ON $\mathbb{R}$-COMPLEX FINSLER SPACES WITH SPECIAL $(\alpha, \beta)$-METRIC

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Abstract: In the present paper, the notion of $\mathbb{R}$-complex Finsler space with special $(\alpha, \beta)$-metric $\alpha + \epsilon\beta + \lambda\beta^2/\alpha$ (where $\epsilon, \lambda \neq 0$ are constants) which is the generalization of Randers metric and Z. Shen’s square metric is defined. The fundamental metric tensor fields $g_{ij}$ and $\bar{g}_{ij}$ and their determinants and inverse tensor fields are obtained. Some examples of non-Hermitian $\mathbb{R}$-complex Finsler spaces with the special $(\alpha, \beta)$-metric are given.

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1. Introduction

The studies of $\mathbb{R}$-complex Finsler spaces are quite new. Munteanu and Purcaru [14] have extended the notion of a complex Finsler space ([1], [2], [13], [15]) to a new class of Finsler space called the $\mathbb{R}$-complex Finsler spaces by reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called the $\mathbb{R}$-complex Finsler spaces.

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In the present paper, following the ideas from real Finsler space with special \((\alpha, \beta)-metric\) \(\alpha + \epsilon\beta + \lambda\frac{\beta^2}{\alpha}\) (where \(\epsilon, \lambda \neq 0\) are constants) ([6], [10], [11], [12], [15]), we introduce the similar notion on \(\mathbb{R}\)-complex Finsler spaces. In this section, we keep the general setting from ([13], [14]) and subsequently we recall only some needed notions.

Let \(M\) be a complex manifold with \(\dim_{\mathbb{C}}M = n\), \((z^k)\) be local complex coordinates in a chart \((U, \phi)\) and \(T'M\) its holomorphic tangent bundle. It has a natural structure of a complex manifold, \(\dim_{\mathbb{C}}T'M = 2n\) and the induced coordinates in a local chart on \(u \in T'M\) are denoted by \(u = (z^k, \eta^k)\). The changes of local coordinates in \(u\) are given by the rules:

\[
z^{ik} = z^{ik}(z); \quad \eta^{ik} = \frac{\partial z^{ik}}{\partial z^j}\eta^j.
\]

The natural frame \(\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}\) of \(T'_u(T'M)\) transforms with the Jacobi matrix of (1) changes, \(\frac{\partial}{\partial z^k} = \frac{\partial z^{ik}}{\partial z^j} + \frac{\partial^2 z^{ij}}{\partial z^k \partial z^m} \eta^m \frac{\partial}{\partial \eta^j}; \frac{\partial}{\partial \eta^k} = \frac{\partial z^{ik}}{\partial z^j} \frac{\partial}{\partial \eta^j}\).

A complex nonlinear connection, briefly (c. n. c.), is a supplementary distribution \(H(T'M)\) to the vertical distribution spanned by \(\left\{\frac{\partial}{\partial \eta^k}\right\}\) and an adapted frame in \(H(T'M)\) is \(\delta_{\overline{z^k}} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\), where \(N^j_k\) are the coefficients of c. n. c. and they have a certain rule of change at (1) so that \(\delta_{\overline{z^k}}\) transform like vectors on the base manifold \(M\) (d-tensor in [14] terminology). Next we use the abbreviations: \(\partial_k = \frac{\partial}{\partial z^k}, \delta_k = \frac{\delta}{\delta z^k}, \hat{\partial}_k = \frac{\partial}{\partial \eta^k}\) and \(\hat{\partial}_k, \delta_k, \hat{\delta}_k\) for their conjugates. The dual adapted basis of \(\{\delta_k, \hat{\partial}_k\}\) are \(\{dz^k, \delta \eta^k = d\eta^k + N^j_k dz^j\}\) and \(\{d\overline{z^k}, d\delta \eta^k\}\) theirs conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space ([1], [2], [13], [15]) is with respect to all complex scalars and the metric tensor of the space, is a Hermitian one.

