**A NOTE ON SOME CLASSES OF FUNCTIONS
RELATED TO THE CLASS OF THE CONVEX FUNCTIONS**

Donka Pashkouleva

Institute of Mathematics and Informatics

Bulgarian Academy of Sciences

Acad. G. Bonchev Str., Block 8, Sofia 1113, BULGARIA

Abstract: In this article the author continues the examination of the classes V_k and C_k^* of functions univalent in the unit disc. In addition to the already obtained sharp growth and distortion results for these classes, we provide a length-area estimate for the class C_k^* .

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1. Introduction and Definitions

Let us first recall the main definitions for the necessary background. By S we denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$, and by C denote the class of convex functions:

$$f(z) \in C \text{ if and only if for } z \in E, \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0.$$

Many authors have examined different classes of functions related to the class C of the convex functions. Some of the new classes are sub-classes of the class C , and others contain this class. Recently, some interesting results have

been obtained by E. Aliago and N. Tuneski [1].

The following definitions will be necessary for the definition of a new class C_k^* , for which a length-area estimate will be obtained.

For $k \geq 2$, denote by V_k the class of normalized functions of bounded boundary rotation at most $k\pi$. Thus $g(z) \in V_k$ if and only if $g(z)$ is analytic in E , $g'(z) \neq 0$, $g(0) = g'(0) - 1 = 0$ and for $z \in E$:

$$\int_0^{2\pi} \left| \Re \frac{(zg'(z))'}{g'(z)} \right| d\theta \leq k\pi. \quad (1.1)$$

Another definition of the class V_k comes from the known fact (see [7]) that for $2 \leq k \leq 4$, V_k consists only of univalent functions (in fact, close-to-convex functions).

For fixed k , let V_k denote the class of functions $g(z)$ normalized so that

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

which are analytic in E and have an integral representation of the form

$$g'(z) = \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \log (1 - ze^{-it})^{-1} d\mu(t) \right\}, \quad (1.2)$$

where $\mu(t)$ is real-valued and bounded variation on $[0, 2\pi]$ with

$$\begin{aligned} \int_0^{2\pi} d\mu(t) &= 2\pi, \\ \int_0^{2\pi} |d\mu(t)| &\leq k\pi. \end{aligned} \quad (1.3)$$

The representation formula (1.2) together with (1.3) is due to Paatero [7] and is equivalent to the definition (1.1) for $g(z) \in V_k$.

In 1917, Lowner [2] was the first to consider functions of bounded boundary rotation. Later Paatero [7, 8] made an exhaustive study of the class. The function $g_k(z)$ defined for $z \in E$ by

$$g_k(z) = \frac{1}{k} \left[\left(\frac{1+z}{1-z} \right)^{\frac{k}{2}} - 1 \right] = \sum_{n=1}^{\infty} B_n(k) z^n \quad (1.4)$$

belongs to V_k and is extremal for many problems.

Paatero [7] proved some sharp distortion theorems for $g(z) \in V_k$.

Let $f(z)$ be analytic in E , $f'(0) \neq 0$ and normalized so that

$$f(0) = 0, \quad f'(0) = 1.$$

Then, for $k \geq 2$, $f(z) \in T_k$ if there exists a function $g(z) \in V_k$, such that for $z \in E$

$$\Re \frac{f'(z)}{g'(z)} > 0. \quad (1.5)$$

Clearly $T_2 = K$, the class of close-to-convex functions.

We now define a new subclass C_k^* which has the same relationship with T_k as C has with S^* (the class of starlike functions).

Let $f(z)$ be analytic in E and normalized so that $f(0) = 0$, $f'(0) = 1$ and $f'(z) \neq 0$. Then $f(z) \in C_k^*$ ($k \geq 2$) if there exists a function $g(z) \in V_k$ such that for $z \in E$

$$\Re \frac{(zf'(z))'}{g'(z)} > 0. \quad (1.6)$$

Clearly, $C_2^* = C^*$ – the class of quasi-convex functions [6].

From definition (1.6) it follows easily that

$$f(z) \in C_k^* \text{ if and only if } zf'(z) \in T_k. \quad (1.7)$$

If we consider the function

$$F_k(z) = \frac{1}{k+2} \left[\left(\frac{1+z}{1-z} \right)^{\frac{k}{2}+1} - 1 \right] = z + \sum_{n=2}^{\infty} A_n(k) z_n, \quad (1.8)$$

it is easy to show that $F_k(z) \in T_k$ ([5]).

Let P be the class of functions $h(z)$ given by $h(z) = \sum_{n=1}^{\infty} c_n z^n$, which are analytic and have positive real part in E .

