LOCAL WELL-POSEDNESS OF THE REGULARIZED
rBO-ZK EQUATION IN SOBOLEV SPACES
OF NEGATIVE INDICES

Fabián Sánchez Salazar
Department of Mathematics
Universidad Nacional de Colombia
Kra. 30 No. 45-03 Bogotá, D.C., COLOMBIA

Abstract: In this paper, we study the local well-posedness of the regularized rBO-ZK equation, defined by

$$u_t + a(u^2)_x + (b\mathcal{H}u_t + u_{yy})_x - \mu(-\Delta)^\alpha u = 0,$$

where $\mathcal{H}$ is the Hilbert transform with respect to x and a, b, μ and α are real numbers, with $b > 0$, $\mu > 0$ and $\alpha > \frac{1}{2}$. We show that the associated Cauchy problem is locally well posed in Sobolev space $H^s(\mathbb{R}^2)$ for $s > -2\alpha + 1$.

AMS Subject Classification: 35A01, 35G25, 42B35

Key Words: regularized Benjamin-Ono-Zakharov-Kuznetsov equation, Cauchy problem, negative indices, global and local well-posedness

1. Introduction

In this paper, we are interested in the proof the local well-posedness in Sobolev space with negative indices to the following Cauchy problem

$$\begin{cases} u_t + a(u^2)_x + (b\mathcal{H}u_t + u_{yy})_x - \mu(-\Delta)^\alpha u = 0 \\ u(0, x, y) = \varphi(x, y) \end{cases}, \quad (1.1)$$

where $\mathcal{H}$ is the Hilbert transform with respect to x , $(-\Delta)^\alpha = (-\partial_x^2 - \partial_y^2)^\alpha$ and a, b, μ, α are real numbers, with $b > 0$, $\mu > 0$, $\alpha > \frac{1}{2}$.

The non-linear evolution equations play a very important role in different areas of science and engineering. It is worth mentioning some of them: fluid mechanics, plasma physics, fiber optics, solid state physics, chemical kinetics, among others. For example, the Korteweg-de Vries equation

$$u_t = u_{xxx} + uu_x, \quad (x, t) \in \mathbb{R}^2, \quad (1.2)$$

that models the behavior of waves in shallow water channels. Other one-dimensional equations are: the one, the introduced by Benjamin in [2] and by Ono in [8],

$$u_t + \mathcal{H}u_{xx} + uu_x = 0, \quad (1.3)$$

which models the internal waves in stratified fluids deep; also the regularized Benjamin-Ono equation

$$u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0, \quad (1.4)$$

where $u = u(x, t)$ is a real valued function, with $x, t \in \mathbb{R}$. This equation is a model for the evolution in time of waves with large crests at the interface between two fluids immiscibles. There are bidimensional versions extending the above equations. In the case of the KdV equation we have the Kadomtsev-Petviashvili equation, see [5],

$$(u_t + auu_x + u_{xxx})_x + u_{yy} = 0, \quad (1.5)$$

which describes waves in thin films of high surface tension. Another one is the Zakharov-Kuznetsov equation

$$u_t = (u_{xx} + u_{yy})_x + uu_x, \quad (1.6)$$

arising in the study of geophysical fluid dynamics in isotropic sets.

For the case of the equations BO (1.3) and rBO (1.4) we have the following bi-dimensional versions, the ZK-BO equation

$$u_t = (\mathcal{H}u_x + u_{yy})_x + uu_x, \quad (1.7)$$

and the rBO-ZK equation

$$u_t + a(u^n)_x + (b\mathcal{H}u_t + u_{yy})_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t > 0. \quad (1.8)$$

Bona in [6], Mammeri in [7] and Angulo in [1] studied the well-posedness of the rBO equation (1.4). Angulo in [1] also examined the existence and stability of travelling periodic waves of rBO equation. In [4], it was studied the well-

and ill-posedness of the rBO in weighted Sobolev spaces, in particular, there it is obtained a result on unique continuation property of this equation. For equation (1.8) has been shown the local well-posedness in Sobolev space and the local well and ill-posedness in weighted Sobolev spaces, also it has been proved a property of unique continuation (see Sánchez [9]). In [10] it is proved a global well-posedness result to this equation for small data and interesting scattering property of these global solutions. For the equation (1.8) we have the following result.