An \(\mathbb{R}\)-complex Finsler metric on \(M\) is a continuous function \(F = T'M \to \mathbb{R}_+\) satisfying:

(i) \(L := F^2\) is smooth on \(T'M\) (except the 0 sections);
(ii) \(F(z, \eta) \geq 0\), the equality holds if and only if \(\eta = 0\);
(iii) \(F(z, \lambda \eta, \bar{z}, \lambda \bar{\eta}) = |\lambda|F(z, \eta, \bar{z}, \bar{\eta})\);

It follows that \(L\) is \((2, 0)\) homogenous with respect to the real scalars \(\lambda, \) and
in [14] we proved that the following identities are fulfilled:

\[
\frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i = 2L; \quad g_{ij} \eta^i + \bar{g}_{ij} \bar{\eta}^i = \frac{\partial L}{\partial \eta^j},
\]

\( (2) \)

\[
\frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0', \quad \frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0,
\]

\( (3) \)

\[
2L = g_{ij} \eta^i \eta^j + \bar{g}_{ij} \bar{\eta}^i \bar{\eta}^j + 2g_{ij} \eta^i \bar{\eta}^j,
\]

\( (4) \)

where

\[
g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \quad \bar{g}_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \quad \bar{g}_{ij} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j},
\]

\( (5) \)

are the metric tensors of the space.

\[\text{2.} \quad \mathbb{R}\text{-complex Finsler space with special } (\alpha, \beta)\text{-metric } \alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha} \]

An \(\mathbb{R}\)-complex Finsler space [14] produces two fundamental tensor fields \(g_{ij}\) and \(g_{ij}\). For a proper Hermitian geometry, the invertibility of \(g_{ij}\) is a compulsory requirement, but for some physicist’s point of view, in which Hermitian condition is an impediment, it seems more appropriate for \(g_{ij}\) to be an invertible metric tensor.

These problems led us to the study of Hermitian \(\mathbb{R}\)-complex Finsler spaces (i.e., \(\text{det}(g_{ij}) \neq 0\)). The present section applies our results to \(\mathbb{R}\)-complex Finsler spaces with special \((\alpha, \beta)\)-metric \(\alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha} \) \((\epsilon, \lambda \neq 0 \text{ are constants})\).

\textbf{Definition 1 (\(\mathbb{R}\)-complex Finsler space)}. An \(\mathbb{R}\)-complex Finsler space with \((\alpha, \beta)\)-metric is a pair \((M, F)\), where the fundamental function \(F(z, \eta, \bar{z}, \bar{\eta})\) is \(\mathbb{R}\)-homogenous by means of functions \(\alpha(z, \eta, \bar{z}, \bar{\eta})\) and \(\beta(z, \eta, \bar{z}, \bar{\eta})\) which are dependent on \(z^i, \eta^i, \bar{z}^i\) and \(\bar{\eta}^i\), \((i = 1, 2, \ldots n\) ), i.e.:

\[
F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})),
\]

\( (6) \)

where

\[
\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(a_{ij} \eta^i \eta^j + a_{ij} \bar{\eta}^i \bar{\eta}^j + 2a_{ij} \eta^i \bar{\eta}^j)
\]

\[
= \text{Re}\{a_{ij} \eta^i \bar{\eta}^j + a_{ij} \eta^i \eta^j\}, \quad \beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(b_i \eta^i + b_i \bar{\eta}^i) = \text{Re}\{b_i \eta^i\},
\]

with:

\[
a_{ij} = a_{ij}(z), \quad a_{ij} = a_{ij}(z), \quad b_i = b_i(z),
\]

\( (7) \)

where \(b_i(z)dz^i\) is a 1-form on the complex manifold \(M\).
We denote

\[ L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \]  

(8)

**Remark 2.** \( L = F^2 \) is \( \mathbb{R} \)-complex Finsler space with \((\alpha, \beta)\)-metric.

**Definition 3.** An \( \mathbb{R} \)-complex Finsler space with special \((\alpha, \beta)\)-metric \( \alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha} \) (\( \epsilon, \lambda \neq 0 \) are constants) is given by:

\[ L(\alpha, \beta) = \left( \alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha} \right)^2. \]  

(9)

It follows that \( F(\alpha, \beta) = \alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha} \), where \( \epsilon, \lambda \neq 0 \) are constants.

