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# Γ-GUARDING OF ORTHOGONAL POLYGONS

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**Abstract:** Guarding of an environment by cameras or guards is a longstanding problem in computational geometry. In this paper, we introduce a new type of guarding problems called Γ-guarding using staircase visibility. A point inside an orthogonal environment is said to be staircase visible for a given camera if there exists a staircase path from camera to the point so that it is completely contained in the environment. In this model, we place a small number of guards inside an orthogonal polygon so that each point of polygon is staircase visible through at least four directions. We present an algorithm for Γ-guarding an arbitrary orthogonal polygon that yields an upper bound for the number of guards that protect an orthogonal polygon through four directions.

**AMS Subject Classification:** 68U05, 68R01
**Key Words:** computational geometry, art gallery problem, visibility, Δ-guarding

## 1. Introduction

Consider the problem of guarding an environment with cameras in order to ensure that every point in the environment is seen from at least one camera [4]. We intend to minimize the total number of cameras required. This is known as the art gallery problem. The art gallery problem posed by V. Klee [3] in 1978. Then V. Chvatal [1] established that $\lceil n/3 \rceil$ guards are always sufficient and sometimes necessary for guarding an $n$-sided simple polygon.

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We guard an environment better when each point in it can be seen from all orientations. In 2003, Smith and Evans [5] defined a new class of Art Gallery Problem called $\Delta$-guarding where the goal is to place a small number of guards in a given polygon such that each point in the polygon belongs to at least one triangle constructed by the guards who see that point. Smith and Evans proved that the problem of finding the minimum required guards to $\Delta$-guard every polygon is NP-hard.

Then Efrat et al. [2] in 2005 presented a random algorithm for $\Delta$-guarding problem. This algorithm was based on the algorithm presented by Bronnimann et al. in 1995. Tokekar and Isler [6] in 2014 presented a lower bound $\Omega(\sqrt{n})$ for the required number of guards to $\Delta$-guard the arbitrary polygons.

In this study, we define a new type of protection called $\Gamma$-guarding and present an algorithm to $\Gamma$-guard an orthogonal polygon.

## 2. Preliminaries

In an orthogonal polygon every edge is either horizontal or vertical, and thus every vertex is a right angle. All of the polygons that we will be considering are simple orthogonal without holes. Let $P$ be a simple orthogonal polygon. An orthogonal path from $x_0$ to $x_n$ in $P$ is a sequence of points $x_0, x_1, ..., x_n$ such that for $i = 1, 2, ..., n$, $x_{i-1}$ and $x_i$ are the endpoints of a horizontal or vertical line segments that is strictly inside $P$.

**Definition 1.** A staircase is an orthogonal path that can be traversed with no two consecutive turns to the same (left or right).

If the segments of a staircase are oriented from $x_0$ to $x_n$, then there are only four types:

1) the segments are either toward the north or toward the east which is called an $NE$ staircase.

2) the segments are either toward the north or toward the west which is called an $NW$ staircase.

3) the segments are either toward the south or toward the east which is called a $SE$ staircase.

4) the segments are either toward the south or toward the east which is called a $SW$ staircase.

We say that two staircases are opposite if they have no common letter in their type, otherwise they are called adjacent.

**Definition 2.** According to the definition of staircase, for two points $p$ and $q$ inside an orthogonal polygon $P$, we say that $q$ is $NE$ staircase visible from $p$ if there exists a $NE$ staircase path from $p$ to $q$ which is completely inside $P$. This definition is also valid for the other types of visibility: $SE$, $SW$ and $NW$.

From now on, we use the above definition for visibility, which is called staircase visibility. Remember that regarding to the type of the staircase path form the guard to the point, there are four types of staircase visibility: $NE$, $NW$, $SE$ and $SW$.

**Definition 3.** A staircase polygon is a simple orthogonal polygon which is formed by joining two staircases. The two highest vertices and the two lowest vertices of a staircase polygon are called apexes of staircase polygon.

This polygon is monotone to both horizontal and vertical directions. Every two arbitrary points inside a staircase polygon are staircase visible to each other. Now by considering the basic definition given above, we define the Γ-guarding of a polygon.

**Definition 4.** Let $G$ be a finite set of point guards in a given orthogonal polygon $P$. We say that $P$ is Γ-guarded by $G$, if for every point $x$ inside $P$, there exist four guards $g_1$, $g_2$, $g_3$ and $g_4$ in $G$ so that $x$ is $NE$ staircase visible from $g_1$, $SE$ staircase visible from $g_2$, $NW$ staircase visible from $g_3$ and $SW$ staircase visible from $g_4$.

