AN ASYMPTOTIC APPROACH FOR DESCRIBING SILTING OF RIVERS

Y.A.S. Wellot¹, C.S. Ngamouyih-Moussata², H. Nkounkou³, T. Djaokamla^{4 §}

^{1,2,3}Department of Mathematics

Marien Ngouabi University
Brazaville, BP: 69, CONGO

⁴Department of Mathematics

Adam Barka University
Abeche, BP 1173, CHAD

Abstract: This paper proposes an asymptotic approach for describing silting of rivers. The proposed approach is based on the hypothesis that rivers silting can be explained by an asymptotic behavior near shock curves of the governing equations of sediment deposits. The computation of the leading term of the asymptotic expansion is then performed and an example which illustrated the proposed approach is presented.

AMS Subject Classification: 65M70, 65N22

Key Words: boundary layer, inner asymptotic expansion, morphological change, sedimentation, silting process

1. Introduction

In the context of climate change, the study of the phenomenon of silting of rivers and lakes is today one of the most important research topics in the fields of environmental and socio-economic. Silting of rivers or lakes can be defined as a state of excess sediment deposits. Such the process is among the most complex and least understood phenomena in nature.

Received: August 20, 2015

© 2015 Academic Publications

[§]Correspondence author

Significant mathematical investigations on rivers sedimentation and their morphological changes can be found in 1970s. Since then, various papers were written but most of them have developed 1-D models ([1, 5, 12]). Also, there are only few papers devoted to modelling silting of rivers or lakes. Numerous existing 2-D and 3-D models describe and simulate sediment transport processes and morphological changes in channels mobile bed and bank ([2, 6, 7, 9, 10, 11]).

This paper is focused on the development of a numerical approach based on an asymptotic expansion of the solutions that are valid in the neighborhood of rivers banks. The approach developed in this paper is very closed to that of limit layers. As in dynamic fluid, the boundary layer under consideration can be defined as the layer where the silting effect is very significant. Thus we shall use the inner asymptotic expansion, a classical technique for the study of boundary layers in fluid dynamic which has been used to a context closed to ours ([3, 4]). Our main objective is then to describe the silting phenomenon on a portion of strip of a river for which the effect of the silting is more important.

The outline of this paper is as follows. In Section 2, we present the equations of the model of our problem, and with the thanks of the asymptotic expansion theory, we study the solution of the problem. Then, numerical simulation is conducted in Section 3.

2. Problem Statement

Let Ω be a $\mathbb{R}^2$ bounded domain which represents the water surface of the part of the studied river. We consider a cartesian axis system (x, y) , and assume that the boundary $\partial\Omega$ of the domain Ω can be splitted by

$$\partial\Omega = \Gamma_1^r \cup \Gamma_2^r \cup \Gamma_1^s \cup \Gamma_2^s, \quad (1)$$

where Γ_i^r , $i = 1, 2$ represent the river banks that we shall assume to be described by equations

$$y = \eta_i(x), \quad i = 1, 2, \quad (2)$$

η_i being known functions. The boundaries Γ_i^s , $i = 1, 2$ are segments that limit the studied river portion. They are given by the equations

$$x = \pm L, \quad (3)$$

where $2L$ is the length of the studied portion of the river (Figure 1a). To be complete in the description of the studied problem, the flow is assumed to be incompressible and is oriented in the x direction. So to describe the sediment

Figure 1: On the left (a) it is illustrated the scheme of the studied portion of the river water free surface while on the right (b) a section along the width of the river bed morphology is shown.

evolution, we denote by $S(t, x, y)$ the height of sediments deposit at the point (x, y) at the time t as it is shown in Figure 1b.