Taking into account the 2-homogeneity condition of \( L \):

\[ L(\alpha(z, \lambda \eta, \bar{z}, \lambda \bar{\eta}), \beta(z, \lambda \eta, \bar{z}, \lambda \bar{\eta})) = \lambda^2 L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \]  

\( \lambda \in \mathbb{R}_+ \).  

(10)

We have the following proposition:

**Proposition 4.** ([5]) In an \( \mathbb{R} \)-complex Finsler space with \((\alpha, \beta)\)-metric the following equalities hold:

\[ \alpha L_\alpha + \beta L_\beta = 2L, \quad \alpha L_{\alpha \alpha} + \beta L_{\alpha \beta} = L_\alpha, \]  

(11)

\[ \alpha L_{\alpha \beta} + \beta L_{\beta \beta} = L_\beta, \quad \alpha^2 L_{\alpha \alpha} + 2\alpha \beta L_{\alpha \beta} + \beta^2 L_{\beta \beta} = 2L, \]  

(12)

where

\[ L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha \beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad L_{\beta \beta} = \frac{\partial^2 L}{\partial \beta^2}. \]  

(13)

**Particular case:** For an \( \mathbb{R} \)-complex Finsler space with special \((\alpha, \beta)\)-metric
\[ L(\alpha, \beta) = \left(\alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha}\right)^2, \] we have:

\[ L\alpha = 2\left(\alpha + \epsilon \beta - \epsilon \lambda \frac{\beta^3}{\alpha^2} - \lambda^2 \frac{\beta^4}{\alpha^3}\right), \] (14)

\[ L\beta = 2\left[\epsilon \alpha + (2\lambda + \epsilon^2)\beta + \frac{3\epsilon \lambda \beta^2}{\alpha} + \frac{2\lambda^2 \beta^3}{\alpha^2}\right], \] (15)

\[ \alpha L\alpha + \beta L\beta = 2L, \] (16)

\[ L\alpha\alpha = 2\left(1 - \frac{2\epsilon \lambda \beta^3}{\alpha^3} + \frac{3\lambda^2 \beta^4}{\alpha^4}\right), \] (17)

\[ L\alpha\beta = 2\left(\epsilon - \frac{3\epsilon \lambda \beta^2}{\alpha^2} - \frac{4\lambda^2 \beta^3}{\alpha^3}\right), \] (18)

\[ L\beta\beta = 2\left[(2\lambda + \epsilon^2) + \frac{6\epsilon \lambda \beta}{\alpha} + \frac{6\lambda^2 \beta^2}{\alpha^2}\right]. \] (19)

In the following, we propose to determine the metric tensors for a \( \mathbb{R} \)-complex Finsler space with special \((\alpha, \beta)\)-metric \[ L(\alpha, \beta) = \left(\alpha + \epsilon \beta + \lambda \frac{\beta^2}{\alpha}\right)^2, \] i.e., \( g_{ij} := \frac{\partial^2 L(z, \eta, \bar{z}, \bar{\eta})}{\partial \eta^i \partial \eta^j}; \ g_{i\bar{j}} = \frac{\partial^2 L(z, \eta, \bar{z}, \bar{\eta})}{\partial \eta^i \partial \bar{\eta}^j}. \)

We consider

\[ \frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} (a_{ij} \eta^j + a_{ij} \bar{\eta}^j) = \frac{1}{2\alpha} l_i, \ \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i, \] (20)

\[ \frac{\partial \alpha}{\partial \bar{\eta}^i} = \frac{1}{2\alpha} (a_{ij} \eta^j + a_{ij} \bar{\eta}^j) = \frac{1}{2\alpha} l_i, \ \frac{\partial \beta}{\partial \bar{\eta}^i} = \frac{1}{2} b_i, \] (21)

where

\[ l_i = a_{ij} \eta^j + a_{ij} \bar{\eta}^j, \ \ l_j = a_{ij} \eta^j + a_{ij} \bar{\eta}^j, \] (22)

and find immediately:

\[ l_i \eta^i + l_j \bar{\eta}^j = 2\alpha^2. \] (23)

