ON $\mathbb{R}$ -COMPLEX FINSLER SPACES
WITH SPECIAL (α, β) -METRIC

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Abstract: In the present paper, the notion of $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$ (where $\epsilon, \lambda \neq 0$ are constants) which is the generalization of Randers metric and Z. Shen's square metric is defined. The fundamental metric tensor fields g_{ij} and $g_{i\bar{j}}$ and their determinants and inverse tensor fields are obtained. Some examples of non-Hermitian $\mathbb{R}$ -complex Finsler spaces with the special (α, β) -metric are given.

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1. Introduction

The studies of $\mathbb{R}$ -complex Finsler spaces are quite new. Munteanu and Purcaru [14] have extended the notion of a complex Finsler space ([1], [2], [13], [15]) to a new class of Finsler space called the $\mathbb{R}$ -complex Finsler spaces by reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called the $\mathbb{R}$ -complex Finsler spaces.

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In the present paper, following the ideas from real Finsler space with special (α, β) -metric $\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$ (where $\epsilon, \lambda \neq 0$ are constants) ([6], [10], [11], [12], [15]), we introduce the similar notion on $\mathbb{R}$ -complex Finsler spaces. In this section, we keep the general setting from ([13], [14]) and subsequently we recall only some needed notions.

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, (z^k) be local complex coordinates in a chart (U, ϕ) and $T'M$ its holomorphic tangent bundle. It has a natural structure of a complex manifold, $\dim_{\mathbb{C}} T'M = 2n$ and the induced coordinates in a local chart on $u \in T'M$ are denoted by $u = (z^k, \eta^k)$. The changes of local coordinates in u are given by the rules:

$$z^{ik} = z^{ik}(z); \quad \eta'^k = \frac{\partial z'^k}{\partial z^j} \eta^j. \quad (1)$$

The natural frame $\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k} \right\}$ of $T'_u(T'M)$ transforms with the Jacobi matrix of (1) changes, $\frac{\partial}{\partial z^k} = \frac{\partial z'^k}{\partial z^k} \frac{\partial}{\partial z'^k} + \frac{\partial^2 z'^j}{\partial z^k \partial z^h} \eta^h \frac{\partial}{\partial \eta'^j}$; $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k} \frac{\partial}{\partial \eta'^j}$.

A complex nonlinear connection, briefly (c. n. c.), is a supplementary distribution $H(T'M)$ to the vertical distribution spanned by $\left\{ \frac{\partial}{\partial \eta^k} \right\}$ and an adapted frame in $H(T'M)$ is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are the coefficients of c. n. c. and they have a certain rule of change at (1) so that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M (d-tensor in [14] terminology). Next we use the abbreviations: $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\partial_k, \dot{\partial}_k, \delta_{\bar{k}}$ for their conjugates. The dual adapted basis of $\{\delta_k, \dot{\partial}_k\}$ are $\{dz^k, \delta\eta^k = d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, d\delta\eta^k\}$ theirs conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space ([1], [2], [13], [15]) is with respect to all complex scalars and the metric tensor of the space, is a Hermitian one.

An $\mathbb{R}$ -complex Finsler metric on M is a continuous function $F = T'M \rightarrow \mathbb{R}_+$ satisfying:

- (i) $L := F^2$ is smooth on $T'M$ (except the 0 sections);
- (ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
- (iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda|F(z, \eta, \bar{z}, \bar{\eta})$;

It follows that L is $(2, 0)$ homogenous with respect to the real scalars λ , and

in [14] we proved that the following identifies are fulfilled:

$$\frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i = 2L; \quad g_{ij} \eta^i + g_{\bar{j}i} \bar{\eta}^i = \frac{\partial L}{\partial \eta^j}, \quad (2)$$

$$\frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0', \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j} \eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0, \quad (3)$$

$$2L = g_{ij} \eta^i \eta^j + g_{\bar{i}\bar{j}} \bar{\eta}^i \bar{\eta}^j + 2g_{i\bar{j}} \eta^i \bar{\eta}^j, \quad (4)$$

where

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j}, \quad (5)$$

are the metric tensors of the space.

2. $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $\alpha + \epsilon\beta + \lambda \frac{\beta^2}{\alpha}$

An $\mathbb{R}$ -complex Finsler space [14] produces two fundamental tensor fields g_{ij} and $g_{i\bar{j}}$. For a proper Hermitian geometry, the invertibility of $g_{i\bar{j}}$ is a compulsory requirement, but for some physicist's point of view, in which Hermitian condition is an impediment, it seems more appropriate for g_{ij} to be an invertible metric tensor.

