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DENSITY OF $C_0^\infty(\mathbb{R}^n)$ IN $W^{1,p(x)}(\mathbb{R}^n)$ WITH ASSUMPTION DENSITY OF $C^1(\mathbb{R}^n)$

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Abstract: In this paper, we present a sufficient condition for the density of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p(x)}(\mathbb{R}^n)$ with assumption that $p(x)$ satisfying a condition such that $C^1(\mathbb{R}^n)$ is dense in$W^{1,p(x)}(\mathbb{R}^n)$. The origin of our work comes from a similar question of Hästö [5] under the density of continuous or Hölder continuous whether it is possible to deduce the density of smooth functions.

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1. Introduction

Variable exponent analysis has become a growing field of interest since Kováčik and Rákosník paper [6]. Many basic properties of Lebesgue and Sobolev spaces were shown in their paper. Variable Lebesgue spaces are a generalization of Lebesgue spaces where we allow the exponent to be a measurable function and thus the exponent may vary, and has found numerous important applications with Sobolev spaces. Examples are fluid dynamics, elasticity theory, differential equations with non-standard growth conditions and image restoration (cf. [2,

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8, 7]). On the basic properties of the variable exponent Lebesgue and Sobolev spaces we refer to [3, 6].

Let $p : \Omega \to [1, \infty)$ be a measurable bounded function, called a variable exponent on Ω. We also define $p^+ = \text{esssup}_{x\in\mathbb{R}^n} p(x)$ and $p^- = \text{essinf}_{x\in\mathbb{R}^n} p(x)$.

The class $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support in Ω is dense in the spaces $L^{p(.)}(\Omega)$, which was established among the first basic properties of these spaces in [6].

Variable exponent Lebesgue spaces do not have the mean continuity property. If p is continuous and non-constant function in an open ball B, then there exists a function $L^{p(.)}(\Omega)$ such that $u(x+h) \notin L^{p(.)}(\Omega$ for $h \in \mathbb{R}^n$ with arbitrary small norm.

We define the *variable exponent Lebesgue space* $L^{p(.)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(.)}(u) = \int_\Omega \mid u(x) \mid^{p(x)} dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(.)}}(\Omega) = \|u\|_{p(.)} = \inf\{\lambda > 0 : \varrho_{p(.)}(u/\lambda) \leq 1\}.$$

If p is a constant function, then the variable exponent Lebesgue spaces coincides with the classical Lebesgue space. One central property of these spaces reads: If $p^+ < \infty$ and (u_i) is a sequence of functions in L^p, then $\|u_i\|_{p(.)} \to 0$ if and only if $\varrho_{p(.)}(u_i) \to 0$. This and many other basic results were proven in [6].

The *variable exponent Sobolev space* $W^{1,p(.)}(\Omega)$ is the subspace of functions $u \in L^{p(.)}((\Omega))$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(.)}(\Omega)$. The norm $\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$ makes $W^{1,p(.)}(\Omega)$ a Banach space. We also define a modular in the Sobolev space by $\varrho_{1,p(.)}(u) = \varrho_{p(.)}(u) + \varrho_{p(.)}(|\nabla u|)$.

It is known that in general, $C^\infty(\Omega)$ may not be dense in $W^{1,p(.)}(\Omega)$. The first example is given by Zhikov [10, 11] as follows:

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\},$$

$$p(x) = \begin{cases} \alpha_1 & \text{if} \quad x_1x_2 > 0, \\ \alpha_2 & \text{if} \quad x_1x_2 < 0, \end{cases}$$

where $1 < \alpha_1 < 2 < \alpha_2$, then $C^\infty(\Omega)$ is not dense in $W^{1,p(.)}(\Omega)$.

Edmunds and Rakosnik [4] proved denseness under some special monotonicity type condition on $p(x)$. In [1, 9] another type of sufficient condition to ensure the density of $C^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ is the so-called logarithmic Holder continuity, which is expressed as

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad \text{for all} \quad x, y \in \Omega, \quad |x-y| \leq \frac{1}{2}.$$

2. Density

After the Zhikov nondensity example it became more significant to look at less smoothness density of functions in variable exponent Sobolev spaces. Hasto [5] made some work in this direction and asked the following.

Question 2.1. *Suppose that $C(\Omega)$ or $C^\alpha(\Omega)$ is dense in $W^{1,p(.)}(\Omega)$. Is it then true that smooth functions are dense in $W^{1,p(.)}(\Omega)$?*

Since the derivative of a Sobolev function may be unbounded it seems for convolution the assumption of density of continuous functions does not help. Similarly, we assume density of $C^1(\mathbb{R}^n)$ in $W^{1,p(.)}(\mathbb{R}^n)$ instead of density $C(\Omega)$ and show the following theorem.

