RESULTS ON THE DILOGARITHM INTEGRAL

Brian Fisher¹, Fatma El-Sirehy^{2 §}

¹Department of Mathematics
University of Leicester
Leicester, LE1 7RH, ENGLAND, UK

²Department of Mathematics
King Abdulaziz University
Jeddah, SAUDI ARABIA

Abstract: The dilogarithm integral $\text{Li}(x)$ and its associated functions $\text{Li}_+(x)$ and $\text{Li}_-(x)$ are defined as locally summable functions on the real line. Some generalization of convolutions and neutrix convolutions of these functions and other functions are then found.

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1. Introduction

The dilogarithm integral $\text{Li}(x)$ is defined by

$$\text{Li}(x) = - \int_0^x \frac{\ln |1-t|}{t} dt, \quad (1)$$

see [1].

The associated functions $\text{Li}_+(x)$ and $\text{Li}_-(x)$ are defined by

$$\text{Li}_+(x) = H(x) \text{Li}(x), \quad \text{Li}_-(x) = H(-x) \text{Li}(x) = \text{Li}(x) - \text{Li}_+(x),$$

where $H(x)$ denotes Heaviside's function.

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[§]Correspondence author

Next, we define the function $\ln|1-x|x^{-1}$ by

$$\ln|1-x|x^{-1} = -\text{Li}'(x)$$

and its associated functions $\ln|1-x|x_+^{-1}$ and $\ln|1-x|x_-^{-1}$ are defined by

$$\ln|1-x|x_+^{-1} = H(x)\ln|1-x|x^{-1} = -\text{Li}'_+(x)$$

$$\ln|1-x|x_-^{-1} = H(-x)\ln|1-x|x^{-1} = -\text{Li}'_-(x).$$

The classical definition of the convolution product of two functions f and g is as follows:

Definition 1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) dt \quad (2)$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists, then $g * f$ exists and

$$f * g = g * f \quad (3)$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \quad (4)$$

Definition 1 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ by the following definition, see Gel'fand and Shilov [5].

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy either of the conditions:

- (a) either f or g has bounded support,
- (b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then equations (2) and (3) are satisfied.

We now define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \ln|1-u| du \quad (5)$$

for $r = 0, 1, 2, \dots$

The following lemma was proved in [8].

Lemma 1.

$$I_r(x) = \frac{1}{r+1}(x^{r+1} - 1) \ln |1 - x| - \frac{1}{r+1} \sum_{i=0}^r \frac{x^{i+1}}{i+1}$$

for $r = 0, 1, 2, \dots$.

The following theorem and corollary were proved in [8].

Theorem 1. *The convolution $\text{Li}_+(x) * x_+^r$ exists and*

$$\text{Li}_+(x) * x_+^r = \frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i} x_+^i I_{r-i}(x) + \frac{1}{r+1} x^{r+1} \text{Li}_+(x),$$

for $r = 0, 1, 2, \dots$.

Corollary 1.1. *The convolution $x^{-1} \ln |1 - x| * x_+^r$ exists and*

$$\begin{aligned} x^{-1} \ln |1 - x| * x_+^r &= \frac{1}{r+1} \sum_{i=1}^r \binom{r+1}{i} (-1)^{r-i} i x_+^{i-1} I_{r-i}(x) \\ &+ \frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i} x_+^{r+i} \ln |1 - x| + x^r \text{Li}_+(x) + \frac{1}{r+1} x^r \ln |1 - x|, \end{aligned}$$

for $r = 0, 1, 2, \dots$.

We now prove the following generalization of Theorem 1.

Theorem 2. *The convolution $x^s \text{Li}_+(x) * x_+^r$ exists and*

$$x^s \text{Li}_+(x) * x_+^r = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{x^{r+s+1} \text{Li}_+(x) + x_+^i I_{r+s-i}(x)}{r+s-i+1}, \quad (6)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. It is obvious that $x^s \text{Li}_+(x) * x_+^r = 0$ if $x < 0$.