We denote \( \eta_i = \frac{\partial l_i}{\partial \eta^i}. \) Consequently, we obtain:

\[ \eta_i = \rho_0 l_i + \rho_1 b_i, \] (24)

where

\[ \rho_0 = \frac{1}{2} \frac{L\alpha}{\alpha} \ (0 - \text{homogeneity}), \ \rho_1 = \frac{1}{2} L\alpha \ (1 - \text{homogeneity}). \] (25)

Differentiating (25) by \( \eta^j \) and \( \bar{\eta}^j \) respectively, we obtain:

\[ \frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2} l_j + \rho_{-1} b_j, \ \ \frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2} l_j + \rho_{-1} b_j, \] (26)

\[ \frac{\partial \rho_1}{\partial \eta^i} = \rho_{-1} l_i + \mu_0 b_i, \ \ \frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-2} l_i + \mu_0 b_i, \] (27)
where
\[ \rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \] (28)

**Proposition 5.** The invariants of \( \mathbb{R} \)-complex Finsler space with special \((\alpha, \beta)\)-metric \( \alpha + \epsilon \beta + \frac{\lambda^2 \beta^2}{\alpha} \) are given by:

\[
\rho_0 = \frac{\alpha^4 + \epsilon \alpha^3 \beta - \lambda^2 \beta^4 - \epsilon \delta \alpha \beta^3}{\alpha^4},
\]

\[
\rho_1 = \epsilon \alpha + (2\delta + \epsilon^2)\beta + \frac{3\epsilon \delta^2}{\alpha^2} + \frac{2\delta^2 \beta^3}{\alpha^2},
\]

\[
\rho_{-2} = \frac{\beta}{2\alpha^6} (4\delta^2 \beta^3 - \epsilon \alpha^3 - \epsilon \delta \alpha \beta^2),
\]

\[
\rho_{-1} = \frac{1}{2\alpha} (\epsilon \alpha^3 - 3\epsilon \delta \alpha \beta^2 - 4\delta^2 \beta^3),
\]

\[
\mu_0 = \frac{1}{2\alpha^2} \left[ (2\lambda + \epsilon^2)\alpha^2 + 6\epsilon \delta \alpha \beta + 6\lambda^2 \beta^2 \right].
\] (29)

Subscripts \(-2, -1, 0, 1\) give us the degree of homogeneity of these invariants. We have immediately:

**Proposition 6.** The fundamental tensor fields of an \( \mathbb{R} \)-complex Finsler space with special \((\alpha, \beta)\)-metric \( \alpha + \epsilon \beta + \frac{\lambda^2 \beta^2}{\alpha} \) are given by:

\[
g_{ij} = \frac{\alpha^4 + \epsilon \alpha^3 \beta - \lambda^2 \beta^4 - \epsilon \delta \alpha \beta^3}{\alpha^4} a_{ij} + \frac{\beta}{2\alpha^6} (4\delta^2 \beta^3 - \epsilon \alpha^3)
\]

\[
- \epsilon \delta \alpha \beta^2) l_i l_j + \frac{1}{2\alpha^2} \left[ (2\lambda + \epsilon^2)\alpha^2 + 6\epsilon \delta \alpha \beta + 6\lambda^2 \beta^2 \right] b_i b_j
\]

\[
+ \frac{1}{2\alpha^4} (\epsilon \alpha^3 - 3\epsilon \delta \alpha \beta^2 - 4\delta^2 \beta^3) (b_j l_i + b_i l_j),
\]

or, in the equivalent form:

\[
g_{ij} = \frac{\alpha^4 + \epsilon \alpha^3 \beta - \lambda^2 \beta^4 - \epsilon \delta \alpha \beta^3}{\alpha^4} a_{ij} + \frac{\beta}{2\alpha^6} (4\delta^2 \beta^3 - \epsilon \alpha^3)
\]

\[
- \epsilon \delta \alpha \beta^2) l_i l_j + \frac{1}{2\alpha^2} \left[ (2\lambda + \epsilon^2)\alpha^2 + 6\epsilon \delta \alpha \beta + 6\lambda^2 \beta^2 \right] b_i b_j
\]