These problems led us to the study of Hermitian $\mathbb{R}$ -complex Finsler spaces (i.e., $\det(g_{i\bar{j}}) \neq 0$). The present section applies our results to $\mathbb{R}$ -complex Finsler spaces with special (α, β) -metric $\alpha + \epsilon\beta + \lambda \frac{\beta^2}{\alpha}$ ($\epsilon, \lambda \neq 0$ are constants).

Definition 1 ($\mathbb{R}$ -complex Finsler space). An $\mathbb{R}$ -complex Finsler space with (α, β) -metric is a pair (M, F) , where the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is $\mathbb{R}$ -homogenous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$ which are dependent on $z^i, \eta^i, \bar{z}^i$ and $\bar{\eta}^i$, ($i = 1, 2, \dots, n$), i.e.:

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \quad (6)$$

where

$$\begin{aligned} \alpha^2(z, \eta, \bar{z}, \bar{\eta}) &= \frac{1}{2}(a_{ij} \eta^i \eta^j + a_{\bar{i}\bar{j}} \bar{\eta}^i \bar{\eta}^j + 2a_{i\bar{j}} \eta^i \bar{\eta}^j) \\ &= \text{Re}\{a_{ij} \eta^i \eta^j + a_{i\bar{j}} \eta^i \bar{\eta}^j\}, \quad \beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(b_i \eta^i + b_{\bar{i}} \bar{\eta}^i) = \text{Re}\{b_i \eta^i\}, \end{aligned}$$

with:

$$a_{ij} = a_{ij}(z), \quad a_{i\bar{j}} = a_{i\bar{j}}(z), \quad b_i = b_i(z), \quad (7)$$

where $b_i(z)dz^i$ is a 1-form on the complex manifold M .

We denote

$$L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \quad (8)$$

Remark 2. $L = F^2$ is $\mathbb{R}$ -complex Finsler space with (α, β) -metric.

Definition 3. An $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$ ($\epsilon, \lambda \neq 0$ are constants) is given by:

$$L(\alpha, \beta) = \left(\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha} \right)^2. \quad (9)$$

It follows that $F(\alpha, \beta) = \alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$, where $\epsilon, \lambda \neq 0$ are constants.

Taking into account the 2-homogeneity condition of L :

$$\begin{aligned} L(\alpha(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}), \beta(z, \lambda\eta, \bar{z}, \lambda\bar{\eta})) &= \lambda^2 L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \\ \lambda &\in \mathbb{R}_+. \end{aligned} \quad (10)$$

We have the following proposition:

Proposition 4. ([5]) *In an $\mathbb{R}$ -complex Finsler space with (α, β) -metric the following equalities hold:*

$$\alpha L_\alpha + \beta L_\beta = 2L, \quad \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \quad (11)$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \quad (12)$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \quad (13)$$

Particular case: For an $\mathbb{R}$ -complex Finsler space with special (α, β) -metric

$L(\alpha, \beta) = \left(\alpha + \epsilon\beta + \lambda \frac{\beta^2}{\alpha} \right)^2$, we have:

$$L_\alpha = 2 \left(\alpha + \epsilon\beta - \epsilon\lambda \frac{\beta^3}{\alpha^2} - \lambda^2 \frac{\beta^4}{\alpha^3} \right), \quad (14)$$

$$L_\beta = 2 \left[\epsilon\alpha + (2\lambda + \epsilon^2)\beta + \frac{3\epsilon\lambda\beta^2}{\alpha} + \frac{2\lambda^2\beta^3}{\alpha^2} \right], \quad (15)$$

$$\alpha L_\alpha + \beta L_\beta = 2L, \quad (16)$$

$$L_{\alpha\alpha} = 2 \left(1 - \frac{2\epsilon\lambda\beta^3}{\alpha^3} + \frac{3\lambda^2\beta^4}{\alpha^4} \right), \quad (17)$$

$$L_{\alpha\beta} = 2 \left(\epsilon - \frac{3\epsilon\lambda\beta^2}{\alpha^2} - \frac{4\lambda^2\beta^3}{\alpha^3} \right), \quad (18)$$

$$L_{\beta\beta} = 2 \left[(2\lambda + \epsilon^2) + \frac{6\epsilon\lambda\beta}{\alpha} + \frac{6\lambda^2\beta^2}{\alpha^2} \right]. \quad (19)$$

