ON THE CONVERGENCE OF
DIRICHLET SERIES WITH RANDOM EXPONENTS

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Abstract: For the Dirichlet series of the form $F(z, \omega) = \sum_{k=0}^{+\infty} f_k(\omega) e^{z\lambda_k(\omega)}$ ($z \in \mathbb{C}$, $\omega \in \Omega$) with pairwise independent real exponents $(\lambda_k(\omega))$ on probability space $(\Omega, \mathcal{A}, P)$ an estimates of abscissas convergence and absolutely convergence are established.

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1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\Lambda = (\lambda_k(\omega))_{k=0}^{+\infty}$ and $\mathbf{f} = (f_k(\omega))_{k=0}^{+\infty}$ sequences of positive and complex-valued random variables on it, respectively. Let $\mathcal{D}$ be the class of formal random Dirichlet series of the form

$$f(z) = f(z, \omega) = \sum_{k=0}^{+\infty} f_k(\omega) e^{z\lambda_k(\omega)} \quad (z \in \mathbb{C}, \omega \in \Omega).$$

Let $\sigma_c(f, \omega)$ and $\sigma(f, \omega)$ be the abscissa of convergence and absolute convergence of this series for fixed $\omega \in \Omega$, respectively. The simple modification of [1]–[3] one has that for the Dirichlet series $f \in \mathcal{D}$ for fixed $\omega \in \Omega$ such that

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$$\lambda_k(\omega) \rightarrow +\infty \quad (k \rightarrow +\infty)$$

$$\begin{aligned} \sigma(f, \omega) &\leq \sigma_c(f, \omega) \leq \alpha_0(\omega) := \lim_{k \rightarrow +\infty} \frac{-\ln |f_k(\omega)|}{\lambda_k(\omega)} \\ &\leq \sigma(f, \omega) + \tau(\omega, \Lambda), \end{aligned} \quad (1)$$

or in the case $-\ln |f_k(\omega)| \rightarrow +\infty \quad (k \rightarrow +\infty)$

$$(1-h)\sigma_c(f, \omega) \leq (1-h)\alpha_0(\omega) \leq \sigma(f, \omega), \quad h = h(\omega, \mathbf{f}), \quad (2)$$

where $\tau(\omega, \Lambda) := \overline{\lim}_{k \rightarrow \infty} \frac{\ln k}{\lambda_k(\omega)}$, $h(\omega, \mathbf{f}) := \overline{\lim}_{k \rightarrow \infty} \frac{\ln k}{-\ln |f_k(\omega)|}$.

Also,

$$\sigma_c(f, \omega) = \sigma(f, \omega) = \alpha_0(\omega) \quad (3)$$

for fixed $\omega \in \Omega$ such that $\tau(\omega) = 0$ or $\ln k / (-\ln |f_k(\omega)|) \rightarrow +0 \quad (k \rightarrow +\infty)$. Remark, that from condition $\tau(\omega) < +\infty$ we get $\lambda_k(\omega) \rightarrow +\infty \quad (k \rightarrow +\infty)$. In the case $\sigma_c(f, \omega) > 0$ the series of the form $\sum_{k=0}^{+\infty} f_k(\omega)$ is convergent, thus $-\ln |f_k(\omega)| \rightarrow +\infty \quad (k \rightarrow +\infty)$.

The following assertion is proved in [3, Corollary 5] (another version [2, Theorem 1]) in the case of the deterministic Dirichlet series with sequence of exponents that increase to infinity, i.e., $f_k(\omega) \equiv f_k \in \mathbb{C} \quad (k \geq 0)$ and $\lambda_k(\omega) \equiv \lambda_k$, $0 \leq \lambda_k < \lambda_{k+1} \rightarrow +\infty \quad (0 \leq k \rightarrow +\infty)$.