It is shown in many papers that sediments deposit depend of course of the flow dynamic of the river which we shall describe its Cartesian velocity field vector at the point (x, y) and at the time t by

$$U = (u(t, x, y), v(t, x, y), w(t, x, y)). \quad (4)$$

To describe the silting phenomena near rivers banks one can neglected the third component w . Thus the equations system of the sediment dynamic are then

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re} \Delta u + g_x, \quad (5)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re} \Delta v + g_y, \quad (6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (7)$$

$$\frac{\partial S}{\partial t} + k_x(u, v) \frac{\partial S}{\partial x} + k_y(u, v) \frac{\partial S}{\partial y} = \Delta(\phi(S)) + f(t, x, y). \quad (8)$$

The equations (5), (6) and (7) are the well known Navier-Stokes equations which describe incompressible flows, while the equation (8) describes sediment

deposit. Functions $k_x(u, v)$ and $k_y(u, v)$ are parameters which describe transport of sediments in x and y directions, and the term $\Delta(\phi(S))$ describes dispersion of sediment particles moving in the river. In addition, we introduce following boundary conditions on the river sides:

$$\frac{\partial v}{\partial n} = g_i \quad \text{on} \quad \Gamma_i^r, \quad i = 1, 2 \quad (9)$$

$$u = 0 \quad \text{on} \quad \Gamma_i^r, \quad i = 1, 2, \quad (10)$$

$$\frac{\partial S}{\partial n} = h_i \quad \text{on} \quad \Gamma_i^r, \quad i = 1, 2, \quad (11)$$

where g_i and h_i are assumed to be known functions and, on the boundary part $x = \pm L$ no condition is imposed. So the considered model is seen as an incomplete data problem and can be investigated in the framework of inverse problems. Our aim in this paper is to provide some solutions which can describe silting of rivers banks by using the partial differential equations asymptotic theory.

3. Inner Asymptotic Expansion Valid on a Neighborhood of a River Side

Remarking that the governing equation of the sediment deposit can be written in the divergence form, we then consider near some side of the river the shock curve described by the equation

$$y = \eta(t, x), \quad (12)$$

along with there exists at time t the solutions which are not continuously differentiable. Also we assume that the function η is sufficiently regular. Then, in order to describe river silting phenomenon we have to consider the following shock layer:

$$\Omega^{\varepsilon, t} = \{(x, y) \in \Omega \mid -L \leq x \leq L, \eta(t, x) - \varepsilon \leq y \leq \eta(t, x) + \varepsilon\}, \quad (13)$$

and we introduce the inner variables

$$x^* = x, \quad y^* = \frac{y - \eta(t, x)}{\delta(\varepsilon)}, \quad t^* = \frac{t - t_0}{\lambda(\varepsilon)}, \quad (14)$$

for appropriate $\delta(\varepsilon)$ and $\lambda(\varepsilon)$ such that $\delta(\varepsilon) \rightarrow 0$ and $\lambda(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$ and, we denote by $u^\varepsilon(t, x, y)$, $v^\varepsilon(t, x, y)$ and $S^\varepsilon(t, x, y)$ the solutions of the system equations (5)-(11) that are valid in the shock layer $\Omega^{\varepsilon, t}$. Afterwards we look for an inner asymptotic expansion of this solution in the following form:

$$S^\varepsilon(t, x, y) = \varepsilon^{-\alpha} S^0(t^*, x^*, y^*) + o(\varepsilon^{-\alpha}), \quad (15)$$

$$u^\varepsilon(t, x, y) = \varepsilon^{-\beta} u^0(t^*, x^*, y^*) + o(\varepsilon^{-\beta}), \quad (16)$$

$$v^\varepsilon(t, x, y) = \varepsilon^{-\gamma} v^0(t^*, x^*, y^*) + o(\varepsilon^{-\gamma}), \quad (17)$$

where for any non null real s ones has

$$\lim_{\varepsilon \searrow 0} \frac{o(\varepsilon^s)}{\varepsilon^s} = 0, \quad (18)$$

and where α , β and γ are some appropriate parameters. It is important to note that, as we are interested in determining solutions expansions that are valid in the vicinity of the river side, normal and tangential velocities may not be considered of the same order. In fact, the tangential velocity can be neglected relatively to the normal ones. This point of view has lead us to assume that

$$0 < \beta < \gamma. \quad (19)$$

while the sediment near the line $y = \eta(t, x)$ can be assumed to be of the same order as the normal velocity.