In the following, we propose to determine the metric tensors for a $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $L(\alpha, \beta) = \left(\alpha + \epsilon\beta + \lambda \frac{\beta^2}{\alpha} \right)^2$, i.e., $g_{ij} := \frac{\partial^2 L(z, \eta, \bar{z}, \bar{\eta})}{\partial \eta^i \partial \eta^j}$, $g_{i\bar{j}} = \frac{\partial^2 L(z, \eta, \bar{z}, \bar{\eta})}{\partial \eta^i \partial \bar{\eta}^j}$.

We consider

$$\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} (a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j) = \frac{1}{2\alpha} l_i, \quad \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i, \quad (20)$$

$$\frac{\partial \alpha}{\partial \bar{\eta}^i} = \frac{1}{2\alpha} (a_{i\bar{j}}\bar{\eta}^j + a_{ij}\eta^j) = \frac{1}{2\alpha} l_i, \quad \frac{\partial \beta}{\partial \bar{\eta}^i} = \frac{1}{2} b_{\bar{i}}, \quad (21)$$

where

$$l_i = a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j, \quad l_{\bar{j}} = a_{i\bar{j}}\bar{\eta}^i + a_{ij}\eta^i, \quad (22)$$

and find immediately:

$$l_i \eta^i + l_{\bar{j}} \bar{\eta}^j = 2\alpha^2. \quad (23)$$

We denote $\eta_i = \frac{\partial l}{\partial \eta^i}$. Consequently, we obtain:

$$\eta_i = \rho_0 l_i + \rho_1 b_i, \quad (24)$$

where

$$\rho_0 = \frac{1}{2} \frac{L_\alpha}{\alpha} \text{ (0-homogeneity)}, \quad \rho_1 = \frac{1}{2} L_\alpha \text{ (1-homogeneity)}. \quad (25)$$

Differentiating (25) by η^j and $\bar{\eta}^j$ respectively, we obtain:

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2} l_j + \rho_{-1} b_j, \quad \frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2} l_{\bar{j}} + \rho_{-1} b_{\bar{j}}, \quad (26)$$

$$\frac{\partial \rho_1}{\partial \eta^i} = \rho_{-1} l_j + \mu_0 b_i, \quad \frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-2} l_{\bar{i}} + \mu_0 b_{\bar{i}}, \quad (27)$$

where

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_{\alpha}}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \quad (28)$$

Proposition 5. *The invariants of $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$ are given by:*

$$\rho_0 = \frac{\alpha^4 + \epsilon\alpha^3\beta - \lambda^2\beta^4 - \epsilon\delta\alpha\beta^3}{\alpha^4}, \quad (29)$$

$$\rho_1 = \epsilon\alpha + (2\delta + \epsilon^2)\beta + \frac{3\epsilon\delta\beta^2}{\alpha} + \frac{2\delta^2\beta^3}{\alpha^2},$$

$$\rho_{-2} = \frac{\beta}{2\alpha^6}(4\delta^2\beta^3 - \epsilon\alpha^3 - \epsilon\delta\alpha\beta^2), \quad (30)$$

$$\rho_{-1} = \frac{1}{2\alpha}(\epsilon\alpha^3 - 3\epsilon\delta\alpha\beta^2 - 4\delta^2\beta^3),$$

$$\mu_0 = \frac{1}{2\alpha^2}[(2\lambda + \epsilon^2)\alpha^2 + 6\epsilon\delta\alpha\beta + 6\lambda^2\beta^2].$$

Subscripts -2, -1, 0, 1 give us the degree of homogeneity of these invariants. We have immediately:

