SUFFICIENT CONDITIONS FOR
STABLE PROPERTIES OF A GRAPH

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Abstract: Using the existing upper bounds for the sum of the squares of degrees of graphs, in this note we present sufficient conditions for stable properties of graphs.

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We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [2]. For a graph $G = (V, E)$, we use $n$ and $e$ to denote its order $|V|$ and size $|E|$, respectively. We use $G^c := (V, E^c)$ to denote its complement, where $E^c := \{ xy : x, y \in V, x \neq y, xy \notin E \}$. We use $\mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$ to denote the eigenvalues of the graph $G$. A graph $G$ is Hamiltonian if $G$ has a Hamiltonian cycle, a cycle containing all the vertices of $G$. A graph $G$ is traceable if $G$ has a Hamiltonian path, a path containing all the vertices of $G$. The concept of stability was introduced by Bondy and Chvátal in [1]. Let $P$ be a property defined on all graphs of order $n$. Let $k$ be a nonnegative integer. The $P$ is said to be $k$-stable if whenever $G + uv$ has property $P$ and $d_G(u) + d_G(v) \geq k(n, P)$, where $uv \notin E$, then $G$ itself has property $P$. It is well known that the Hamiltonicity
and traceability are respectively $n$ - stable and $(n - 1)$ - stable. The $k$ - closure of a graph $G$, denoted $cl_k(G)$, is a graph obtained from $G$ by recursively joining two nonadjacent vertices such that their degree sum is at least $k$. For a graph $G$ and a real number $r$, we define

$$S_r(G) := \sum_{v \in V(G)} d_G^r(v).$$

We use $C(n, r)$ to denote the number of $r$-combinations of a set with $n$ distinct elements.

The following results were obtained by Fiedler and Nikiforov in [5].

**Theorem 1.** Let $G$ be a graph of order $n$.

(i) If $\mu_n(G^c) \leq \sqrt{n - 1}$, then $G$ contains a Hamiltonian path unless $G = K_{n-1} + v$.

(ii) If $\mu_n(G^c) \leq \sqrt{n - 2}$, then $G$ contains a Hamiltonian cycle unless $G = K_{n-1} + e$.

Zhou [9] and Li [6] further obtained more spectral sufficient conditions for the Hamiltonian properties of graphs. Motivated by the results in [5], [9], and [6], Li in [7] presented spectral sufficient conditions for the stable properties of graphs. In this note, we will present new sufficient conditions for the stable properties of graphs.

We need the following results in our proofs. The first one is a theorem proved by Székely, Clark, and Entringer in [8].

**Theorem 2.** [8] Let $G$ be a graph of order $n$ and suppose $p$ is a positive integer. Then

$$S_p(G) \leq \left( S_{\frac{1}{p}}(G) \right)^p.$$

The second one is a theorem proved by de Caen in [3].

**Theorem 3.** [3] Let $G$ be a graph of order $n$ and size $e$. Then

$$S_2(G) \leq e \left( \frac{2e}{n-1} + n - 2 \right).$$

The last one is a theorem (Theorem 3.7, Page 42) proved by Das in [4].
Theorem 4. [4] Let $G$ be a graph of order $n$ and size $e$. Then
\[ S_2(G) \leq e(e + 1). \]

Below is the theorem we will prove in this paper.

Theorem 5. Let $G$ be a graph of order $n$ and size $e$. Suppose that $P$ is a $r(n, P)$ - stable property and the complete graph $K_n$ of order $n$ has property $P$. Then:

1. If
\[ S_2^1(G^c) < \sqrt{(2n - r(n, P) - 1)(2n - r(n, P) - 2)}, \]
then $G$ has property $P$.

2. If
\[ e(G) > C(n, 2) - (n - 1)(n - r(n, P) + 1)/2, \]
then $G$ has property $P$.

3. If
\[ e(G) > C(n, 2) - (2n - r(n, P) - 2), \]
then $G$ has property $P$.

Next, we will present the proof of Theorem 5. Some ideas and techniques from [5] will be used in our proofs.