If $x > 0$, we have on using equation (2)

$$\begin{aligned} x^s \text{Li}_+(x) * x_+^r &= - \int_0^x t^s (x-t)^r \int_0^t u^{-1} \ln |1-u| du dt \\ &= - \int_0^x u^{-1} \ln |1-u| \int_u^x t^s (x-t)^r dt du \\ &= - \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \int_0^x \frac{x^{r+s+1} u^{-1} - x^i u^{r+s-i}}{r+s-i+1} \ln |1-u| du \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{x^{r+s+1} \text{Li}(x) + x_+^i I_{r+s-i}(x)}{r+s-i+1}, \end{aligned}$$

proving equation (6).

Corollary 2.1. *The convolution $x_+^s \ln |1-x| * x_-^r$ exists and*

$$\begin{aligned} x_+^s \ln |1-x| * x_+^r &= (s+1) \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{x^{r+s+1} \text{Li}(x) + x^i \text{I}_{r+s-i}(x)}{r+s-i+1} \\ &\quad - \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{(r+s+2)x_+^{r+s+1} \text{Li}_+(x) - x^{r+s+1} \ln |1-x|}{r+s-i+2} \\ &\quad - \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{ix_+^{i-1} \text{I}_{r+s-i+1}(x) + x^{r+s+1} \ln |1-x|}{r+s-i+2}, \end{aligned}$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. Replacing s by $s+1$ in equation (6) and differentiating using equation (4), we have

$$\begin{aligned} (s+1)x^s \text{Li}_+(x) * x_+^r - x_+^s \ln |1-x| * x_+^r &= \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{(r+s+2)x_+^{r+s+1} \text{Li}_+(x) - x^{r+s+1} \ln |1-x|}{r+s-i+2} \\ &\quad + \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \frac{ix_+^{i-1} \text{I}_{r+s-i+1}(x) + x^{r+s+1} \ln |1-x|}{r+s-i+2}, \end{aligned}$$

proving equation (7).

Theorem 3. *The convolution $x^s \text{Li}_-(x) * x_-^r$ exists and*

$$x^s \text{Li}_-(x) * x_-^r = \sum_{i=0}^r \binom{r}{i} (-1)^i \frac{x^{r+s+1} \text{Li}_+(x) + x^i \text{I}_{r+s-i}(x)}{r+s-i+1}, \quad (7)$$

for $r = 0, 1, 2, \dots$ and $m, s = 1, 2, \dots$.

Proof. If $x > 0$, then $x^s \text{Li}_-(x) * x_-^r = 0$ and if $x < 0$, we have

$$\begin{aligned} x^s \text{Li}_-(x) * x_-^r &= \int_x^0 t^s (t-x)^r \int_t^0 u^{-1} \ln |1-u| du dt \\ &= \int_x^0 u^{-1} \ln |1-u| \int_x^u t^s (t-x)^r dt du \\ &= - \sum_{i=0}^r \binom{r}{i} (-1)^i \int_x^0 \frac{x^{r+s+1} u^{-1} - x^i u^{r+s-i}}{r+s-i+1} \ln |1-u| du \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^i \frac{x^{r+s+1} \text{Li}(x) + x^i \text{I}_{r+s-i}(x)}{r+s-i+1}, \end{aligned}$$

proving equation (8).

In order to extend Definition 2 to distributions which do not satisfy conditions (a) or (b), we let τ be a function in $\mathcal{D}$, see [9], satisfying the conditions:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
- (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \dots$

The following definition of the non-commutative neutrix convolution was given in [3].

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \dots$. Then the non-commutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$N\text{-}\lim_{n \rightarrow \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [2], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

In particular, if

$$\lim_{n \rightarrow \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

exists, we say that the non-commutative convolution $f \circledast g$ exists.

It is easily seen that any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution. Note also that because of the lack of symmetry in the definition of $f \circledast g$ the neutrix convolution is in general non-commutative.

The following result proved in [3] holds, showing that the neutrix convolution is a generalization of the convolution.

