HAUSDORFF PROPERTY OF CARTESIAN AND WREATH PRODUCT OF HYPERGRAPHS

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Abstract: A hypergraph $H = (V, \mathcal{E})$ is said to be a *Hausdorff hypergraph* if for any two distinct vertices u and v of V there exist hyperedges $e_1, e_2 \in \mathcal{E}$ such that $u \in e_1, v \in e_2$ and $e_1 \cap e_2 = \emptyset$. In this paper we derive sufficient conditions for cartesian and wreath products of two hypergraphs to be Hausdorff.

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1. Introduction

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [7]. A *hypergraph* [3] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges*. Therefore, $\mathcal{E}$ is a subset of $P(X) \setminus \{\emptyset\}$, where $P(X)$ is the power set of X . In drawing hypergraphs, each vertex is a point in the plane and each edge is a closed curve separating the respective subset from the remaining vertices. The cardinality of the finite set V , is denoted by $|V|$, is called the *order* [9] of the hypergraph. The number of edges is usually denoted by m or $m(H)$ [9].

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A *simple hypergraph* [2] is a hypergraph with the property that if e_i and e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. Two vertices in a hypergraph are *adjacent*[7] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident*[7] if their intersection is nonempty.

A *k-uniform hypergraph* [4] or a *k-hypergraph* is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [7] $r(H)$ of a hypergraph is the maximum of the cardinalities of the edges in the hypergraph. The *co-rank* [7] $cr(H)$ of a hypergraph is the minimum of the cardinalities of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is *k-uniform*. The *degree* [8] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v . H is *k-regular* if every vertex has degree k . The *degree*[1] of a hyperedge, $e \in \mathcal{E}$ is its cardinality, that is $d(e) = |e|$.

A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyper-edge $e = \{v\}$ is a loop at the vertex v . A vertex of degree 1 is called a pendant vertex .

A simple hypergraph H with $|E_i| = 2$ for each $E_i \in \mathcal{E}$ is a simple graph.

Definition 1. A hypergraph $H = (V, \mathcal{E})$ is said to be a Hausdorff hypergraph if for any two distinct vertices u and v of V there exist hyperedges $e_1, e_2 \in \mathcal{E}$ such that $u \in e_1$ and $v \in e_2$; and $e_1 \cap e_2 = \emptyset$.

Throughout this paper we consider only simple hypergraph.

2. Cartesian Product

Let us start with the Cartesian product of two hypergraphs.

Definition 2. [6] The *Cartesian product* $H_1 \square H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is the hypergraph $H = (V, \mathcal{E})$ with vertex set $V = V_1 \times V_2$ and edge set $\mathcal{E} = \{\{u\} \times f : u \in V_1, f \in \mathcal{E}_2\} \cup \{e \times \{v\} : e \in \mathcal{E}_1, v \in V_2\}$.

Proposition 3. Let H_1 and H_2 be two hypergraphs with all of its edges are loops. Then $H_1 \square H_2$ is a hypergraph with all of its edges are loops.

Theorem 4. Let H_1 and H_2 be two hypergraphs. Then the Cartesian

product $H = H_1 \square H_2$ of H_1 and H_2 is a Hausdorff hypergraph.

Proof. Let $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_m\}$ be the set of vertices of H_1 and H_2 respectively. Let the hyperedges of H_1 be $e_1, e_2, \dots, e_p$ and that of H_2 be $f_1, f_2, \dots, f_q$. Let us denote the edges $\{u_i\} \times f_j$ and $e_r \times \{v_s\}$ by $f_{i,j}$ and $e_{r,s}$ respectively.

Consider two distinct vertices (u_i, v_j) and (u_r, v_s) of H .

Case 1. $v_j = v_s$

Let f_j be an edge of H_2 containing v_j . Then $f_{i,j}$ and $f_{r,j}$ are two nonadjacent edges of $H_1 \square H_2$ such that $(u_i, v_j) \in f_{i,j}$ and $(u_r, v_s) \in f_{r,j}$.