Proposition 6. *The fundamental tensor fields of an $\mathbb{R}$ -complex Finsler space with special (α, β) - metric $\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}$ are given by:*

$$\begin{aligned} g_{ij} = & \frac{(\alpha^4 + \epsilon\alpha^3\beta - \lambda^2\beta^4 - \epsilon\delta\alpha\beta^3)}{\alpha^4}a_{ij} + \frac{\beta}{2\alpha^6}(4\delta^2\beta^3 - \epsilon\alpha^3 \\ & - \epsilon\delta\alpha\beta^2)l_i l_j + \frac{1}{2\alpha^2}[(2\lambda + \epsilon^2)\alpha^2 + 6\epsilon\delta\alpha\beta + 6\lambda^2\beta^2]b_i b_j \\ & + \frac{1}{2\alpha^4}(\epsilon\alpha^3 - 3\epsilon\delta\alpha\beta^2 - 4\delta^2\beta^3)(b_j l_i + b_i l_j), \end{aligned}$$

$$\begin{aligned} g_{i\bar{j}} = & \frac{(\alpha^4 + \epsilon\alpha^3\beta - \lambda^2\beta^4 - \epsilon\delta\alpha\beta^3)}{\alpha^4}a_{i\bar{j}} + \frac{\beta}{2\alpha^6}(4\delta^2\beta^3 - \epsilon\alpha^3 \\ & - \epsilon\delta\alpha\beta^2)l_i l_{\bar{j}} + \frac{1}{2\alpha^2}[(2\lambda + \epsilon^2)\alpha^2 + 6\epsilon\delta\alpha\beta + 6\lambda^2\beta^2]b_i b_{\bar{j}} \\ & + \frac{1}{2\alpha^4}(\epsilon\alpha^3 - 3\epsilon\delta\alpha\beta^2 - 4\delta^2\beta^3)(b_{\bar{j}} l_i + b_i l_{\bar{j}}), \end{aligned}$$

or, in the equivalent form:

$$g_{ij} = \rho_0[a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j], \quad (31)$$

$$g_{i\bar{j}} = \rho_0[a_{i\bar{j}} - p_1 l_i l_{\bar{j}} + p_2 b_i b_{\bar{j}} + p_3 \eta_i \eta_{\bar{j}}], \quad (32)$$

where

$$\begin{aligned}
 p_1 &= \frac{4\lambda^3\beta^5 - \epsilon\alpha\lambda^2\beta^4 + 4\beta^3\alpha^2\lambda^2 - \beta^3\epsilon^2\alpha^2\lambda + 2\epsilon\alpha^3\lambda\beta^2 - \epsilon^2\alpha^4\beta - \alpha^5\epsilon}{2\alpha^2(\epsilon\alpha^3 + 2\alpha^2\beta\lambda + \alpha^2\beta\epsilon^2 + 3\epsilon\lambda\alpha\beta^2 + 2\lambda^2\beta^3)(\beta^2\lambda - \alpha^2)}, \\
 p_2 &= \frac{1}{(\beta^2\lambda - \alpha^2)(-\alpha^4 - \epsilon\alpha^3\beta + \lambda^2\beta^4 + \epsilon\lambda\alpha\beta^3)} \left[\alpha^2(8\lambda^3\beta^4 + \right. \\
 &\quad 10\epsilon\alpha\lambda^2\beta^3 - \beta^2\lambda[2\alpha^2\lambda + \epsilon^2\alpha^2 + 6\epsilon\lambda\alpha\beta + 6\lambda^2\beta^2] + \\
 &\quad 3\beta^2\epsilon^2\alpha^2\lambda - 2\epsilon\alpha^3\lambda\beta + \alpha^2[2\alpha^2\lambda + \epsilon^2\alpha^2 + 6\epsilon\lambda\alpha\beta + \\
 &\quad \left. 6\lambda^2\beta^2] - \epsilon^2\alpha^4) \right], \\
 p_3 &= [-(-\epsilon\alpha^3 + 4\lambda^2\beta^3 + 3\epsilon\lambda\alpha\beta^2)\alpha^6] / [2(-\alpha^4 - \epsilon\alpha^3\beta + \lambda^2\beta^4 + \\
 &\quad \epsilon\lambda\alpha\beta^3)^2(\epsilon\alpha^3 + 2\alpha^2\beta\lambda + \alpha^2\beta\epsilon^2 + 3\epsilon\lambda\alpha\beta^2 + 2\lambda^2\beta^3)].
 \end{aligned}$$

Proof. Using the relations (30) in Theorem 2.1, [5] by direct calculations we obtain the results. $\square$

The next objective is to obtain the determinant and the inverse of the tensor field g_{ij} . The solution is obtained by adapting Proposition 2.2 from [4] for an arbitrary non singular non-Hermitian matrix (Q_{ij}) . The result is given by the following proposition.