Next, for the sake of the simplicity, we shall assume that the diffusion term in equation (8) is expressed as

$$\Delta\phi(S) = \mu(u, v)\Delta S. \quad (20)$$

For an asymptotic description, following assumption may be done

$$k_x(u^\varepsilon, v^\varepsilon) \ll k_x(u^\varepsilon, v^\varepsilon) \text{ in } \Omega^{(\varepsilon, t)}. \quad (21)$$

This means that tangential component of the velocity is neglected relatively to the normal ones. In fact it is natural to assume that near the river side convection in y direction is least important than the convection in x direction.

Furthermore we shall assume that following expansions hold:

$$k_x(u^\varepsilon, v^\varepsilon) = k_x^0 u^0 \varepsilon^{-\beta} + o(\varepsilon^{-\beta}), \quad (22)$$

$$k_y(u^\varepsilon, v^\varepsilon) = k_y^0 v^0 \varepsilon^{-\gamma} + o(\varepsilon^{-\gamma}), \quad (23)$$

$$\mu(u^\varepsilon, v^\varepsilon) = \mu_0 v^0 \varepsilon^{-\gamma} + o(\varepsilon^{-\gamma}), \quad (24)$$

where k_x^0 , k_y^0 and μ^0 are constant parameters. We have the following result.

Theorem 3.1. *If we set*

$$\lambda(\varepsilon) = \delta(\varepsilon)^2 \varepsilon^\gamma, \quad (25)$$

then, passing to the limit as $\varepsilon \searrow 0$ (t^ , x^* and y^* fixed), we have*

$$\frac{\partial S^0}{\partial t^*} + \mu^0 v^0 \frac{\partial^2 S^0}{\partial y^{*2}} = \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\alpha} f, \quad (26)$$

$$\frac{\partial u^0}{\partial t^*} + \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} \frac{\partial p}{\partial x^*} = \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} g_x, \quad (27)$$

$$\frac{\partial v^0}{\partial t^*} + \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} \frac{\partial p}{\partial y^*} = \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} g_y. \quad (28)$$

Assuming that the equation (26) holds and if in addition we suppose that

$$\lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\alpha} f^\varepsilon = \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} g_x = \lim_{\varepsilon \searrow 0} \delta(\varepsilon)^2 \varepsilon^{\gamma+\beta} \frac{\partial p}{\partial x^*} = 0, \quad (29)$$

then

$$\frac{\partial S^0}{\partial t^*} + \mu^0 v^0 \frac{\partial^2 S^0}{\partial y^{*2}} = 0, \quad (30)$$

$$\frac{\partial u^0}{\partial t^*} = 0, \quad (31)$$

$$\frac{\partial v^0}{\partial t^*} = 0. \quad (32)$$

Thus we deduce

$$u^0 = u^0(x^*, y^*), \quad v^0 = v^0(x^*, y^*).$$

4. A Numerical Example

Let us consider the problem of searching S^ε which satisfies the following initial condition

$$S^\varepsilon(0, x, y) = \begin{cases} \varepsilon^{-\alpha} & \text{if } x^3 \leq y \leq x^3 + \varepsilon \\ 0 & \text{if } x^3 - \varepsilon \leq y \leq x^3 \end{cases}$$

Then we have to look for an asymptotic expansion of the sedimentation deposit valid in the vicinity of the shock that we assume to be defined by the following

$$y = \eta(t, x) := \sqrt{t} + x^3. \quad (33)$$

With the assumptions (25) and (29) we have to solve a the leading term the following partial differential equation:

$$\frac{\partial S^0}{\partial t^*} + \mu^0 v^0 \frac{\partial^2 S^0}{\partial y^{*2}} = 0, \quad -\infty \leq y^* \leq \infty, \quad 0 \leq t^* \leq \infty, \quad (34)$$

with the initial condition

$$S^0(0, x^*, y^*) = \begin{cases} 1 & \text{if } -\infty \leq y^* \leq 0 \\ 0 & \text{else} \end{cases} \quad (35)$$