## 3. Γ-Guarding Problem

Let $P$ be a nonempty and orthogonal polygon. Note that the boundary and interior of $P$ is represented by $\delta P$ and *int* $P$ respectively. We want to place a small number of guards on the boundary of $P$ so that all the points of $P$ are Γ-guarded. We assume that all the guards we use to Γ-guard $P$ lie on the edges and have staircase visibility.

**Definition 5.** We assume that $D \in \{N, S, W, E\}$ is one of four directions and $e$ is an arbitrary edge of the polygon $P$. We call $e$ as a $D$-edge if the normal vector of $e$ towards outside of polygon is in direction $D$. Two directions of $C$

and $D$ are called opposite if $\{N, S\} = \{C, D\}$ or $\{E, W\} = \{C, D\}$, otherwise they are called adjacent. A vertex of $P$ is called $CD$-vertex for two adjacent directions $D, C \in \{N, S, E, W\}$, if its incident edges are $C$-edge and $D$-edge.

**Observation 6.** *Let $S$ be a staircase polygon with apexes $u$ and $v$. Suppose that $u$ is an $NE$-vertex and $v$ is a $SW$-vertex, then all points inside $S$ are both $NE$ and $SW$ staircase visible from $u$ and $v$. this holds also for $NW$- and $SE$- vertices.*

**Definition 7.** Let $D \in \{N, S, E, W\}$ and $e$ be an edge of $P$. Edge $e$ is called a $D$-valley if its both endpoints are concave vertices and it is called a $D$-mountain if both endpoints are convex vertices.

### 3.1. $N$-State and $S$-State

From now on, for two arbitrary points $p$ and $q$, the smallest coordinate-aligned rectangle containing $p$ and $q$ is denoted by $bd(p, q)$. Let $u$ and $v$ are two vertices of $P$ so that $u$ is higher than $v$ and let $A = bd(u, v) \cap P$. Now we move counterclockwise on the boundary of $A$. The boundary of $A$ from $u$ to $v$ is called the upper border and from $v$ to $u$ called the lower border.

Let $u$ be a convex $NE$-vertex and $v$ be a convex $SW$-vertex. Assume that an $N$-valley is on the upper border of $A$ and a $S$-valley is on the lower border so that $N$-valley is able to see $S$-valley and $N$-valley is closer to $u$ with regard to $x$ coordinate and lower than $S$-valley with regard to $y$ coordinate, then we say that $N$-state has happened and $u$ does not see $v$. See Figure 1.

Similarly assume that an $E$-valley is on the lower border of $A$ and a $W$-valley is on the upper border, so that W-valley is able to see E-valley and W-valley is closer to $u$ with regard to $x$ coordinate and higher than $W$-valley with regard to $y$ coordinate, then we can say $S$-state has happened and point $u$ does not see $v$. See Figure 1.

If $u$ is an $NW$-vertex and $v$ is a $SE$-vertex, $S$- and $N$-state are similarly defined.

**Propsition 8.** *Two points $p$ and $q$ with staircase visibility in an orthogonal polygon $P$ see each other if $bd(p, q) \cap P$ is a simple orthogonal polygon and $N$- and $S$-state does not happen.*

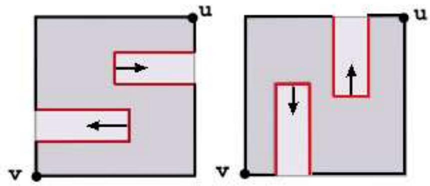


Figure 1: $S$-state and $N$-state.

### 3.2. Guarding $P$ by Limited Guards

In this section we intend to guard the simple orthogonal polygon $P$ by a set of vertex guards with only $NE$ and $SW$ staircase visibility. We will show that if we place a set of guards $G$ only at the convex $SW$- and $NE$-vertices of $P$, it will be $NE$ and $SW$ staircase visible from $G$. This holds for $SE$ and $NW$ visibility.

For two convex vertices $u$ and $v$ of $P$ which are $NE$- and $SW$-vertices respectively, the staircase polygon from $u$ to $v$ which is a subpolygon of $P$ is denoted by $S(u, v)$.