Proposition 1. *Let $f \in \mathcal{D}$. Then $\sigma_a(f, \omega) \leq \sigma_c(f, \omega) \leq \alpha_0(\omega) \quad (\forall \omega \in \Omega)$, and*

$$\sigma_a(f, \omega) \geq \gamma(\omega)\alpha_0(\omega) - \delta(\omega) \geq \gamma(\omega)\sigma_c(f, \omega) - \delta(\omega) \quad (4)$$

for arbitrary real random variables γ, δ and for all $\omega \in \Omega$ such that $\gamma(\omega) > 0$ and

$$\sum_{k=0}^{+\infty} |f_k(\omega)|^{1-\gamma(\omega)} e^{-\delta(\omega)\lambda_k(\omega)} < +\infty. \quad (5)$$

Remark 2. Condition (5) implies, that $(\gamma(\omega)-1) \ln |f_k(\omega)| + \delta(\omega)\lambda_k(\omega) \rightarrow +\infty \quad (k \rightarrow +\infty)$ for such ω . But, in general, from this condition don't follows neither $\lambda_k(\omega) \rightarrow +\infty$ nor $\ln |f_k(\omega)| \rightarrow \infty \quad (k \rightarrow +\infty)$.

Proof of Proposition 1. It is obvious that $\sigma(f, \omega) \leq \sigma_c(f, \omega)$.

We prove now that $\sigma_c(f, \omega) \leq \alpha_0(\omega)$. Indeed, assume first that $\alpha_0(\omega) \neq \infty$ and put $x_0 = \alpha_0(\omega) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Then, $|f_k(\omega)|e^{x_0\lambda_k(\omega)} = \exp\{\lambda_k(\omega)(\ln |f_k(\omega)|/\lambda_k(\omega) + x_0)\}$. But by definition of $\alpha_0(\omega)$ there exists a

sequence $k_j \rightarrow +\infty$ ($j \rightarrow +\infty$) such that $\ln |f_k(\omega)|/\lambda_k(\omega) > -(\alpha_0(\omega) + \varepsilon/2)$ ($k = k_j$, $j \geq 1$). Thus, $\ln |f_k(\omega)|/\lambda_k(\omega) + x_0 > \varepsilon/2$ ($k = k_j$, $j \geq 1$), and

$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} \geq e^{\lambda_k(\omega)\varepsilon/2} \geq 1 \quad (k = k_j, j \geq 1),$$

therefore $\sigma_c(f, \omega) \leq \alpha_0(\omega) + \varepsilon$, but $\varepsilon > 0$ is arbitrary.

The case $\alpha_0(\omega) = +\infty$ is trivial. In the case $\alpha_0(\omega) = -\infty$ for every $E > 0$ and for some sequence $k_j \rightarrow +\infty$ ($j \rightarrow +\infty$) by definition $\alpha_0(\omega)$ we obtain $\ln |f_k(\omega)|/\lambda_k > E$ ($k = k_j$, $j \geq 1$). Therefore $|f_k(\omega)| \exp\{-E\lambda_k\} > 1$ ($k = k_j$, $j \geq 1$), i.e. the Dirichlet series diverges at the point $z = -E$, but $E > 0$ is arbitrary. Thus, $\sigma_c = -\infty$.

Let now $x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega)$ for arbitrary $\varepsilon > 0$. Then,

$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} = |f_k(\omega)|^{1-\gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}\left(|f_k(\omega)|e^{(\alpha_0(\omega)-\varepsilon)\lambda_k(\omega)}\right)^{\gamma(\omega)}. \quad (6)$$

By definition of $\alpha_0(\omega)$, we obtain $\alpha_0(\omega) < \frac{-\ln f_k(\omega)}{\lambda_k(\omega)} + \varepsilon/2$ for $k \geq k_0(\omega)$, and thus $|f_k(\omega)|e^{(\alpha_0(\omega)-\varepsilon)\lambda_k(\omega)} < \exp\{-\lambda_k\varepsilon/2\} \leq 1$ ($k \geq k_0(\omega)$). Hence by (6) one has $|f_k(\omega)|e^{x_0\lambda_k(\omega)} \leq |f_k(\omega)|^{1-\gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}$ and by condition (5) we obtain $\sigma(f, \omega) \geq x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega)$. But, $\varepsilon > 0$ is arbitrary. $\square$

From Proposition 1 it simply follows such a statement.