\[
+ \frac{1}{2\alpha^4} (\epsilon \alpha^3 - 3\epsilon \delta \alpha \beta^2 - 4\delta^2 \beta^3) (b_j l_i + b_i l_j),
\]

or, in the equivalent form:

\[
g_{ij} = \rho_0 [a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j], \quad (31)
g_{ij} = \rho_0 [a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j], \quad (32)\]
where

\[
p_1 = \frac{4\lambda^3 \beta^5 - \epsilon \alpha \lambda^2 \beta^4 + 4\beta^3 \alpha^2 \lambda^2 - \beta^3 \epsilon^2 \alpha^2 \lambda + 2\epsilon \alpha^3 \lambda \beta^2 - \epsilon^2 \alpha^4 \beta - \alpha^5 \epsilon}{2\alpha^2(\epsilon\alpha^3 + 2\alpha^2 \beta \lambda + \alpha^2 \beta \epsilon + 3\epsilon \lambda \alpha \beta^2 + 2\lambda^2 \beta^3)(\beta^2 \lambda - \alpha^2)},
\]

\[
p_2 = \frac{1}{(\beta^2 \lambda - \alpha^2)(-\alpha^4 - \epsilon \alpha^3 \beta + \lambda^2 \beta^4 + \epsilon \lambda \alpha \beta^3)} \left[ \alpha^2(8\lambda^3 \beta^4 + 10\epsilon \alpha \lambda^2 \beta^3 - \beta^2 \lambda[2\alpha^2 \lambda + \epsilon^2 \alpha^2 + 6\epsilon \lambda \alpha \beta + 6\lambda^2 \beta^2] + 3\beta^2 \epsilon^2 \alpha^2 \lambda - 2\epsilon \alpha^3 \lambda \beta + \alpha^2[2\alpha^2 \lambda + \epsilon^2 \alpha^2 + 6\epsilon \lambda \alpha \beta + 6\lambda^2 \beta^2] - \epsilon^2 \alpha^4) \right],
\]

\[
p_3 = \frac{[-(-\epsilon \alpha^3 + 4\lambda^2 \beta^3 + 3\epsilon \lambda \alpha \beta^2) \alpha^6]/[2(-\alpha^4 - \epsilon \alpha^3 \beta + \lambda^2 \beta^4 + \epsilon \lambda \alpha \beta^3) \epsilon^2 \alpha^3 + 2\alpha^2 \beta \lambda + \alpha^2 \beta \epsilon + 3\epsilon \lambda \alpha \beta^2 + 2\lambda^2 \beta^3]}.\]

**Proof.** Using the relations (30) in Theorem 2.1, [5] by direct calculations we obtain the results.

The next objective is to obtain the determinant and the inverse of the tensor field \(g_{ij}\). The solution is obtained by adapting Proposition 2.2 from [4] for an arbitrary non-singular non-Hermitian matrix \((Q_{ij})\). The result is given by the following proposition.

**Proposition 7.** Suppose

1. \((Q_{ij})\) is a non-singular \(n \times n\) complex matrix with inverse \((Q^i_j)\);
2. \(C_i\) and \(C_i = \bar{C}_i\), \(i = 1, 2, \ldots, n\) are complex numbers;
3. \(C^i = Q'^i_j C_j\) and its conjugates \(C^2 = C_i C_i = \bar{C}_i \bar{C}_i\); \(H_{ij} = Q_{ij} \pm C_i C_j\).

Then \(\text{det}(H_{ij}) = (1 \pm C^2)\text{det}(Q_{ij})\).

And whenever \(1 \pm C^2 \neq 0\), the matrix \((H_{ij})\) is invertible and in this case its inverse is \(H^{ij} = Q^{ij} \mp \frac{1}{1 \pm C^2} C^i C^j\).