Theorem 4. *Let f and g be distributions in $\mathcal{D}'$, satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

Theorem 5. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note however that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$ but we do have the following theorem.

Theorem 6. *Let f and g be distributions in $\mathcal{D}'$ and suppose that convolution $f \circledast g$ exists. If $\text{N-lim}_{n \rightarrow \infty} \langle (f \tau'_n) * g, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all φ in $\mathcal{D}$, then $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

The following lemma was proved in [8].

Lemma 2.

$$\text{N-lim}_{n \rightarrow \infty} \text{I}_r(n) = -\frac{1}{(r+1)^2}, \quad (8)$$

for $r = 0, 1, 2, \dots$

We also need the following lemma.

Lemma 3.

$$\text{N-lim}_{n \rightarrow \infty} n^s \text{Li}(n) = -s^{-2}, \quad (9)$$

for $s = 1, 2, \dots$

Proof. For $n > 1$, we have

$$\begin{aligned} \text{Li}(n) &= -\int_0^n \frac{\ln|1-t|}{t} dt \\ &= -\int_0^1 \frac{\ln|1-t|}{t} dt - \int_1^n \frac{\ln|1-t|}{t} dt. \end{aligned} \quad (10)$$

We have

$$\int_1^n \frac{\ln |1-t|}{t} dt = \int_1^n \frac{\ln t + \ln(1-1/t)}{t} dt, \quad (11)$$

where

$$\int_1^n \frac{\ln t}{t} dt = \frac{\ln^2 n}{2} \quad (12)$$

and

$$\int_1^n \frac{\ln |1-1/t|}{t} dt = - \sum_{i=1}^{\infty} \int_1^n \frac{t^{-i-1}}{i} dt = \sum_{i=1}^{\infty} \frac{n^{-i}}{i^2} - \zeta(2), \quad (13)$$

where ζ denotes the zeta function.

Further, from equation (14), we have

$$n^s \int_1^n \frac{\ln |1-1/t|}{t} dt = \sum_{i=1}^{\infty} \frac{n^{s-i}}{i^2} - n^s \zeta(2),$$

and so

$$\text{N-lim}_{n \rightarrow \infty} n^s \int_1^n \frac{\ln |1-1/t|}{t} dt = \frac{1}{s^2}. \quad (14)$$

It now follows from equations (11), (12), (13) and (15) that

$$\text{N-lim}_{n \rightarrow \infty} n^s \text{Li}(n) = \frac{1}{s^2},$$

proving equation (10) for $s = 1, 2, \dots$.

We now prove the following theorem.

Theorem 7. *The neutrix convolution $x^s \text{Li}_+(x) \circledast x^r$ exists and*

$$x^s \text{Li}_+(x) \circledast x^r = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2x^i}{(r+s-i+1)^3}, \quad (15)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. We put $[x^s \text{Li}_+(x)]_n = x^s \text{Li}_+(x) \tau_n(x)$. Then the convolution $[x^s \text{Li}_+(x)]_n * x^r$ exists by Definition 1 and

$$\begin{aligned} [x^s \text{Li}_+(x)]_n * x^r &= \int_0^n t^s \text{Li}(t)(x-t)^r dt + \int_n^{n+n^{-n}} \tau_n(t) t^s \text{Li}(t)(x-t)^r dt \\ &= J_{1,1} + J_{2,1}, \end{aligned} \quad (16)$$

where

$$\begin{aligned}
 J_{1,1} &= \int_0^n t^s \operatorname{Li}(t)(x-t)^r dt \\
 &= - \int_0^n t^s (x-t)^r \int_0^t u^{-1} \ln |1-u| du dt \\
 &= - \int_0^n u^{-1} \ln |1-u| \int_u^n t^s (x-t)^r dt du \\
 &= - \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} x^i \int_0^n \frac{n^{r+s-i+1} u^{-1} - u^{r+s-i}}{r+s-i+1} \ln |1-u| du \\
 &= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} x^i \frac{n^{r+s-i+1} \operatorname{Li}(n) + \operatorname{I}_{r+s-i}(n)}{r+s-i+1}.
 \end{aligned}$$