Proof of Theorem 5. Let $G$ be a graph satisfying the conditions in Theorem 5 and $G$ does not have property $P$. Then $H := d_{r(n, P)}(G)$ does not have property $P$ and therefore $H$ is not $K_n$. Thus there exist two vertices $x$ and $y$ in $V(H)$ such that $xy \notin E(H)$ and for any pair of nonadjacent vertices $u$ and $v$ in $V(H)$ we have $d_H(u) + d_H(v) \leq r(n, P) - 1$. Hence for any pair of adjacent vertices $u$ and $v$ in $V(H^c)$ we have that $d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq 2n - r(n, P) - 1$. So
\[
\sum_{uv \in E(H^c)} (d_{H^c}(u) + d_{H^c}(v)) \geq (2n - r(n, P) - 1)e(H^c).
\]

Moreover, we have that
\[
\sum_{v \in V(H^c)} d_{H^c}^2(v) = \sum_{uv \in E(H^c)} (d_{H^c}(u) + d_{H^c}(v)) \geq (2n - r(n, P) - 1)e(H^c).
\]
Since $H$ is not complete and there exist two nonadjacent vertices, say $x$ and $y$, in $V(H)$ such that $d_H(x) + d_H(y) \leq r(n, P) - 1$. Then $e(H) \leq C(n - 2, 2) + r(n, P) - 1$. Thus

$$e(H^c) \geq C(n, 2) - C(n - 2, 2) - r(n, P) + 1 = 2n - r(n, P) - 2.$$ 

(1) Applying Theorem 2 with $p = 2$ to the graph $H^c$, we have that

$$\left( S_1^2(H^c) \right)^2 = \left( \sum_{v \in V(H^c)} d_{H^c}^{1/2}(v) \right)^2 \geq \sum_{v \in V(H^c)} d_{H^c}^2(v) \geq (2n - r(n, P) - 1)e(H^c) \geq (2n - r(n, P) - 1)(2n - r(n, P) - 2).$$

Since $H^c$ is a subgraph of $G^c$,

$$\left( S_1^2(G^c) \right)^2 \geq \left( S_1^2(H^c) \right)^2 \geq (2n - r(n, P) - 1)(2n - r(n, P) - 2).$$

Namely,

$$S_1^2(G^c) \geq \sqrt{(2n - r(n, P) - 1)(2n - r(n, P) - 2)},$$

a contradiction.

(2) Applying Theorem 3 to the graph $H^c$, we have that

$$e(H^c) \left( \frac{2e(H^c)}{n - 1} + n - 2 \right) \geq (2n - r(n, P) - 1)e(H^c).$$

Since $H$ is not complete, $e(H^c) > 0$. Thus

$$\frac{2e(H^c)}{n - 1} + n - 2 \geq (2n - r(n, P) - 1).$$

Namely,

$$e(H^c) \geq (n - 1)(n - r(n, P) + 1)/2.$$ 

Since $H^c$ is a subgraph of $G^c$,

$$e(G^c) \geq e(H^c) \geq (n - 1)(n - r(n, P) + 1)/2.$$ 

Thus

$$e(G) = C(n, 2) - e(G^c) \leq C(n, 2) - (n - 1)(n - r(n, P) + 1)/2,$$

a contradiction.
(3) Applying Theorem 4 to the graph $H^c$, we have that
$$e(H^c)(e(H^c) + 1) \geq (2n - r(n, P) - 1)e(H^c).$$

Since $H$ is not complete, $e(H^c) > 0$. Thus
$$e(H^c) + 1 \geq (2n - r(n, P) - 1).$$

Namely,
$$e(H^c) \geq (2n - r(n, P) - 2).$$

Since $H^c$ is a subgraph of $G^c$,
$$e(G^c) \geq e(H^c) \geq (2n - r(n, P) - 2).$$

Thus,
$$e(G) = C(n, 2) - e(G^c) \leq C(n, 2) - (2n - r(n, P) - 2),$$
a contradiction.

Therefore, we complete the proof of Theorem 5.

Since the Hamiltonicity and traceability of a graph of order $n$ are respectively $n$-stable and $(n - 1)$-stable, item (1) in Theorem 5 has the following corollaries.

**Corollary 6.** Let $G$ be a graph of order $n$. If
$$S_{\frac{1}{2}}(G^c) < \sqrt{(n - 1)(n - 2)},$$
then $G$ is Hamiltonian.

**Corollary 7.** Let $G$ be a graph of order $n$. If
$$S_{\frac{1}{2}}(G^c) < \sqrt{n(n - 1)},$$
then $G$ is traceable.

Notice that from (2) and (3) in Theorem 5 we can also get corollaries for the Hamiltonicity and traceability of a graph. However, the results in the corollaries are not stronger than the existing edge conditions (see [2]) for the Hamiltonicity and traceability of a graph. So it is not necessary for us to list the corollaries here.

Bondy and Chvátal in [1] proved a variety of theorems on stable properties of a graph. Using those theorems and Theorem 5, we can obtain sufficient conditions for those stable properties of a graph. Those results show that Theorem 5 has a lot of applications when we investigate on the stable properties of a graph. Since the sufficient conditions for the stable properties of a graph can be easily obtained, the details for those results are omitted here.
References