Theorem 1.1. *The initial value problem associated to (1.8) is locally well-posed in $H^s(\mathbb{R}^2)$ for $s > 1$.*

Proof. We consider the integral equation associated to (1.8)

$$u = E(t)\varphi + \int_0^t E(t - \tau)B(u^n(\tau)) d\tau, \quad (1.9)$$

where

$$E(t) = e^{tA}, \quad A = -\partial_x(1 + \mathcal{H}\partial_x)^{-1}\partial_y^2, \quad \text{and} \quad B = -\partial_x(1 + \mathcal{H}\partial_x)^{-1}. \quad (1.10)$$

Since B is a bounded operator in $H^s(\mathbb{R}^2)$, and it is a Banach algebra for $s > 1$, using the Banach fixed point theorem we conclude the result. $\square$

The main results of this paper read as follows.

Theorem 1.2. *Let $\mu > 0$ fixed and $s > -2\alpha + 1$, then for all $\varphi \in H^s(\mathbb{R}^2)$, there exist $T = T(\|\varphi\|_{H^s(\mathbb{R}^2)}) > 0$, a space $\mathcal{X}_T^s \subset C([0, T]; H^s(\mathbb{R}^2))$ with continuous inclusion, and a unique solution $u(t)$ of (1.1) such that $u(0) = \varphi(x, y)$. Moreover, $u \in C([0, T] : H^\infty(\mathbb{R}^2))$ and the map solution*

$$H^s(\mathbb{R}^2) \longrightarrow \mathcal{X}_T^s \cap C([0, T] : H^s(\mathbb{R}^2))$$

is smooth.

Theorem 1.3. *Let $\alpha = 2$, $s > 1$, $\mu > 0$. Then, the equation (1.1) is globally well-posed for data initial in $H^s(\mathbb{R}^2)$.*

2. Estimates

In order to prove Theorem 1.2, we make the assumption $s \in (-2\alpha + 1, 0]$. The proof by $s \in [0, 2\alpha - 1)$ is analogous. The proof idea is to use a contraction argument on the integral equation associated to (1.1)

$$u(t) = \Phi(u_\mu(t)) := U_\mu(t)\widehat{u_0} + \int_0^t U_\mu(t-t') \frac{i\xi}{1+|\xi|} \widehat{u^2(t')} dt', \quad (2.1)$$

in some well-adapted function space. The semigroup associated to the linear part of (1.1) it is defined by

$$U_\mu(t) = e^{\frac{i\xi\eta^2}{1+|\xi|}t - \mu t(\xi^2 + \eta^2)^\alpha}.$$

For this purpose we use dissipative methods as applied by Dix [3] to study the initial value problem associated to the Burgers equation. For $0 < T \leq 1$, we define

$$\mathcal{X}_T^s(\mathbb{R}^2) = \{u \in C([0, T] : H^s(\mathbb{R}^2)) \mid \|u\|_{\mathcal{X}_T^s} < \infty\}, \quad (2.2)$$

where

$$\|u\|_{\mathcal{X}_T^s} = \sup_{t \in [0, T]} (\|u(t)\|_{H^s(\mathbb{R}^2)} + t^{|s|/(2\alpha)} \|u(t)\|_{L^2(\mathbb{R}^2)}). \quad (2.3)$$

First, we show the following lemmas.