For $z = re^{i\theta} \in E$ and $f(z) \in S$ we will denote

$$L(r, f) = \int_0^{2\pi} r \left| f'(re^{i\theta}) \right| d\theta.$$

Also, we will use the following denotation

$$A(r, f) = \int_0^{2\pi} \int_0^r \left| f'(re^{i\theta}) \right|^2 \rho d\rho d\theta = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}.$$

2. Some Known Results

Theorem 2.1. (see [7]) *If $g(z) \in V_k$, then for $z = re^{i\theta} \in E$*

$$\frac{1}{k} \left[1 - \left(\frac{1-r}{1+r} \right)^{\frac{k}{2}} \right] \leq |g(z)| \leq \frac{1}{k} \left[\left(\frac{1+r}{1-r} \right)^{\frac{k}{2}} - 1 \right],$$

$$\frac{1}{1-r^2} \left(\frac{1-r}{1+r} \right)^{\frac{k}{2}} \leq g'(z) \leq \frac{1}{1-r^2} \left(\frac{1+r}{1-r} \right)^{\frac{k}{2}}.$$

The function $g_k(z)$ defined for $z \in E$ by

$$g_k(z) = \frac{1}{k} \left[\left(\frac{1+z}{1-z} \right)^{\frac{k}{2}} - 1 \right] = \sum_{n=2}^{\infty} B_n z^n$$

shows that these inequalities are sharp.

Theorem 2.2. ([10]) *Let $g(z) \in V_k$ and $\xi \in E$. Then $F(z) \in V_k$, where $F(z)$ is given by*

$$F(z) = \frac{g\left(\frac{\xi+z}{1+\xi z}\right) - g(\xi)}{g'(\xi)[1-|\xi|^2]}.$$

Theorem 2.3. ([5]) *Let $f(z) \in T_k$, then for $z = re^{i\theta} \in E$:*

$$\frac{(1-r)^{\frac{k}{2}}}{(1+r)^{\frac{k}{2}+2}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{k}{2}}}{(1-r)^{\frac{k}{2}+2}}.$$

These bounds are sharp, and equality is attained for the functions $F_k(z)$ defined by

$$F_k(z) = \frac{1}{k+2} \left[\left(\frac{1+z}{1-z} \right)^{\frac{k}{2}+1} - 1 \right].$$

Theorem 2.4. ([3]) *If $h(z) \in P$, then for $z = re^{i\theta} \in E$:*

$$\frac{1-r}{1+r} \leq |h(z)| \leq \frac{1+r}{1-r}, \quad (i)$$

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2r}{1-r^2}, \quad (ii)$$

$$h'(z) \leq \frac{2\Re h(z)}{1-r^2}. \quad (iii)$$

Theorem 2.5. ([5]) Let $H(z)$ be analytic in E and be defined for $z \in E$ by $H(z)g'(z) = (zg'(z))'$ for $g(z) \in V_k$. Then

$$H(z) = \left(\frac{k}{2} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z),$$

where $h_1(z), h_2(z) \in P$.

Also for $z = re^{i\theta} \in E$,

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta \leq \frac{1 + (k^2 - 1)r^2}{1 - r^2}.$$

3. Some Basic Properties of Functions in the Classes V_k and C_k^*

Theorem 3.1. Let $g(z) \in V_k$, then for $z = re^{i\theta} \in E$

$$\frac{k(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1} \left[1 - \left(\frac{1-r}{1+r}\right)^{\frac{k}{2}}\right]} \leq \left| \frac{g'(z)}{g(z)} \right| \leq \frac{k(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1} \left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}} - 1\right]}.$$

The result is sharp.

Theorem 3.2. Let $f(z) \in C_k^*$, then for $z = re^{i\theta} \in E$ and $k \geq 2$

$$\frac{1}{k+2} \left[1 - \left(\frac{1-r}{1+r}\right)^{\frac{k}{2}+1}\right] \leq |zf'(z)| \leq \frac{1}{k+2} \left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}+1} - 1\right],$$

$$\frac{1}{k+2} \int_0^r \left[1 - \left(\frac{1-t}{1+t}\right)^{\frac{k}{2}+1}\right] \frac{dt}{t} \leq |f(z)| \leq \frac{1}{k+2} \int_0^r \left[\left(\frac{1+t}{1-t}\right)^{\frac{k}{2}+1} - 1\right] \frac{dt}{t}.$$

These results have been obtained by the author [9] by means of classic methods by using the already known results mentioned in Section 2.

4. A Length-Area Estimate for the Class C_k^*

Theorem 4.1. Let $f(z) \in C_k^*$ ($k \geq 2$). Then for $0 < r < 1$,

$$L(r) < C(k) [A(\sqrt{r})]^{\frac{1}{2}} \log \frac{1}{1-r},$$

where $C(k)$ is a constant depending only upon k .