Proposition 7. Suppose

1. (Q_{ij}) is a non-singular $n \times n$ complex matrix with inverse (Q^{ji}) ;
2. C_i and $\bar{C}_i = \bar{C}_i$, $i = 1, 2, \dots, n$ are complex numbers;
3. $C^i = Q^{ji}C_j$ and its conjugates $C^2 = C^iC_i = \bar{C}^iC_{\bar{i}}$; $H_{ij} = Q_{ij} \pm C_iC_j$.
Then $\det(H_{ij}) = (1 \pm C^2)\det(Q_{ij})$.

And whenever $1 \pm C^2 \neq 0$, the matrix (H_{ij}) is invertible and in this case its inverse is $H^{ji} = Q^{ji} \mp \frac{1}{1 \pm C^2} C^iC^j$.

Theorem 8. For a non-Hermitian $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $L(\alpha, \beta) = (\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})^2$, $\beta \neq 0$ with $a_{i\bar{j}} = 0$, we have:

1. The contravariant tensor g^{ij} of the fundamental tensor field g_{ij} is:

$$\begin{aligned}
 g^{ij} &= \frac{1}{\rho_0} \left[a^{ji} + \left(\frac{p_1}{1 - \gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1 - \gamma p_1)^2} \right) \eta^i \eta^j - \frac{p_2 b^i b^j}{\tau} \right. \\
 &\quad \left. - \frac{\epsilon p_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau(1 - \gamma p_1)} - \frac{P^2 \eta^i \eta^j + PQ(\eta^i b^j + \eta^j b^i) + Q^2 b^i b^j}{1 + (P\gamma + Q\epsilon)\sqrt{p_3}} \right],
 \end{aligned}$$

where

$$\begin{aligned} l_i &= a_{ij}\eta^j, \quad \gamma = a_{ij}\eta^j\eta^k = l_k\eta^k, \quad \epsilon = b_j\eta^j, \quad \omega = b_jb^j, \\ b^k &= a^{jk}b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk}\eta^j b^k = l_k b^k, \quad l^j = a^{ji}l_i = \eta^j. \end{aligned}$$

2. $\det(g_{ij}) = (\rho_0)^n \left[1 + (P\gamma + Q\epsilon)\sqrt{p_3} \right] \left[1 + \omega + \frac{p_1\epsilon^2}{1-p_1\gamma} \right] (1-p_1\gamma)\det(a_{ij})$, where:

$$\begin{aligned} \rho_0 &= \frac{\alpha^4 + \epsilon\alpha^3\beta - \lambda^2\beta^4 - \epsilon\delta\alpha\beta^3}{\alpha^4}, \\ P &= 1 + \left[\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right] \gamma - \left[\frac{p_1 p_2}{\tau(1-\gamma p_1)} \right] \epsilon^2, \\ Q &= -\frac{p_2}{\tau}\epsilon + \left[\frac{p_1 p_2 \epsilon}{\tau(1-\gamma p_1)} \right] \gamma, \\ p_1 &= \frac{4\lambda^3\beta^5 - \epsilon\alpha\lambda^2\beta^4 + 4\beta^3\alpha^2\lambda^2 - \beta^3\epsilon^2\alpha^2\lambda + 2\epsilon\alpha^3\lambda\beta^2 - \epsilon^2\alpha^4\beta - \alpha^5\epsilon}{2\alpha^2(\epsilon\alpha^3 + 2\alpha^2\beta\lambda + \alpha^2\beta\epsilon^2 + 3\epsilon\lambda\alpha\beta^2 + 2\lambda^2\beta^3)(\beta^2\lambda - \alpha^2)}, \\ p_2 &= \frac{1}{(\beta^2\lambda - \alpha^2)(-\alpha^4 - \epsilon\alpha^3\beta + \lambda^2\beta^4 + \epsilon\lambda\alpha\beta^3)} \left[\alpha^2(8\lambda^3\beta^4 + \right. \\ &\quad 10\epsilon\alpha\lambda^2\beta^3 - \beta^2\lambda[2\alpha^2\lambda + \epsilon^2\alpha^2 + 6\epsilon\lambda\alpha\beta + 6\lambda^2\beta^2] + \\ &\quad 3\beta^2\epsilon^2\alpha^2\lambda - 2\epsilon\alpha^3\lambda\beta + \alpha^2[2\alpha^2\lambda + \epsilon^2\alpha^2 + 6\epsilon\lambda\alpha\beta + \\ &\quad \left. 6\lambda^2\beta^2] - \epsilon^2\alpha^4) \right], \\ p_3 &= [(-\epsilon\alpha^3 + 4\lambda^2\beta^3 + 3\epsilon\lambda\alpha\beta^2)\alpha^6]/[2(-\alpha^4 - \epsilon\alpha^3\beta + \lambda^2\beta^4 + \epsilon\lambda\alpha\beta^3)^2(\epsilon\alpha^3 + 2\alpha^2\beta\lambda + \alpha^2\beta\epsilon^2 + 3\epsilon\lambda\alpha\beta^2 + 2\lambda^2\beta^3)]. \end{aligned}$$