**Theorem 8.** For a non-Hermitian \(\mathbb{R}\)-complex Finsler space with special \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = (\alpha + \epsilon \beta + k \frac{\beta^2}{\alpha})^2\), \(\beta \neq 0\) with \(a_{ij} = 0\), we have:

1. The contravariant tensor \(g^{ij}\) of the fundamental tensor field \(g_{ij}\) is:

\[
g^{ij} = \frac{1}{\rho_0} \left[ a^{ij} + \left( \frac{p_1}{1 - \gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau(1 - \gamma p_1)^2} \right) \eta^i \eta^j - \frac{p_2 b^i b^j}{\tau} \right] - \frac{\epsilon p_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau(1 - \gamma p_1)} - \frac{P^2 \eta^i \eta^j + P Q (\eta^i b^j + \eta^j b^i) + Q^2 b^i b^j}{1 + (P \gamma + Q \epsilon) \sqrt{\rho_3}} \right],
\]
where

\[ l_i = a_{ij} \eta^j, \quad \gamma = a_{ij} \eta^j \eta^k = l_k \eta^k, \quad \epsilon = b_j \eta^j, \quad \omega = b_j b^j, \]

\[ b^k = a^{jk} b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk} \eta^j b^k = l_k b^k, l^i = a^{ji} l_i = \eta^i. \]

2. \( \det(g_{ij}) = (\rho_0)^n \left[ 1 + (P \gamma + Q \epsilon) \sqrt{p_3} \left[ 1 + \omega + \frac{p_{12}}{1 - p_1 \gamma} \right] (1 - p_1 \gamma) \det(a_{ij}), \right.\]

\[
\rho_0 = \frac{\alpha^4 + \epsilon \alpha^3 \beta - \lambda^2 \beta^4 - \epsilon \delta \alpha \beta^3}{\alpha^4},
\]

\[ P = 1 + \left[ \frac{p_1}{1 - \gamma p_1} - \frac{\epsilon^2 p_1^2 p_2}{\tau (1 - \gamma p_1)^2} \right] \gamma - \left[ \frac{p_1 p_2}{\tau (1 - \gamma p_1)} \right] \epsilon^2, \]

\[ Q = -\frac{p_2}{\tau} \epsilon + \left[ \frac{p_1 p_2 \epsilon}{\tau (1 - \gamma p_1)} \right] \gamma, \]

\[ p_1 = \frac{4 \lambda^3 \beta^5 - \epsilon \alpha \lambda^2 \beta^4 + 4 \beta^3 \alpha^2 \lambda^2 - \beta^3 \epsilon \alpha^2 \lambda + 2 \epsilon \alpha^3 \lambda \beta^2 - \epsilon^2 \alpha^4 \beta - \alpha^5 \epsilon}{2 \alpha^2 (\epsilon \alpha^3 + 2 \alpha \beta \lambda + \alpha^2 \epsilon \beta^2 + 3 \epsilon \lambda \alpha \beta^2 + 2 \lambda^2 \beta^3)(\beta^2 \lambda - \alpha^2)}, \]

\[ p_2 = \frac{1}{(\beta^2 \lambda - \alpha^2)(-\alpha^4 - \epsilon \alpha^3 \beta + \lambda^2 \beta^4 + \epsilon \lambda \alpha \beta^3)} \left[ \alpha^2 (8 \lambda^3 \beta^4 + 10 \epsilon \alpha \lambda^2 \beta^3 - \beta^2 \lambda [2 \alpha^2 \lambda + \epsilon^2 \alpha^2 + 6 \epsilon \lambda \alpha \beta + 6 \lambda^2 \beta^2]) + 3 \beta^2 \epsilon^2 \alpha^2 \lambda - 2 \epsilon \alpha^3 \lambda \beta + \alpha^2 [2 \alpha^2 \lambda + \epsilon^2 \alpha^2 + 6 \epsilon \lambda \alpha \beta + 6 \lambda^2 \beta^2] - \beta^2 \alpha^4 \right], \]

\[ p_3 = \frac{[-(\epsilon \alpha^3 + 4 \lambda^2 \beta^3 + 3 \epsilon \lambda \alpha \beta^2) \alpha^6] / [2(-\alpha^4 - \epsilon \alpha^3 \beta + \lambda^2 \beta^4 + \epsilon \lambda \alpha \beta^3) (\epsilon \alpha^3 + 2 \alpha^2 \beta \lambda + \alpha^2 \beta \epsilon^2 + 3 \epsilon \lambda \alpha \beta^2 + 2 \lambda^2 \beta^3)]}{2}. \]