Proposition 3. *Let $f \in \mathcal{D}$. Then equalities (3) hold for all $\omega \in \Omega$ such, that*

$$\ln k = o(\ln |f_k(\omega)|) \quad (k \rightarrow +\infty). \quad (7)$$

Remark 4. If the sequences Λ and $\mathbf{f}$ such that $(|f_k(\omega)|e^{x\lambda_k(\omega)})$ are the sequences of independent random variables for every $x \in \mathbb{R}$, then by Kolmogorov's Zero-One Law ([4]) random variable $\sigma(f, \omega)$ is almost surely (a.s.) constant. That is, $\sigma(f, \omega) = \sigma \in [-\infty, +\infty]$ a.s. In the book [4] it is written when Λ monotonic increasing to infinity sequence $\lambda_k(\omega) \equiv \lambda_k$. The same we get when $(\frac{-\ln |f_k(\omega)|}{\lambda_k(\omega)})$ is the sequence of independent random variables, and $\tau(\omega, \Lambda) = 0$ or $h(\omega, \mathbf{f}) = 0$. It follows from Proposition 3 and equalities (3).

In the papers [5]–[10] considered question about abscissas of convergence random Dirichlet series from the class $\mathcal{D}$ in case, when $\Lambda_+ = (\lambda_k)$ is increasing sequence of positive numbers, i.e., $0 = \lambda_0 < \lambda_k < \lambda_{k+1} \rightarrow +\infty$ ($1 \leq k \rightarrow +\infty$) and $\tau(\omega, \Lambda) \equiv \tau(\Lambda) < +\infty$.

We have such elementary assertion.

Proposition 5. *Let $f \in \mathcal{D}(\Lambda)$ be a Dirichlet series of the form $f(z) = f(z, \omega) = \sum_{k=0}^{+\infty} a_k Z_k(\omega) e^{z\lambda_k(\omega)}$, where $(Z_k(\omega))$ is a sequence of random complex-valued variables.*

1⁰. *If the condition $\tau(\omega, \Lambda) = 0$ holds and*

$$\lim_{k \rightarrow +\infty} \frac{-\ln |Z_k(\omega)|}{\lambda_k(\omega)} = 0 \quad \text{a.s.}, \quad (8)$$

then $\sigma_c(f, \omega) = \sigma(f, \omega) = \alpha_0^(\omega) := \varliminf_{k \rightarrow +\infty} -\ln |a_k| / \lambda_k(\omega)$ a.s.*

2⁰. *If $\alpha_0(\omega) = +\infty$ and the conditions $\tau(\omega, \Lambda) < +\infty$,*

$$\varliminf_{k \rightarrow +\infty} \frac{-\ln |Z_k(\omega)|}{\lambda_k(\omega)} > -\infty \quad \text{a.s.} \quad (9)$$

hold, then $\sigma(f, \omega) = +\infty$ a.s.

We obtain Proposition 5 immediately from inequalities (1).

In the paper [6], it is considered only 1⁰ for the case of the Dirichlet series $f \in \mathcal{D}(\Lambda_+)$ of the form $f(z) = f(z, \omega) = \sum_{k=0}^{+\infty} a_k Z_k(\omega) e^{z\lambda_k}$.