Lemma 2.1. *For any $s, \mu > 0$, we have that*

$$\sup_{(x,y) \in \mathbb{R}^2} \left| (x^2 + y^2)^s e^{-\mu t(x^2 + y^2)^\alpha} \right| \leq \left(\frac{s}{\mu t \alpha} \right)^{\frac{s}{\alpha}} e^{-\frac{s}{\alpha}}.$$

Proof. Let $r = x^2 + y^2$ and consider $f(r) = r^s e^{-\mu t r^\alpha}$. □

Lemma 2.2. *Let $\mu > 0$, $s \in \mathbb{R}$ y $\varphi \in H^s(\mathbb{R}^2)$, then*

$$\|U_\mu(t)\varphi\|_{H^{s+\lambda}}^2 \leq \left(1 + \left(\frac{\lambda}{\mu t \alpha} \right)^{\lambda/\alpha} e^{-\lambda/\alpha} \right) \|\varphi\|_{H^s}^2.$$

Proof. By Lemma 2.1, we have that

$$\begin{aligned}\|U_\mu(t)\varphi\|_{H^{s+\lambda}}^2 &= \int_{\mathbb{R}^2} (1 + \xi^2 + \eta^2)^{s+\lambda} \left| e^{\frac{i\xi\eta^2}{1+|\xi|}t - \mu(\xi^2 + \eta^2)^\alpha t} \widehat{\varphi}(\xi, \eta) \right|^2 d\xi d\eta \\ &\leq \left(1 + \sup_{(\xi, \eta) \in \mathbb{R}^2} (\xi^2 + \eta^2)^\lambda e^{-t\mu(\xi^2 + \eta^2)^\alpha} \right)^2 \|\varphi\|_{H^s}^2 \\ &\leq \left(1 + \left(\frac{\lambda}{\mu t \alpha} \right)^{\lambda/\alpha} e^{-\lambda/\alpha} \right)^2 \|\varphi\|_{H^s}^2.\end{aligned}$$

□

We will turn our attention to estimate the linear part.

Proposition 2.1. *Let $\mu > 0$, $0 < T < 1$, $s \leq 0$ and $\varphi \in H^s(\mathbb{R}^2)$, then*

$$\sup_{t \in [0, T]} \|U_\mu(t)\varphi\|_s \leq \|\varphi\|_{H^s} \quad (2.4)$$

and

$$\sup_{t \in [0, T]} t^{|s|/(2\alpha)} \|U_\mu(t)\varphi\|_{L^2} \leq C(s, \mu, \alpha) \|\varphi\|_{H^s}. \quad (2.5)$$

Proof. The proof of (2.4) is immediately from Lemma 2.2 with $\lambda = 0$. Now, for proof the inequality (2.5), we first observe, since $t < 1$ y $s < 0$, that

$$t^{|s|/2\alpha} \leq (1 + t^{1/\alpha}(\xi^2 + \eta^2))^{\frac{|s|}{2}} (1 + \xi^2 + \eta^2)^{\frac{s}{2}}, \quad (2.6)$$

for all $t \in [0, T]$. From Lemma 2.2, the inequality (2.6) and the Plancherel theorem, we deduce that

$$\begin{aligned}t^{|s|/(2\alpha)} \|U_\mu(t)\varphi\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} \left| (1 + t^{\frac{1}{\alpha}}(\xi^2 + \eta^2))^{\frac{|s|}{2}} e^{-t\mu(\xi^2 + \eta^2)^\alpha} \right|^2 \\ &\quad \times \left| (1 + \xi^2 + \eta^2)^{\frac{s}{2}} \widehat{\varphi} \right|^2 d\xi d\eta \\ &\leq \left[1 + t^{|s|/\alpha} \left(\sup_{(\xi, \eta) \in \mathbb{R}^2} (\xi^2 + \eta^2)^{\frac{|s|}{2}} e^{-t\mu(\xi^2 + \eta^2)^\alpha} \right)^2 \right] \|\varphi\|_{H^s}^2 \\ &\leq \left(1 + t^{|s|/\alpha} \left[\frac{|s|}{2\mu t \alpha} \right]^{|s|/\alpha} \right) \|\varphi\|_{H^s}^2.\end{aligned}$$

□

Next step is to derive the bilinear estimates.