**Proof.** To prove the claims we apply the above proposition in a recursive algorithm in three steps. We write \( g_{ij} \) from (31) in the form:

\[ g_{ij} = \rho_0 [a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta^i \eta^j]. \tag{33} \]

1. In the first step, we set \( Q_{ij} = a_{ij} \) and \( c_i = \sqrt{p_1 l_i} \). By applying the proposition 7, we obtain \( Q^{ji} = a^{ji}, \ c^2 = \gamma p_1 \). So, the matrix \( H_{ij} = a_{ij} - p_1 l_i l_j \) is invertible with

\[ H^{ji} = a^{ji} + \frac{p_1 \eta^i \eta^j}{1 - p_1 \gamma}. \tag{34} \]

and \( \det(a_{ij} - p_1 l_i l_j) = (1 - p_1 \gamma) \det(a_{ij}) \).
2. Now, we consider $Q_{ij} = a_{ij} - p_1 l_i l_j$ and $c_i = \sqrt{p_2} b_i$. By applying Proposition 7, we obtain this time: $Q^{ji} = a^{ji} + \frac{p_1 \eta_j \eta_i}{1 - p_1 \gamma}, c^2 = p_2 \left( \omega + \frac{p_1 c^2}{1 - p_1 \gamma} \right)$, $1 + c^2 = 1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma} \neq 0$ and $c^i = \sqrt{p_2} \left( b^i + \frac{p_1 \eta_j}{1 - p_1 \gamma} \right)$.

So, the matrix $H_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j$ is invertible with

$$H^{ji} = a^{ji} + \left[ \frac{p_1}{1 - \gamma p_1} - \frac{c_i^2 p_2}{(1 - \gamma p_1)^2} \right] \eta_j \eta^i - \frac{p_2 b^i b^j}{\tau}$$

$$= \frac{c_i^2 p_2}{\tau (1 - \gamma p_1)}$$

where $\tau = 1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma}$ and $\text{det}(a_{ij} - p_1 l_i l_j + p_2 b_i b_j) = \left[ 1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma} \right] (1 - p_1 \gamma) \text{det}(a_{ij})$.

3. Now, we consider $Q_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j$ and $c_i = \sqrt{p_3} \eta_i$. By applying Proposition 7, we obtain this time: $Q^{ji} = a^{ji} + \left[ \frac{p_1}{1 - \gamma p_1} - \frac{c_i^2 p_2}{(1 - \gamma p_1)^2} \right] \eta_j \eta^i - \frac{p_2 b^i b^j}{\tau} - \frac{ep_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau (1 - \gamma p_1)}, c^i = \left[ \eta^i + \left( \frac{p_1}{1 - \gamma p_1} - \frac{c_i^2 p_2}{(1 - \gamma p_1)^2} \right) \eta^i \gamma - \frac{p_2 b^i}{\tau} - \frac{ep_1 p_2 (b^i \gamma + \eta^i)}{\tau (1 - \gamma p_1)} \right] \sqrt{p_3} = P \eta^i + Q b^i$, where $P = 1 + \left[ \frac{p_1}{1 - \gamma p_1} - \frac{c_i^2 p_2}{(1 - \gamma p_1)^2} \right] \gamma - \left[ \frac{p_2 p_2}{\tau (1 - \gamma p_1)} \right] \epsilon^2, Q = - \frac{p_2 \epsilon + \frac{p_1 p_2}{\tau (1 - \gamma p_1)} \gamma}{1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma}}$.

So, the matrix $H_{ij} = a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j$ is invertible with

$$H^{ji} = a^{ji} + \left[ \frac{p_1}{1 - \gamma p_1} - \frac{c_i^2 p_2}{(1 - \gamma p_1)^2} \right] \eta_j \eta^i - \frac{p_2 b^i b^j}{\tau} - \frac{ep_1 p_2 (b^i \eta^j + b^j \eta^i)}{\tau (1 - \gamma p_1)}$$

$$= \frac{P \eta^i + Q b^i + Q^2 b^i b^j}{1 + (P \gamma + Q \epsilon) \sqrt{p_3}}, (36)$$

and $\text{det}(a_{ij} - p_1 l_i l_j + p_2 b_i b_j + p_3 \eta_i \eta_j) = \left[ 1 + (P \gamma + Q \epsilon) \sqrt{p_3} \right] \left[ 1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma} \right] (1 - p_1 \gamma) \text{det}(a_{ij})$.