From Proposition 5, in particular, they follow Theorem 1 (when $\alpha_0 := \alpha_0^* = +\infty$) and Theorem 3 (when $\alpha_0 = 0$) from [6], which are proved under such condition for expectation:

$$(\exists \alpha > 0, \beta > 0): \sup\{\mathbf{E}|Z_k|^\alpha, \mathbf{E}|Z_k|^{-\beta}: k \geq 0\} < +\infty. \quad (10)$$

By the Bienayme-Chebyshev inequality ([11, 12]) and the Borel-Cantelli Lemma ([4], also about refined Second Borel-Cantelli lemma see [13]) from condition (10) it easy follows, that a.s. for all enough large k inequalities $k^{-\gamma} \leq |Z_k(\omega)| < k^\gamma$ with $\gamma = \max\{2/\alpha, 2/\beta\}$ hold, and if $\tau(\Lambda) = 0$, then and condition (8). Similarly, if $\tau(\Lambda) < +\infty$, then from condition $(\exists \beta > 0): \sup\{\mathbf{E}|Z_k|^{-\beta}: k \geq 0\} < +\infty$ follows condition (9).

It should be noted, that condition (8) follows from such condition (see [10]) on sequence of distribution functions of random variables $(|Z_k(\omega)|)$,

$$(\forall \varepsilon > 0): \sum_{k=0}^{+\infty} (1 - F_k^*(e^{\varepsilon\lambda_k}) + F_k^*(e^{-\varepsilon\lambda_k})) < +\infty,$$

where $F_k^*(x) := P\{\omega: |Z_k(\omega)| < x\}$. In particular, from this condition it follows $\lim_{k \rightarrow +\infty} F_k^*(+0) = 0$.

In the papers [7]–[9] in the case of independent random variables $\mathbf{f} = (f_k)$, besides, generalized on class $\mathcal{D}(\Lambda)$ assertion of known Blackwell's conjecture on power series with random coefficients, proved in [14] (see also [4]).

In the general case, for Dirichlet series from the class $\mathcal{D}(\Lambda_+)$ in [10] (see also similar results for random gap power series in [15]–[18]) two theorems are proved. In particular, we find ([10]) the following theorem.

Theorem 6 ([10]). *Let $f \in \mathcal{D}(\Lambda_+)$ and $\mathbf{f} = (f_k(\omega))$ be a sequence such that $(|f_k(\omega)|)$ is the sequence of pairwise independent random variables with functions of distribution $F_k(x) := P\{\omega : |f_k(\omega)| < x\}$, $x \in \mathbb{R}$, $k \geq 0$. The following assertions are true:*

- a) *If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, +\infty)$ a.s., then*

$$(\forall \varepsilon > 0): \sum_{k=0}^{+\infty} (1 - F_k((e^{-\rho} + \varepsilon)^{\lambda_k})) < \infty.$$
- b) *If there exists a sequence $(\delta_k): \delta_k > -\infty$ ($k \geq 0$), $\lim_{k \rightarrow +\infty} \delta_k = e^{-\rho}$, $\rho \in (-\infty, +\infty]$, and $\sum_{k=0}^{+\infty} (1 - F_k(\delta_k^{\lambda_k})) = +\infty$, then $\sigma(f, \omega) \leq \rho$ a.s.*

Another theorem in [10] contains the converse statements.

In this paper we prove similar theorems for Dirichlet series with random exponents $(\lambda_k(\omega))$ and deterministic coefficients $\mathbf{f} = (f_k)$, $f_k \in \mathbb{C}$, $k \geq 0$. Note that in paper [19] a power series of the form $\sum_{k=0}^{+\infty} z^{X_k(\omega)}$ is studied, where $(X_k(\omega))$ is a strictly increasing integer-valued stochastic process.

2. The Main Results: Series with Random Exponents

In this section we assume that $f_k(\omega) \equiv f_k \in \mathbb{C}$ ($k \geq 0$) and condition $\ln k = o(\ln |f_k|)$ ($k \rightarrow +\infty$) holds, that condition (7) is satisfied for all $\omega \in \Omega$, therefore by Proposition 3 equalities (3) for every $\omega \in \Omega$ hold.