Theorem 2.1. *Let $0 < t \leq T < 1$ and $s \in (-2\alpha + 1)$, then*

$$\left\| \int_0^t U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(uv) dt' \right\|_{\mathcal{X}_T^s} \leq C(s, \mu, \alpha) T^{\frac{s+2\alpha-1}{2\alpha}} \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}.$$

Proof. Let $0 \leq t \leq T \leq 1$ and $s < 0$. The Young inequality implies that

$$\|\widehat{u} * \widehat{v}\|_{L^\infty} \leq \frac{\|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}}{t^{\frac{|s|}{\alpha}}}. \quad (2.7)$$

By (2.7) and the Minkowski inequality, we have that

$$\begin{aligned} & \left\| \int_0^t U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(uv) dt' \right\|_{H^s} \\ & \leq \int_0^t \|U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(uv)\|_{H^s} dt' \\ & \leq \int_0^t \|(1 + \xi^2 + \eta^2)^{\frac{s}{2}} e^{(t-t')(\frac{i\xi\eta^2}{1+|\xi|} - \mu(\xi^2 + \eta^2))^\alpha} \frac{i\xi}{1+|\xi|} \widehat{u(t')v(t')}\|_{L^2} dt' \\ & \leq \int_0^t \|(\xi^2 + \eta^2)^{\frac{s}{2}} e^{-(t-t')\mu(\xi^2 + \eta^2)^\alpha} \widehat{u(t')v(t')}\|_{L^2} dt' \\ & \leq \int_0^t \|(\xi^2 + \eta^2)^{\frac{s}{2}} e^{-(t-t')\mu(\xi^2 + \eta^2)^\alpha}\|_{L^2} \frac{\|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}}{t^{\frac{|s|}{\alpha}}} dt' \\ & \leq \left(\int_0^t \|(\xi^2 + \eta^2)^{\frac{s}{2}} e^{-t'\mu(\xi^2 + \eta^2)^\alpha}\|_{L^2} \frac{1}{|t-t'|^{\frac{|s|}{\alpha}}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}. \end{aligned} \quad (2.8)$$

We use a change of variable to deduce that

$$\begin{aligned} \|(\xi^2 + \eta^2)^{\frac{s}{2}} e^{-t'\mu(\xi^2 + \eta^2)^\alpha}\|_{L^2}^2 &= \int_{\mathbb{R}^2} |(\xi^2 + \eta^2)^{s/2} e^{-\mu t'(\xi^2 + \eta^2)^\alpha}|^2 d\xi d\eta \\ &= \int_{\mathbb{R}^2} |(\xi^2 + \eta^2)^{s/2} e^{-\mu((t^{1/(2\alpha)}\xi)^2 + (t^{1/(2\alpha)}\eta)^2)^\alpha}|^2 d\xi d\eta \\ &\leq t'^{\frac{-s-1}{\alpha}} \int_{\mathbb{R}^2} |(\xi^2 + \eta^2)^{s/2} e^{-\mu(\xi^2 + \eta^2)^\alpha}|^2 d\xi d\eta \\ &\leq C(s, \mu, \alpha) t'^{\frac{-s-1}{\alpha}}. \end{aligned} \quad (2.9)$$

Therefore, we get from (2.8) and (2.9) that

$$\begin{aligned}
 & \left\| \int_0^t U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(uv) dt' \right\|_{H^s} \\
 & \leq C \left(\int_0^t (t')^{\frac{-s-1}{2\alpha}} |t-t'|^{\frac{-|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s} \\
 & \leq t^{\frac{s+2\alpha-1}{2\alpha}} \left(\int_0^1 (t')^{\frac{-s-1}{2\alpha}} |1-t'|^{\frac{-|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s} \\
 & \leq CT^{\frac{s+2\alpha-1}{2\alpha}} \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}.
 \end{aligned} \tag{2.10}$$