Proof. To prove the claims we apply the above proposition in a recursive algorithm in three steps. We write g_{ij} from (31) in the form:

$$g_{ij} = \rho_0[a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j]. \quad (33)$$

1. In the first step, we set $Q_{ij} = a_{ij}$ and $c_i = \sqrt{p_1}l_i$. By applying the proposition 7, we obtain $Q^{ji} = a^{ji}$, $c^2 = \gamma p_1$. So, the matrix $H_{ij} = a_{ij} - p_1 l_i l_j$ is invertible with

$$H^{ji} = a^{ji} + \frac{p_1 \eta^i \eta^j}{1 - p_1 \gamma} \quad (34)$$

and $\det(a_{ij} - p_1 l_i l_j) = (1 - p_1 \gamma)\det(a_{ij})$.

2. Now, we consider $Q_{ij} = a_{ij} - p_1 l_i l_j$ and $c_i = \sqrt{p_2} b_i$. By applying Proposition 7, we obtain this time: $Q^{ji} = a^{ji} + \frac{p_1 \eta^i \eta^j}{1-p_1 \gamma}$, $c^2 = p_2 \left(\omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma} \right)$, $1 + c^2 = 1 + \omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma} \neq 0$ and $c^i = \sqrt{p_2} \left(b^i + \frac{\epsilon p_1 \eta^i}{1-p_1 \gamma} \right)$.

So, the matrix $H_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j$ is invertible with

$$\begin{aligned} H^{ji} = & a^{ji} + \left[\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right] \eta^i \eta^j - \frac{p_2 b^i b^j}{\tau} \\ & - \frac{\epsilon p_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau(1-\gamma p_1)}, \end{aligned} \quad (35)$$

where $\tau = 1 + \omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma}$ and $\det(a_{ij} - p_1 l_i l_j + p_2 b_i b_j) = \left[1 + \omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma} \right] (1 - p_1 \gamma) \det(a_{ij})$.

3. Now, we consider $Q_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j$ and $c_i = \sqrt{p_3} \eta_i$. By applying Proposition 7, we obtain this time: $Q^{ji} = a^{ji} + \left[\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right] \eta^i \eta^j - \frac{p_2 b^i b^j}{\tau} - \frac{\epsilon p_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau(1-\gamma p_1)}$, $c^i = \left[\eta^i + \left(\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right) \eta^i \gamma - \frac{p_2 b^i \epsilon}{\tau} - \frac{\epsilon p_1 p_2 (b^i \gamma + \epsilon \eta^i)}{\tau(1-\gamma p_1)} \right] \sqrt{p_3} = P \eta^i + Q b^i$, where $P = 1 + \left[\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right] \gamma - \left[\frac{p_1 p_2}{\tau(1-\gamma p_1)} \right] \epsilon^2$, $Q = -\frac{p_2}{\tau} \epsilon + \left[\frac{p_1 p_2 \epsilon}{\tau(1-\gamma p_1)} \right] \gamma$ and $c^2 = (P \gamma + Q \epsilon) \sqrt{p_3}$, $1 + c^2 = 1 + (P \gamma + Q \epsilon) \sqrt{p_3} \neq 0$, where $c^i c^j = P^2 \eta^i \eta^j + P Q (\eta^i b^j + \eta^j b^i) + Q^2 b^i b^j$.