For the sake of the simplicity we set $\mu v^0 = -1$. Then the equations system (34)-(35) admits the solution which is explicitly given by

$$S^0(t^*, x^*, y^*) = \frac{\sqrt{t^*}}{\sqrt{\pi}} \left(1 - \operatorname{erf} \left(\frac{y^*}{2\sqrt{t^*}} \right) \right). \quad (36)$$

Therefore, the expansion of the solution near the shock given by (33) is

$$S^\varepsilon = \frac{\sqrt{t^*}}{\sqrt{\pi}} \left(1 - \operatorname{erf} \left(\frac{y^*}{2\sqrt{t^*}} \right) \right) \varepsilon^{-\alpha} + \dots \quad (37)$$

We have illustrated this solution in Figure 2 by taking $\delta(\varepsilon) = \varepsilon^{\frac{1}{2}}$, $\alpha = \gamma = \frac{1}{2}$, and $\varepsilon = 0.1$. Figure 2 clearly shows the dynamic of the silting of the bed which becomes more and more important near the shock line.

Figure 2: *Illustration of the asymptotic behaviour of the bed silting near a shock line for increasing time.*

5. Concluding Remarks

In this paper we have presented expansion solutions which give rise to a complete description of sediment process near rivers banks. Particularly, we estab-

lished that these solutions can be expressed in terms of inner variables described with moving shock lines. However, this work was not interested in determining the exact equation of the shock line. In our future work we shall identify this function using data variational assimilation. Note that to compute some constants in the solution, matching properties will be needed. This required to determine the outer expansion of the solution.

References

- [1] J.A. Cunge , F.M. Jr. Holly and A. Verwey, *Practical Aspects of Computational River Hydraulics*, Pitman Advanced Publishing Program (1980), 420.
- [2] Y. Jia and S.S.Y. Wang, Numerical model for channel flow and morphological change studies, *Journal of Hydraulic Engineering, ASCE*, **125**, No 9 (1999), 924-933.
- [3] D. Leguillon and E. Sanchez-Palencia, *Computation of Singular Solution in Elasticity and Elliptic Problems*, Wiley, New York (1987).
- [4] B. Mampassi, Etude asymptotique de la jonction d'un Massif tridimensionnel et d'une tige élancée en Flexion, *Revista Mathematica de la Universidad Complutense de Madrid*, **7**, No 2 (1994).
- [5] J.L. Rahuel, F.M. Holly, J.P. Chollet, P.J. Belleudy and F. Yang, Modeling of riverbed evolution for bed load sediment mixtures, *J. Hydr. Engrg., ASCE*, **115**, No 11 (1989), 1521-1542.
- [6] Y. P. Sheng, *Mathematical Modeling of Three-Dimensional Coastal Currents and Sediment Dispersion: Model Development and Application*, Technical Report CERC-83-2, Aeronautical Research Associates of Princeton, Inc., N.J., 1983.
- [7] M. Spasojevic and F.M. Holly, *Three-dimensional Numerical Simulation of Mobile-Bed Hydrodynamics*, Iowa Institute of Hydraulic Research, The University of Iowa, Technical Report No 367, 1993.
- [8] W.A. Thomas, Chapter 18: Mathematical modeling of sediment movement, In: *Gravel-bed Rivers*, Eds: R.D. Hey and al., John Wiley & Sons, Ltd., New York (1982).

- [9] S.S.-Y. Wang and S.E. Adeff, *Three-Dimensional Modelling of River Sedimentation Processes*, Proc. of the 3rd Int. Symp. on River Sedimentation, The Univ. of Mississippi, USA (1986).
- [10] W. Wu, W. Rodi and T. Wenka, 3-D numerical modeling of water flow and sediment transport in open channels, *J. Hydr. Engrg., ASCE*, **126**, No 1 (2000), 4-15.
- [11] W. Wu, S.S.-Y. Wang and Y. Jia, Nonuniform sediment transport in alluvial rivers, *J. Hydr. Res., IAHR*, **38**, No 6 (2000), 427-434.
- [12] W. Wu and D.A. Vieira, *One-Dimensional Channel Network Model CCHE1D 3.0 – Technical Manual*, Technical Report No. NCCHE-TR-2002-1, National Center for Computational Hydroscience and Engineering, The University of Mississippi (2002).