Now, we will estimate the second term of (2.3). From the following inequality

$$\|e^{-\mu t'(\xi^2+\eta^2)^\alpha}\|_{L^2}^2 = t^{\frac{-1}{\alpha}} \int_{\mathbb{R}^2} |e^{-\mu(\xi^2+\eta^2)^\alpha}|^2 d\xi d\eta \leq Ct^{\frac{-1}{\alpha}}, \tag{2.11}$$

and using a change of variable, we deduce that, for all $0 \leq t \leq T$,

$$\begin{aligned}
 & t^{\frac{|s|}{2\alpha}} \left\| \int_0^t U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(uv) dt' \right\|_{L^2} \\
 & \leq t^{\frac{|s|}{2\alpha}} \int_0^t \|e^{-\mu(t-t')(\xi^2+\eta^2)^\alpha}\|_{L^2} \|\widehat{u(t')} * \widehat{v(t')}\|_{L^\infty} dt' \\
 & \leq t^{\frac{|s|}{2\alpha}} \left(\int_0^t \|e^{-\mu(\xi^2+\eta^2)^\alpha} t'\|_{L^2} \frac{1}{|t-t'|^{\frac{|s|}{\alpha}}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s} \\
 & \leq t^{\frac{|s|}{2\alpha}} \left(\int_0^t (t')^{\frac{-1}{2\alpha}} |t-t'|^{\frac{-|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s} \\
 & \leq t^{\frac{s-1+2\alpha}{2\alpha}} \left(\int_0^1 (t')^{\frac{-1}{2\alpha}} |1-t'|^{\frac{-|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s} \\
 & \leq CT^{\frac{s-1+2\alpha}{2\alpha}} \|u\|_{\mathcal{X}_T^s} \|v\|_{\mathcal{X}_T^s}.
 \end{aligned}$$

This complete the proof. $\square$

Next, we derive a regularity property.

Theorem 2.2. *Let $\mu > 0$, $0 \leq t \leq T \leq 1$, $s \in (-2\alpha + 1, 0]$ and $k \in [0, s + 2\alpha - 1)$, then*

$$\mathcal{F} : t \longrightarrow \int_0^t U_\mu(t-t')\partial_x(1 + \mathcal{H}\partial_x)^{-1}(u^2)(t') dt', \tag{2.12}$$

is in $C([0, T] : H^{s+k}(\mathbb{R}^2))$, for all $u \in \mathcal{X}_T^s$.

Proof. Let $t_0, t_1 \in [0, T]$ be fixed such that $t_0 < t_1$. By the Minkowski inequality, we have

$$\begin{aligned} \|\mathcal{F}(t_1) - \mathcal{F}(t_0)\|_{H^{s+k}} &\leq \int_{t_0}^{t_1} \|U_\mu(t_1 - t')u^2(t')\|_{H^{s+k}} dt' \\ &\quad + \int_0^{t_0} \|U_\mu(t_1 - t') - U_\mu(t_0 - t')u^2(t')\|_{H^{s+k}} dt'. \end{aligned} \quad (2.13)$$

We suppose that $s + k < 0$. In this case, $(1 + \xi^2 + \eta^2)^{s+k} < (\xi^2 + \eta^2)^{s+k}$. For the other case, if $s + k > 0$, then $(t^{\frac{1}{\alpha}} + \xi^2 + \eta^2)^{s+k} < (1 + \xi^2 + \eta^2)^{s+k}$ and the proof is analogous:

$$\begin{aligned} &\int_{t_0}^{t_1} \|U_\mu(t_1 - t')u^2(t')\|_{H^{s+k}} dt' \\ &\leq \left(\int_{t_0}^{t_1} \|(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_1-t')\mu(\xi^2+\eta^2)^\alpha}\|_{L^2(\mathbb{R}^2)} \frac{1}{|t' - t_0|^{\frac{|s|}{\alpha}}} dt' \right) \|u\|_{\mathcal{X}_T^2}^2 \\ &\leq \left(\int_{t_0}^{t_1} (t_1 - t')^{-\frac{s+k+1}{2\alpha}} (t' - t_0)^{-\frac{|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^2}^2 \\ &\leq (t_1 - t_0)^{\frac{s-k+2\alpha-1}{2\alpha}} \left(\int_0^1 (1-t')^{-\frac{s+k+1}{2\alpha}} (t')^{-\frac{|s|}{\alpha}} dt' \right) \|u\|_{\mathcal{X}_T^2}^2. \end{aligned}$$

