[x^{-r}e^{-x}\ln^i x + \gamma_i(-r+1,x)\right]\\ &\quad +\frac{(-1)^r(s+1)!}{r^{s+2}r!}\end{aligned}$$

and so equation (27) holds for $s+1$ and $r = 1, 2, \ldots$. Equation (27) now follows by induction.

Equation (28) follows on taking $r = 1$ in equation (27) and using equation (4). □

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References

[1] J.G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.*, **7** (1959-60), 291-398.

[2] B. Fisher and A. Kılıçman, Some results on the Gamma function for negative integers, *Appl. Math. Inf. Sci.*, **6**, No 2 (2012), 173-176.

[3] B. Fisher and Y. Kuribayashi, Neutrices and the Gamma function, *J. Fac. Ed. Tottori Univ.*, **36**, No 1-2 (1987), 1-7.

[4] I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry,* Oliver & Boyd (1956).