But $g_{ij} = \rho_0 H_{ij}$, with $H_{ij}$ from Step 3. Thus, $g^{ij} = \frac{1}{\rho_0} H^{ij}$ and $\text{det}(g_{ij}) = (\rho_0)^n \left[ 1 + (P \gamma + Q \epsilon) \sqrt{p_3} \right] \left[ 1 + \omega + \frac{p_1 c^2}{1 - p_1 \gamma} \right] (1 - p_1 \gamma) \text{det}(a_{ij})$. Hence the statement holds.

\textbf{Proposition 9.} In a non-Hermitian $\mathbb{R}$-complex Finsler space with special $(\alpha, \beta)$-metric $\alpha + \epsilon \beta + k^2 \frac{\alpha^2}{\alpha}$, we have the following properties:

$$\gamma + \bar{\gamma} = l_i \eta^i + l_j \eta^j = a_{ij} \eta^i \eta^j + a_{jk} \eta^k \eta^j = 2 \alpha^2,$$

$$\theta + \bar{\theta} = b_j \eta^i + b_j \eta^j = 2 \beta, \delta = \theta,$$
where:

\[ l_i = a_{ij} \eta^i, \quad \gamma = a_{ij} \eta^i \eta^j = l_k \eta^k, \quad \theta = b_j \eta^j, \quad \omega = b_j b^j, \]

\[ b^k = a^j b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk} \eta^j b^k = l_k b^k, l^j = a^{ji} l_i = \eta^i. \]

**Example 10.** We set \( \alpha \) as

\[
\alpha^2 = \frac{(1 + \theta |z|) \sum_{k=1}^{n} \Re(\eta^i)^2 - \theta \Re < z, \eta >^2}{(1 + \theta |z|^2)^2}, \tag{39}
\]

where \( |z|^2 = \sum_{k=1}^{n} z^k \bar{z}^k, \quad < z, \eta > = \sum_{k=1}^{n} z^k \bar{\eta}^k \), defined over the disk \( \Delta^n_r = \{ z \in \mathbb{C}^n, |z| < r, r = \sqrt{\frac{1}{|\theta|}} \} \) if \( \theta = 0 \) and on the complex projective space \( P^n(C) \) if \( \theta > 0 \). By computation, we obtain \( a_{ij} = \frac{1}{1 + \theta |z|^2} \left( \delta_{ij} - \frac{\theta \bar{z}^i z^j}{1 + \theta |z|^2} \right) \) and \( a_{ij} = 0 \) and so, \( \alpha^2(z, \eta) = \frac{1}{2} (a_{ij} \eta^i \eta^j + a_{jk} \eta^j \eta^k) \). Now, taking \( \beta(z, \eta) = \Re \frac{<z, \eta>}{1 + \theta |z|^2} \), where \( b_i = \frac{\bar{z}^i}{1 + \theta |z|^2} \), we obtain some examples of non-Hermitian \( \mathbb{R} \)-complex special \( (\alpha, \beta) \)-metric \( \alpha + \theta \beta + k \frac{\bar{\alpha}^2}{\alpha} \):

\[
F_\theta = \sqrt{\frac{(1 + \theta |z|) \sum_{k=1}^{n} \Re(\eta^i)^2 - \theta \Re < z, \eta >^2}{(1 + \theta |z|^2)^2} + \epsilon \Re \frac{<z, \eta>}{1 + \theta |z|^2}} + k \frac{\left( \Re \frac{<z, \eta>}{1 + \theta |z|^2} \right)^2}{\sqrt{(1 + \theta |z|) \sum_{k=1}^{n} \Re(\eta^i)^2 - \theta \Re < z, \eta >^2}}. \tag{40}
\]

Similarly, one can obtain the inverse for \( g_{ij} \).

**References**