Now, using the hypotheses, we get that

$$\lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \|U_\mu(t_1 - t')u^2(t')\|_{H^{s+k}} dt' = 0 \quad (2.14)$$

and for the integral on the right-hand side of (2.13)

$$\begin{aligned} &\int_0^{t_0} \|U_\mu(t_1 - t') - U_\mu(t_0 - t')u^2(t')\|_{H^{s+k}} dt' \\ &\leq \left(\int_0^{t_0} \|(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_0-t)\mu(\xi^2+\eta^2)^\alpha} (e^{-(t_1-t_0)\mu(\xi^2+\eta^2)^\alpha} - 1)\|_{L^2(\mathbb{R})} \frac{1}{|t'|^{\frac{|s|}{\alpha}}} dt' \right) \\ &\quad \times \|u\|_{\mathcal{X}_T^s}^2. \end{aligned}$$

The function $(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_0-t)\mu(\xi^2+\eta^2)^\alpha} (e^{-(t_1-t_0)\mu(\xi^2+\eta^2)^\alpha} - 1)$ tend to zero pointwise for almost every $(\xi, \eta) \in \mathbb{R}^2$ and for all $t' \in [0, t_0]$. Furthermore,

$$\begin{aligned} &|(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_0-t)\mu(\xi^2+\eta^2)^\alpha} (e^{-(t_1-t_0)\mu(\xi^2+\eta^2)^\alpha} - 1)| \\ &\leq 2(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_0-t')\mu(\xi^2+\eta^2)^\alpha}. \end{aligned} \quad (2.15)$$

We deduce from (2.15) and Lebesgue dominated convergence theorem that

$$\lim_{t_1 \rightarrow t_0} \|(\xi^2 + \eta^2)^{\frac{s+k}{2}} e^{-(t_0-t)\mu(\xi^2+\eta^2)^\alpha} (e^{-(t_1-t_0)\mu(\xi^2+\eta^2)^\alpha} - 1)\|_{L^2(\mathbb{R}^2)} = 0. \quad (2.16)$$

On the other hand, since $s > -\alpha$, then $t^{-\frac{|s|}{\alpha}}$ is integrable around of zero and using again the Lebesgue dominate convergence, we have that

$$\lim_{t_1 \rightarrow t_0} \int_0^{t_0} \|U(t_1 - t') - U(t_0 - t')u^2(t')\|_{H^{s+k}} dt' = 0. \quad (2.17)$$

This completes the proof. $\square$

3. Local Well-Posedness

In this section we give the proof of Theorem 1.2.

Proof. (Theorem 1.2) Let $\mu > 0$, $s \in (-2\alpha + 1, 0]$ and $\varphi \in H^s(\mathbb{R}^2)$. We will show that there exist $T > 0$ and $M > 0$ such that the operator Φ defined in (2.1) is a contraction in the closed ball

$$B(T, M) = \{u \in C([0, T] : \mathcal{X}_T^s) \mid \|U_\mu \varphi - u(t)\|_{\mathcal{X}_T^s} \leq M\}, \quad (3.1)$$