Proof. With $z = re^{i\theta}$, (1.6) gives

$$\begin{aligned}
 L(r) &= \int_0^{2\pi} |zf'(z)| d\theta \leq \int_0^r \int_0^{2\pi} \left| (\rho e^{i\theta}) \left[\rho e^{i\theta} f'(\rho e^{i\theta}) \right]' \right| d\theta d\rho \\
 &= \int_0^r \int_0^{2\pi} \left| (\rho e^{i\theta}) g'(\rho e^{i\theta}) h(\rho e^{i\theta}) \right| d\theta d\rho \\
 &\leq \int_0^r \int_0^{\rho_1} \int_0^{2\pi} \left| \left[\rho e^{i\theta} g'(\rho e^{i\theta}) \right]' h(\rho e^{i\theta}) \right| d\theta d\rho d\rho_1 \\
 &+ \int_0^r \int_0^{\rho_1} \int_0^{2\pi} \left| (\rho e^{i\theta}) g'(\rho e^{i\theta}) h'(\rho e^{i\theta}) \right| d\theta d\rho d\rho_1 \\
 &= J_1(r) + J_2(r), \text{ say.}
 \end{aligned}$$

Now $J_1(r) = \int_0^r \int_0^{\rho_1} \int_0^{2\pi} \left| \left[\rho e^{i\theta} f'(\rho e^{i\theta}) \right]' H(\rho e^{i\theta}) \right| d\theta d\rho d\rho_1$, where

$$H(z) = \frac{(zg'(z))'}{g'(z)}.$$

Thus

$$\begin{aligned}
 J_1(r) &\leq \int_0^r \int_0^{\rho_1} \left\{ \int_0^{2\pi} \left| \left[(\rho e^{i\theta}) f'(\rho e^{i\theta}) \right]' \right|^2 d\theta \right\}^{\frac{1}{2}} d\rho d\rho_1 \\
 &\leq \int_0^r \int_0^{\rho_1} \left\{ \int_0^{2\pi} \left[\sum_{n=1}^{\infty} n^4 (a_n)^2 \rho^{2n-2} \right] d\theta \right\}^{\frac{1}{2}} \\
 &\times \left\{ 2\pi \int_0^{2\pi} \frac{1 + (k^2 - 1)\rho^2}{1 - \rho^2} d\theta \right\}^{\frac{1}{2}} d\rho d\rho_1,
 \end{aligned} \tag{4.1}$$

where we have used the Cauchy-Schwartz inequality and Theorem 2.5.

To estimate the first expression in (4.1) we proceed as follows:

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^4 |a_n|^2 \rho^{2n-2} &= \sum_{n=1}^{\infty} n |a_n|^2 \rho^n \times n^3 \rho^{n-2} \\
 &\leq \frac{A(\sqrt{\rho})}{\pi} \times \sum_{n=1}^{\infty} n^3 \rho^{n-2},
 \end{aligned} \tag{4.2}$$

where $A_\rho = \pi \sum_{n=1}^{\infty} n |a_n|^2 \rho^{2n}$.

To estimate the second part in (4.1), let

$$\Phi(n) = n^3 \rho^{n-2}.$$

Then $\Phi'(n) = 0$ if $n = n_0 = \frac{3}{\log \frac{1}{\rho}}$. Thus,

$$n^3 \rho^{n-2} \leq n_0^3 \rho^{n_0-2} \leq \frac{27}{\rho^2 e^3 (1-\rho)^3}. \quad (4.3)$$

On using (4.2) and (4.3) we obtain

$$\sum_{n=1}^{\infty} n^4 |a_n|^2 \rho^{2n-2} \leq \frac{A(\sqrt{\rho})}{\pi} \cdot \frac{27}{\rho^2 e^3 (1-\rho)^3}. \quad (4.4)$$

Thus from (4.1) and (4.4) we have

$$J_1(r) \leq C_1(k) [A(\sqrt{r})]^{\frac{1}{2}} \log \frac{1}{1-r}.$$

To estimate $J_2(r)$, we note from Theorem 2.4 that since $\Re h(z) > 0$ for $z \in E$:

$$|h'(\rho e^{i\theta})| \leq \frac{2\Re h(\rho e^{i\theta})}{1-\rho^2}.$$

Thus,

$$\begin{aligned} J_2(r) &\leq \int_0^r \int_0^{\rho_1} \left[\int_0^{2\pi} |(\rho e^{i\theta}) g'(\rho e^{i\theta})| \Re h(\rho e^{i\theta}) \right] \frac{d\rho d\rho_1}{1-\rho^2} \\ &\leq \frac{2k\pi}{\sqrt{\pi}} \int_0^r \int_0^{\rho_1} \frac{[A(\sqrt{r})]^{\frac{1}{2}}}{\rho(1-\rho)(1-\rho^2)} d\rho d\rho_1 \leq C_2(k) [A(\sqrt{r})]^{\frac{1}{2}} \log \frac{1}{1-r}. \end{aligned}$$

Combination of the estimates for $J_1(r)$ and $J_2(r)$ gives Theorem 4.1. $\square$

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