Hence, using Lemmas 2 and 3, we have

$$\operatorname{N-lim}_{n \rightarrow \infty} J_{1,1} = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2x^i}{(r+s-i+1)^3}. \quad (17)$$

Further, it is easily seen that, for each fixed x and $K = \sup\{|\operatorname{Li}_+(x)|\}$ we have

$$\left| \int_n^{n+n^{-n}} \tau_n(t) t^s \operatorname{Li}(t)(x-t)^r dt \right| \leq K n^{-n} (n + n^{-n} + |x|)^{r+s}$$

and so

$$\lim_{n \rightarrow \infty} J_{1,2} = \lim_{n \rightarrow \infty} \int_n^{n+n^{-n}} \tau_n(t) t^s \operatorname{Li}(t)(x-t)^r dt = 0. \quad (18)$$

Equation (16) now follows from equations (17), (18) and (19).

Corollary 7.1. *The neutrix convolution $x_+^s \ln |1-x| \circledast x^r$ exists and*

$$x_+^s \ln |1-x| \circledast x^r = \sum_{i=1}^r \binom{r}{i} \frac{(-1)^{r-i} 2ix^{i-1}}{(r+s-i+2)^3} + \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2(s+1)x^i}{(r+s-i+1)^3}, \quad (19)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. Replacing s by $s+1$ in equation (16), differentiating and using Theorem 6, we get

$$(s+1)x^s \operatorname{Li}_+(x) \circledast x^r - x_+^s \ln |1-x| \circledast x^r$$

$$\sum_{i=1}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2ix^{i-1}}{(r+s-i+2)^3} + \operatorname{N-lim}_{n \rightarrow \infty} [x^{s+1} \operatorname{Li}_+(x) \tau_n'(x)] * x^r, \quad (20)$$

where on integration by parts, we have

$$\begin{aligned}
 [x^{s+1} \text{Li}_+(x) \tau'_n(x)] * x^r &= \int_n^{n+n^{-n}} t^{s+1} \text{Li}(t)(x-t)^r d\tau_n(t) \\
 &= -n^{s+1} \text{Li}(n)(x-n)^r + \int_n^{n+n^{-n}} \ln |1-t| t^s (x-t)^r \tau_n(t) dt \\
 &\quad - (-1)^r r \int_n^{n+n^{-n}} t^{s+1} \text{Li}(t)(x-t)^{r-1} \tau_n(t) dt \\
 &\quad - (s+1) \int_n^{n+n^{-n}} t^s \text{Li}(t)(x-t)^r \tau_n(t) dt.
 \end{aligned} \tag{21}$$

It is clear that

$$\lim_{n \rightarrow \infty} \int_n^{n+n^{-n}} \ln |1-t| t^s (x-t)^r \tau_n(t) dt = 0, \tag{22}$$

$$\lim_{n \rightarrow \infty} \int_n^{n+n^{-n}} t^{s+1} \text{Li}(t)(x-t)^{r-1} \tau_n(t) dt = 0, \tag{23}$$

$$\lim_{n \rightarrow \infty} \int_n^{n+n^{-n}} t^s \text{Li}(t)(x-t)^r \tau_n(t) dt = 0 \tag{24}$$

and it now follows from equations (22) to (25) that

$$\text{N-lim}_{n \rightarrow \infty} [x^{s+1} \text{Li}_+(x) \tau'_n(x)] * x^r = 0. \tag{25}$$

Equation (20) now follows from equations (16), (21) and (26).