It is obvious that $B(T, M)$ is a complete metric space with the metric $d_{T, M}$ defined by

$$d_{T, M}(u, v) = \sup_{t \in [0, T]} \|u(t) - v(t)\|_{\mathcal{X}_T^s}.$$

By Proposition 2.1 and Theorem 2.1, for $u \in B(T, M)$, we have

$$\begin{aligned} \|\Phi(u)(t) - U_\mu \varphi\|_{\mathcal{X}_T^s} &\leq \int_0^t \|U_\mu(t - t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(u^2(t'))\|_{\mathcal{X}_T^s} dt' \\ &\leq \int_0^t CT^{\frac{s+2\alpha-1}{2\alpha}} \|u\|_{\mathcal{X}_T^s}^2 dt' \\ &\leq c(T) \int_0^t (M + \|\varphi\|_{\mathcal{X}_T^s})^2 dt' \\ &\leq c(T)T(M + \|\varphi\|_{\mathcal{X}_T^s})^2, \end{aligned}$$

where $c(t)$ is a increasing continuous function. Let $T_1 > 0$ such that

$$T_1 c(T_1) \leq \frac{M}{(M + \|\varphi\|_{\mathcal{X}_{T_1}^s})^2}, \quad (3.2)$$

thus, $\Phi(u) \in B(M, T_1)$. On the other hand, since

$$\begin{aligned}
 \|\Phi(u)(t) - \Phi(v)(t)\|_{\mathcal{X}_T^s} &\leq \int_0^t \|U_\mu(t-t')(1 + \mathcal{H}\partial_x)^{-1}\partial_x(u^2(t') - v^2(t'))\|_{\mathcal{X}_T^s} dt' \\
 &\leq \int_0^t CT^{\frac{s+2\alpha-1}{2\alpha}} \|u+v\|_{\mathcal{X}_T^s} \|u-v\|_{\mathcal{X}_T^s} dt' \\
 &\leq \int_0^t CT^{\frac{s+2\alpha-1}{2\alpha}} (\|u\|_{\mathcal{X}_T^s} + \|v\|_{X_T^s}) \|u-v\|_{\mathcal{X}_T^s} dt' \\
 &\leq \tilde{c}(T)(M + \|\varphi\|_{\mathcal{X}_T^s}) d_{T,M}(u, v) dt' \\
 &\leq \tilde{c}(T)T(M + \|\varphi\|_{\mathcal{X}_T^s}) d_{T,M}(u, v),
 \end{aligned}$$

for some increasing continuous function $\tilde{c}(t)$. If we take $T_2 > 0$, such that

$$\tilde{c}(T_2)(M + \|\varphi\|_{\mathcal{X}_{T_2}^s})T_2 < 1, \quad (3.3)$$

then, we have that Φ is a contraction in $B(T_2, M)$. Let $T = \min\{T_1, T_2\}$, from Banach fixed point theorem there exist a unique solution $u \in B(T, M)$ of the integral equation (2.1) with the initial data $u(0) = \varphi$.

Let u and v be solutions of (1.1) in $[0, T]$. We have that

$$\begin{aligned}
 \|u(t) - v(t)\|_{\mathcal{X}_T^s} &\leq \|\varphi - \psi\|_{H^s} \\
 &+ \int_0^t c(T)(M + \|\varphi\|_{\mathcal{X}_T^s})(M + \|\psi\|_{\mathcal{X}_T^s}) \|u(t) - v(t)\|_{\mathcal{X}_T^s} dt'. \quad (3.4)
 \end{aligned}$$

By the Gronwall Lemma and inequality (3.4) we conclude that $u = v$.

Finally, the solution map $t \rightarrow U_\mu(t)\varphi$ is continuous in the time interval $(0, T]$ with respect to the topology of H^∞ . Since $u \in \mathcal{X}_T^s$, we deduce from Theorem 2.2 that there exist $k > 0$ such that $u \in C((0, T] : H^{s+k}(\mathbb{R}^2))$. Observe of the proof the theorem 1.1 that T depends only that $\|\varphi\|_{H^s(\mathbb{R}^2)}$. Thus, we conclude that $u \in C([0, T] : H^\infty(\mathbb{R}^2))$ $\square$

4. Global Well Posedness

In this section we will show Theorem 1.3.