Theorem 2.2. *Let $p(.) \in P(\mathbb{R}^n)$ be a bounded variable exponent. Suppose that $C^1(\mathbb{R}^n)$ is dense in $W^{1,p(.)}(\mathbb{R}^n)$, then $C_0^\infty(\mathbb{R}^n)$ is dense $W^{1,p(.)}(\mathbb{R}^n)$.*

Proof. We first show that $C_0^1(\mathbb{R}^n)$ is dense in $C^1(\mathbb{R}^n)$ on the norm of $W^{1,p(.)}(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n)$ and $\eta \in C_0^1(\mathbb{R}^n)$ be a compactly supported function which equals 1 near the origin, and consider the functions $u_R(x) = u(x)\eta(\frac{x}{R})$ for $R > 0$. It is easy to see $u_R \in C_0^1(\mathbb{R}^n)$. Since $p^+ < \infty$, instead of $\|u_R - u\|_{p(.)} \to 0$ norm convergence, by using Lebesgue dominated convergence in the modular we see that

$$\int_{\mathbb{R}^n} |u_R - u|^{p(x)}\, dx \to 0$$

holds as $R \to \infty$.

We now show $\|(\nabla u_R)(x) - (\nabla u)(x)\|_{p(.)} \to 0$ as $R \to \infty$. By product rule we get

$$\nabla u_R = (\nabla u)(x)\eta(\frac{x}{R}) + \frac{1}{R}u(x)(\nabla\eta)(\frac{x}{R}).$$

Let us prove the convergence:

$$\begin{aligned}\|(\nabla u_R)(x) - (\nabla u)(x)\|_{p(.)} &= \|(\nabla u)(x)\eta(\frac{x}{R}) + \frac{1}{R}u(x)(\nabla\eta)(\frac{x}{R}) - (\nabla u)(x)\|_{p(.)}\\ &\leq \|(\nabla u)(x)\eta(\frac{x}{R}) - (\nabla u)(x)\|_{p(.)}\\ &+ \frac{1}{R}\|u(x)(\nabla\eta)(\frac{x}{R})\|_{p(.)}.\end{aligned}$$

Again since $p+ < \infty$, we use modular convergence for the first term and by using Lebesgue dominated convergence the first term goes to zero as $R \to \infty$ and it is obvious the second term goes to zero as $R \to \infty$.

We now show that $C_0^\infty(\mathbb{R}^n)$ is dense in $\in C^1(\mathbb{R}^n)$.

Since $u \in C_0^1(\mathbb{R}^n)$ has compact support, we denote this set by spt $u = K$ and from K we define larger compact set as follows:

$$K_{\delta(\epsilon)} = \{x \in \mathbb{R}^n : dist(x, K) \leq \delta(\epsilon)\},$$

where $\delta(\epsilon)$ will be chosen later and will be taken such that $0 < \delta(\epsilon) < 1$.

With this condition ∇u and u are uniformly continuous on $K_{\delta(\epsilon)}$.

Let $0 < \epsilon < 1$, then we get

$$|u(x-y) - u(x)| < \epsilon \quad \text{as} \quad |y| < \delta_1(\epsilon)$$

and

$$|\nabla u(x-y) - \nabla u(x)| < \epsilon \quad \text{as} \quad |y| < \delta_1(\epsilon)$$

in which $\delta_1(\epsilon), \delta_2(\epsilon) < \epsilon$ can be chosen. We get

$$\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\},$$

as we promised.

Let $\phi_{\phi_{\delta(\epsilon)}}$ be a standard mollifier. Then

$$\|u * \phi_{\delta(\epsilon)} - u\|_{W^{1,p(.)}(\mathbb{R}^n)} = \|u * \phi_{\delta(\epsilon)} - u\|_{p(.)} + \|\nabla u * \phi_{\delta(\epsilon)} - \nabla u\|_{p(.)}.$$

Since $p^+ < \infty$, we show convergence in the modular:

$$\int_{\mathbb{R}^n} \left| \int_{B(0,\delta(\epsilon))} |u(x-y) - u(x)| \phi_{\delta(\epsilon)}(y) dy \right|^{p(x)} dx \leq \epsilon^{p^-} |K_{\delta(\epsilon)}|$$

and

$$\int_{\mathbb{R}^n} \left| \int_{B(0,\delta(\epsilon))} |\nabla u(x-y) - \nabla u(x)| \phi_{\delta(\epsilon)}(y) dy \right|^{p(x)} dx \leq \epsilon^{p^-} |K_{\delta(\epsilon)}|$$

are obtained. Since $|K_{\delta(\epsilon)}| < \infty$ the convergence holds.

Thus we show that $C_0^1(\mathbb{R}^n)$ is dense in $C^1(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is dense $C_0^1(\mathbb{R}^n)$. By the assumption of theorem, the density that we have showed and triangular inequality of norm easily show that $C_0^\infty(\mathbb{R}^n)$ is also dense in $W^{1,p(.)}(\mathbb{R}^n)$. $\square$

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