Theorem 8. *The neutrix convolution $x^s \text{Li}_-(x) \circledast x^r$ exists and*

$$x^s \text{Li}_-(x) \circledast x^r = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2x^i}{(r+s-i+1)^3}, \tag{26}$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. This time we put $[x^s \text{Li}_-(x)]_n = x^s \text{Li}_-(x) \tau_n(x)$. Then the convolution $[x^s \text{Li}_-(x)]_n * x^r$ exists by Definition 1 and

$$\begin{aligned}
 [x^s \text{Li}_-(x)]_n * x^r &= \int_{-n}^0 t^s \text{Li}(t)(x-t)^r dt \\
 &\quad + \int_{-n-n^{-n}}^{-n} \tau_n(t) t^s \text{Li}(t)(x-t)^r dt = J_{2,1} + J_{2,2},
 \end{aligned} \tag{27}$$

where

$$\begin{aligned}
J_{2,1} &= \int_{-n}^0 t^s (x-t)^r \int_t^0 u^{-1} \ln |1-u| du dt \\
&= - \int_0^n u^{-1} \ln |1-u| \int_u^n t^s (x-t)^r dt du \\
&= - \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} x^i \int_0^n \frac{n^{r+s-i+1} u^{-1} - u^{r+s-i}}{r+s-i+1} \ln |1-u| du \\
&= \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} x^i \frac{n^{r+s-i+1} \text{Li}(n) + \text{I}_{r+s-i}(n)}{r+s-i+1}.
\end{aligned}$$

It follows as in the proof of Theorem 7 that

$$\text{N-lim}_{n \rightarrow \infty} J_{2,1} = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2x^i}{(r+s-i+1)^3}. \quad (28)$$

Again, as in the proof of Theorem 7, we have

$$\lim_{n \rightarrow \infty} J_{2,2} = 0. \quad (29)$$

Equation (27) now follows from equations (28), (29) and (30).

Corollary 8.1. *The neutrix convolution $x^s \text{Li}(x) \circledast x^r$ exists and*

$$x^s \text{Li}(x) \circledast x^r = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i+1} 4x^i}{(r+s-i+1)^3}, \quad (30)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. Equation (31) follows on noting that

$$x^s \text{Li}(x) \circledast x^r = x^s \text{Li}_+(x) \circledast x^r + x^s \text{Li}_-(x) \circledast x^r.$$

Corollary 8.2. *The neutrix convolution $x_-^s \ln |1-x| \circledast x^r$ exists and*

$$x_-^s \ln |1-x| \circledast x^r = \sum_{i=1}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2ix^{i-1}}{(r+s-i+2)^3} + \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i} 2(s+1)x^i}{(r+s-i+1)^3}, \quad (31)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. Replacing s by $s+1$ in equation (27), differentiating and using Theorem 6, we get

$$(s+1)x^s \text{Li}_-(x) \circledast x^r + x_-^s \ln |1-x| \circledast x^r$$

$$\sum_{i=1}^r \binom{r}{i} \frac{(-1)^{r-i+1} 2ix^{i-1}}{(r+s-i+2)^3} + \text{N-}\lim_{n \rightarrow \infty} [x^{s+1} \text{Li}_-(x) \tau'_n(x)] * x^r. \quad (32)$$

It follows as in the proof of equation (20) that

$$\text{N-}\lim_{n \rightarrow \infty} [x^{s+1} \text{Li}_+(x) \tau'_n(x)] * x^r = 0, \quad (33)$$

and equation (32) now follows from equations (27), (33) and (34).

Corollary 8.3. *The neutrix convolution $x^s \ln |1-x| \circledast x^r$ exists and*

$$\begin{aligned} x^s \ln |1-x| \circledast x^r &= \sum_{i=1}^r \binom{r}{i} \frac{(-1)^{r-i} [1 - (-1)^s] 2ix^{i-1}}{(r+s-i+2)^3} \\ &\quad - \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r-i} [1 - (-1)^s] 2(s+1)x^i}{(r+s-i+1)^3}, \end{aligned} \quad (34)$$

for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

Proof. Equation (35) follows on noting that

$$x^s \ln |1-x| \circledast x^r = x_+^s \ln |1-x| \circledast x^r + (-1)^s x_-^s \ln |1-x| \circledast x^r.$$

For further results involving the neutrix convolution, see [4], [6] and [7].

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