So, the matrix $H_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j$ is invertible with

$$\begin{aligned} H^{ji} = & a^{ji} + \left[\frac{p_1}{1-\gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1-\gamma p_1)^2} \right] \eta^i \eta^j - \frac{p_2 b^i b^j}{\tau} - \frac{\epsilon p_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau(1-\gamma p_1)} \\ & - \frac{P^2 \eta^i \eta^j + P Q (\eta^i b^j + \eta^j b^i) + Q^2 b^i b^j}{1 + (P \gamma + Q \epsilon) \sqrt{p_3}}, \end{aligned} \quad (36)$$

and $\det(a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j) = \left[1 + (P \gamma + Q \epsilon) \sqrt{p_3} \right] \left[1 + \omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma} \right] (1 - p_1 \gamma) \det(a_{ij})$. But $g_{ij} = \rho_0 H_{ij}$, with H_{ij} from Step 3. Thus, $g^{ij} = \frac{1}{\rho_0} H^{ji}$ and $\det(g_{ij}) = (\rho_0)^n \left[1 + (P \gamma + Q \epsilon) \sqrt{p_3} \right] \left[1 + \omega + \frac{p_1 \epsilon^2}{1-p_1 \gamma} \right] (1 - p_1 \gamma) \det(a_{ij})$. Hence the statement holds. $\square$

Proposition 9. *In a non-Hermitian $\mathbb{R}$ -complex Finsler space with special (α, β) -metric $\alpha + \epsilon \beta + k \frac{\beta^2}{\alpha}$, we have the following properties:*

$$\gamma + \bar{\gamma} = l_i \eta^i + \bar{l}_{\bar{j}} \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}\bar{k}} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \quad (37)$$

$$\theta + \bar{\theta} = b_j \eta^j + \bar{b}_{\bar{j}} \eta^{\bar{j}} = 2\beta, \delta = \theta, \quad (38)$$

where:

$$\begin{aligned} l_i &= a_{ij}\eta^j, \quad \gamma = a_{ij}\eta^j\eta^k = l_k\eta^k, \quad \theta = b_j\eta^j, \quad \omega = b_jb^j, \\ b^k &= a^{jk}b_j, \quad b_l = b^ka_{kl}, \quad \delta = a_{jk}\eta^jb^k = l_kb^k, \quad l^j = a^{ji}l_i = \eta^j. \end{aligned}$$

Example 10. We set α as

$$\alpha^2 = \frac{(1 + \theta|z|) \sum_{k=1}^n \operatorname{Re}(\eta^j)^2 - \theta \operatorname{Re} \langle z, \eta \rangle^2}{(1 + \theta|z|^2)^2}, \quad (39)$$

where $|z|^2 = \sum_{k=1}^n z^k \bar{z}^k$, $\langle z, \eta \rangle = \sum_{k=1}^n z^k \bar{\eta}^k$, defined over the disk $\Delta_r^n = \{z \in \mathbb{C}^n, |z| < r, r = \sqrt{\frac{1}{|\theta|}}\}$ if $\theta = 0$ and on the complex projective space $P^n(C)$ if $\theta > 0$. By computation, we obtain $a_{ij} = \frac{1}{1+\theta|z|^2} \left(\delta_{ij} - \frac{\theta \bar{z}^i \bar{z}^j}{1+\theta|z|^2} \right)$ and $a_{i\bar{j}} = 0$ and so, $\alpha^2(z, \eta) = \frac{1}{2}(a_{ij}\eta^j\eta^i + a_{j\bar{k}}\eta^{\bar{i}}\eta^{\bar{j}})$. Now, taking $\beta(z, \eta) = \operatorname{Re} \frac{\langle z, \eta \rangle}{1+\theta|z|^2}$, where $b_i = \frac{\bar{z}^1}{1+\theta|z|^2}$, we obtain some examples of non- Hermitian $\mathbb{R}$ -complex special (α, β) -metric $\alpha + \theta\beta + k\frac{\beta^2}{\alpha}$:

$$\begin{aligned} F_\theta &= \sqrt{\frac{(1 + \theta|z|) \sum_{k=1}^n \operatorname{Re}(\eta^j)^2 - \theta \operatorname{Re} \langle z, \eta \rangle^2}{(1 + \theta|z|^2)^2}} + \epsilon \operatorname{Re} \frac{\langle z, \eta \rangle}{1 + \theta|z|^2} \\ &\quad + k \frac{\left(\operatorname{Re} \frac{\langle z, \eta \rangle}{1 + \theta|z|^2} \right)^2}{\sqrt{\frac{(1 + \theta|z|) \sum_{k=1}^n \operatorname{Re}(\eta^j)^2 - \theta \operatorname{Re} \langle z, \eta \rangle^2}{(1 + \theta|z|^2)^2}}}. \end{aligned} \quad (40)$$

Similarly, one can obtain the inverse for $g_{i\bar{j}}$.

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