**Theorem 9.** *For every arbitrary point $p \in int\ P$, there are a pair of convex $NE$- and $SW$-vertices $r$ and $q$ so that $p \in S(r, q)$.*

*Proof.* For an arbitrary point $p \in intP$, We draw two horizontal and vertical lines through $p$ to hit the edges $P$. So the plane will be divided to four quarter. Note that there is at least one $NE$-vertex in the first quarter. Now to prove that there is an $NE$-vertex in the first quarter which can see point $p$, first we show that there is at least one $NE$ like $r$ in the first quarter that $bd(p, r) \cap P$ is a simple orthogonal polygon. We assume for a contradiction that for every $NE$-vertex like $r$ in the first quarter $bd(p, r) \cap P$ is not a simple orthogonal polygon, so $p$ is not a member of $P$, that is a contradiction. So there is at least one $NE$ like $r$ that $bd(p, r) \cap P$ is a simple orthogonal polygon. Now if $N$- and $S$-states in $bd(p, r)$ do not occur, so point $p$ sees $r$ and if $N$- or $S$-state occur in $bd(p, r)$, then there is a new $NE$-vertex like $u$ which can see $p$, as illustrated in the figure. Similarly it can be shown that there is at least one $SW$ like $q$ in first quarter which can see $p$. so, since $p$ is always higher than $q$, then $u$ can see $q$, as well. Thus there is a sub-polygon $S(p, q)$ that $p \in S(p, q)$ and the theorem is proved. □

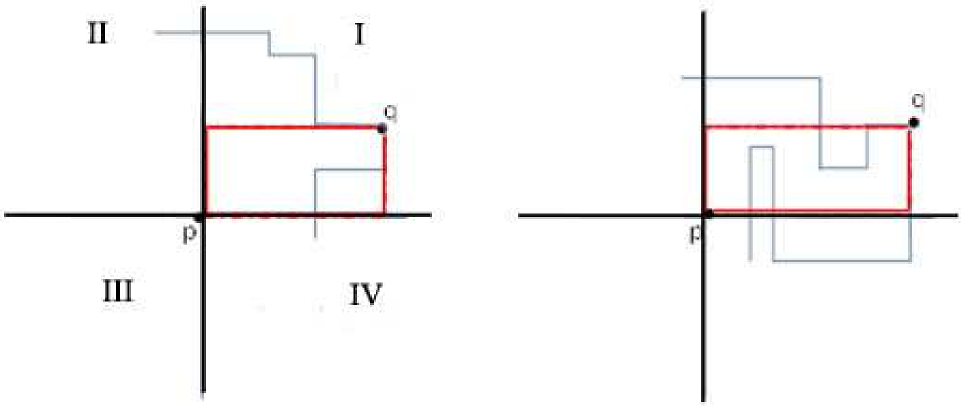


Figure 2: Guarding of an arbitrary point inside polygon: a) Division of plane around $p$, b) $N$-state occurs.

**Corollary 10.** *All the points inside $P$ are $NE$ and $SE$ visible from the guards located in convex $NE$- and $SW$-vertices of $P$.*

### 3.3. Γ-Guarding P

In this section we present a method to Γ-guard an orthogonal polygon $P$ by a set of point guards which have the four types of $NW$, $NE$, $SE$ and $SW$ visibility which see all their surroundings in four directions.

**Theorem 11.** *Let $P$ be a simple orthogonal polygon. It will be Gamma-guarded by a set of guards which are placed on each $D$-mountain and $D$-valley, where $D \in \{N, S, E, W\}$.*

*Proof.* We consider an arbitrary point $p$ belonging to int $P$. We draw two horizontal and vertical lines through $p$ to hit the edges $P$. So the plane will be divided to four quarter. Now we are going to show that there are at least two guards $g_1$ and $g_2$, one is in the first quarter and the other one is in the third quarter (and similarly one in the second quarter and the other one is in the fourth quarter) sees $p$. Suppose for a contradiction that there are not any guards in the first and second or third and fourth. Without loss of generality, we assume that there are not any guards in first and second quarter. Starting at the leftmost common point of horizontal line and $P$ and going clockwise to achieve the first point under the horizontal line, we will have a 180 degrees turn. So we reached at least one $N$-mountain on this path, i.e., there is at least one guard in the first and the second quarter. Now if this $N$-mountain sees point $p$, we reach a contradiction, otherwise if $N$-mountain does not see point $p$, so an $E$- or $W$-valley has certainly blocked the visibility. That is a contradiction because there is a guard on every $D$-valley and the theorem is proved. □

## Algorithm of Γ-Guarding an Orthogonal Polygon

**Input:** An orthogonal polygon $P$.
**Output:** A set $G$ of guards which can Γ-guard $P$.

1) If $P$ is a staircase, let $G$ be the set of apexes of $P$ and return $G$. else it has at least one edge which is a valley or mountain.

2) Let $v_0$ be one of the vertices of $P$. Traverse the boundary of $P$ counter-clockwise. Set $i = 0$ and $G = \emptyset$.

3) If the angles of $v_i$ and $v_{i+1}$ have the same value, place a guard somewhere on edge $v_i v_{i+1}$, add this guard to $G$ and set $i = i + 2$. else set $i = i + 1$.

4) If $i < n$, go to 2, else return $G$.

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