Case 2. $v_j \neq v_s$

Let e_i and e_r be edges of H_1 containing u_i and u_r respectively. Then $e_{i,j}$ and $e_{r,s}$ are two nonadjacent edges of $H_1 \square H_2$ such that $(u_i, v_j) \in e_{i,j}$ and $(u_r, v_s) \in e_{r,s}$.

Now, suppose $u_i = u_r$. Then $v_j \neq v_s$. This case can be dealt as in Case 1. Hence the theorem. $\square$

Proposition 5. Let H_1 and H_2 be two hypergraphs with no loops. Then $\deg(u, v) \geq 2$ for every vertex $(u, v) \in H_1 \square H_2$.

Proof. Consider the vertex (u_i, v_j) of $H_1 \square H_2$. Let e_i be an edge of H_1 containing u_i and f_j be an edge of H_2 containing v_j then (u_i, v_j) is contained in both $e_{i,j}$ and $f_{i,j}$. Therefore, $\deg(u, v) \geq 2$. $\square$

Now we consider the case of wreath product of hypergraphs.

3. Wreath Product

Let A and B be two sets. For each $e \in P(A \times B)$ we denote the set of first coordinates of elements of e by e^1 and the set of second coordinates of elements of e by e^2 .

Definition 6. [5] Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with disjoint vertex sets. The wreath product $H = (V, \mathcal{E}) = H_1[H_2]$ of H_1 and

H_2 is a hypergraph with vertex set $V = V_1 \times V_2$ and a subset e of $V \times V$ belongs to $\mathcal{E}$ if at least one of the following condition holds.

1. $e^1 \in \mathcal{E}_1$ and $|e \cap (\{u\} \times V_2)| \leq 1$ for each $u \in V_1$, or
2. $|e^1| = 1$ and $e^2 \in \mathcal{E}_2$.

Theorem 7. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then their wreath product $H_1[H_2]$ is Hausdorff.*

Proof. Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1[H_2]$

Case 1. $u_1 = u_2$ and $v_1 \neq v_2$

Let $e = \{u_1, u_3, u_4, \dots, u_{n+1}\}$ be an edge of H_1 . Then the edges $e_1 = \{(u_1, v_1), (u_3, v_1), (u_4, v_1), \dots, (u_{n+1}, v_1)\}$ and $e_2 = \{(u_1, v_2), (u_3, v_2), (u_4, v_2), \dots, (u_{n+1}, v_2)\}$ of $H_1[H_2]$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$ and $v_1 = v_2$

Let $f = \{v_1, v_3, v_4, \dots, v_{p+1}\}$ be an edge of H_2 . Now, the edges $e_1 = \{(u_1, v_1), (u_1, v_3), (u_1, v_4), \dots, (u_1, v_{p+1})\}$ and $e_2 = \{(u_2, v_1), (u_2, v_3), (u_2, v_4), \dots, (u_2, v_{p+1})\}$ of $H_1[H_2]$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_2, v_1) \in e_2$.

Case 3. $u_1 \neq u_2$ and $v_1 \neq v_2$

Subcase 1. There exists an edge of H_2 containing both v_1 and v_2 .

Let $f = \{v_1, v_2, \dots, v_p\}$ be an edge of H_2 . Now, the edges

$$e_1 = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), \dots, (u_1, v_p)\}$$

and

$$e_2 = \{(u_2, v_1), (u_2, v_2), (u_2, v_3), \dots, (u_2, v_p)\}$$

of $H_1[H_2]$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$.

Subcase 2. There exists no edge of H_2 containing both v_1 and v_2 .

Let $f_1 = \{v_1, w_2, w_3, \dots, w_p\}$ and $f_2 = \{v_2, z_2, z_3, \dots, z_q\}$ be edges H_2 containing v_1 and v_2 respectively. Then the edges

$$e_1 = \{(u_1, v_1), (u_1, w_2), (u_1, w_3), \dots, (u_1, w_p)\}$$

and

$$e_2 = \{(u_2, v_2), (u_2, z_2), (u_2, z_3), \dots, (u_2, z_q)\}$$

of $H_1[H_2]$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$. $\square$

4. Conclusion

In this paper we have discussed conditions under which Cartesian product of two hypergraphs to be Hausdorff. It is proved that wreath product of any two hypergraphs is always Hausdorff.

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