Proof. (Theorem 1.3) First we consider $s = 1$. The key point is to obtain a priori estimates of the Sobolev norm in $H^1(\mathbb{R}^2)$. Let $u \in C([0, T] : H^1(\mathbb{R}^2))$ be the solution of (1.1),

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leq \|\varphi\|_{L^2(\mathbb{R}^2)}. \quad (4.1)$$

Now, we derive (1.1) with respect to variable x , calculate the product in L^2 with u_x , using integration by parts and the Cauchy-Schwarz inequality, then we get

$$\begin{aligned} \frac{d}{dt} \|u_x(t)\|_{L^2}^2 &= -2\mu \|\Delta u_x(t)\|_{L^2}^2 - 2 \int_{\mathbb{R}^2} u(t) u_x(t) u_{xx}(t) dx dy \\ &\leq -2\mu \|\Delta u_x(t)\|_{L^2}^2 + 2 \|\varphi\|_{L^2} \|u_x(t)\|_{L^\infty} \|u_{xx}(t)\|_{L^2}. \end{aligned} \quad (4.2)$$

By the Gagliardo-Nirenberg inequality

$$\|u_x(t)\|_{L^\infty}^2 \leq C \|\Delta u_x(t)\|_{L^2} \|u_x(t)\|_{L^2} \quad (4.3)$$

and

$$\|u_{xx}(t)\|_{L^2}^2 \leq C \|\Delta u_x(t)\|_{L^2} \|u_x(t)\|_{L^2}. \quad (4.4)$$

From (4.2), (4.3), (4.4), Young inequality and Gronwall inequality, we obtain

$$\|u_x(t)\|_{L^2}^2 \leq c(T) \|\varphi_x\|_{L^2}^2. \quad (4.5)$$

Analogously,

$$\|u_y(t)\|_{L^2}^2 \leq c(T) \|\varphi_y\|_{L^2}^2. \quad (4.6)$$

This complete the proof for $u \in C([0, T] : H^1(\mathbb{R}^2))$. Now, from Lemma 2.2 and Proposition 2.1 we can extend the result for $s > 1$. $\square$

Acknowledgments

The author was supported by Colciencias-Colombia.

References

- [1] J. Angulo, M. Scialom and C. Banquet, The regularized Benjamin-Ono and BBM equations: well-posedness and nonlinear stability, *J. Differential Equations*, **250**, No 11 (2011), 4011-4036.
- [2] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.*, **29** (1967), 559-592.
- [3] D. Dix, Nonuniqueness and uniqueness in the initial value problem for Burgers' equation, *J. Math. Anal.*, **27** (1996), 708-724.

- [4] G.E. Fonseca, G. Rodriguez and W. Sandoval, Well posedness and Ill posedness result for the regularized Benjamin-Ono equation in weighted Sobolev spaces, *Comm. Pure and App. Anal.*, **14**, No 4 (2015), 1327-1341.
- [5] B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Soviet Physics Doklady*, **15** (Dec. 1970), 539.
- [6] H. Kalisch and J.L. Bona, Models for internal waves in deep water, *Discrete Contin. Dynam. Systems*, **6**, No 1 (2000), 1-20.
- [7] Y. Mammeri, Long time bounds for the periodic Benjamin-Ono-BBM equation, *Nonlinear Anal.*, **71**, No 10 (2009), 5010-5021.
- [8] H. Ono, Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan*, **39**, No 4 (1975), 1082-1091.
- [9] F. Sánchez and F.H. Soriano, On the Cauchy problem associated to a regularized Benjamin-Ono-Zakharov-Kuznetsov (rBO-ZK) type equation, *Submitted*, 2015.
- [10] F. Sánchez and F.H. Soriano, On the existence and analycity of solitary waves solutions to regularized Benjamin-Ono-Zakharov-Kuznetsov type equation, *Submitted*, 2015.