Theorem 7. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x) := P\{\omega : \lambda_k(\omega) < x\}$, $x \in \mathbb{R}$, $k \geq 0$. The following assertions hold:*

- i) *If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s. then*

$$(\forall \varepsilon \in (0, \rho)): \sum_{k=0}^{+\infty} (1 - F_k(\ln |f_k| / (-\rho + \varepsilon))) < \infty.$$
- ii) *If $0 \geq \sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, 0]$ a.s. then*

$$(\forall \varepsilon > 0): \sum_{k=0}^{+\infty} F_k(\ln |f_k| / (-\rho + \varepsilon)) < \infty.$$

Proof of Theorem 7. i) If $\sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s., then from (3) we have $(\exists B \in \mathcal{A}, P(B) = 1)(\forall \omega \in B): \lim_{k \rightarrow +\infty} -\ln |f_k| / \lambda_k(\omega) \geq \rho$, and by definition of

lim,

$$(\forall \omega \in B)(\forall \varepsilon \in (0, \rho))(\exists k^*(\omega) \in \mathbb{N})(\forall k \geq k^*(\omega)):$$

$$\lambda_k(\omega) < \ln |f_k| / (-\rho + \varepsilon). \quad (11)$$

We denote

$$A_k := \left\{ \omega : \lambda_k(\omega) \geq \frac{\ln |f_k|}{(-\rho + \varepsilon)} \right\}.$$

It is clear, that $B \subset \overline{C} := \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_k$, hence $P(\overline{C}) = 1$, and $C = \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k$ is the event “ (A_k) infinitely often”, i.e. $\overline{C}$ is the event “ (A_k) finitely often”. From pairwise independence of random variables $(\lambda_k(\omega))$ follows pairwise independence of events (A_k) . Therefore, by refined Second Borel-Cantelli Lemma ([13, p.84])

$$\sum_{k=0}^{+\infty} (1 - F_k(\ln |f_k| / (-\rho + \varepsilon))) = \sum_{k=0}^{+\infty} P(A_k) < +\infty.$$

ii) If $0 \geq \sigma(\omega, f) \geq \rho \in (-\infty, 0]$ a.s., then instead of (11) we obtain

$$(\exists B, P(B) = 1)(\forall \omega \in B)(\forall \varepsilon > 0)(\exists k^*(\omega) \in \mathbb{N})(\forall k \geq k^*(\omega)):$$

$$\lambda_k(\omega) > \ln |f_k| / (-\rho + \varepsilon).$$

Therefore, for $A_k := \left\{ \omega : \lambda_k(\omega) \leq \ln |f_k| / (-\rho + \varepsilon) \right\}$ by the refined Second Borel-Cantelli lemma we obtain again

$$\sum_{k=0}^{+\infty} F_k(\ln |f_k| / (-\rho + \varepsilon)) = \sum_{k=0}^{+\infty} P(A_k) < +\infty.$$

This completes the proof of Theorem 7. $\square$

Remark 8. If $\sigma(f, \omega) > \rho \in [0, +\infty)$ a.s., then from (3) by definition of lim we have $(\forall \omega \in B)(\exists \varepsilon^* = \varepsilon^*(\omega) > 0)(\exists k^*(\omega) \in \mathbb{N})(\forall k \geq k^*(\omega)):$ $\lambda_k(\omega) < \ln |f_k| / -(\rho + \varepsilon^*)$, and similarly as in proof of i) we obtain

$$\sum_{k=0}^{+\infty} (1 - F_k(-\ln |f_k| / \rho)) < +\infty$$

in the case $\rho > 0$ and in the case $\rho = 0$ one has

$$\sum_{k=0}^{+\infty} (1 - F_k(+0)) < +\infty,$$

i.e., in particular, $\lim_{k \rightarrow +\infty} F_k(+0) = 1$. Namely, if $\lim_{k \rightarrow +\infty} F_k(+0) < 1$, then $\sigma(f, \omega) \leq 0$ a.s.

Theorem 9. Let $\Lambda = (\lambda_k(\omega))$ be a sequence of random variables with distribution functions $F_k(x) := P\{\omega : \lambda_k(\omega) < x\}$, $x \in \mathbb{R}$, $k \geq 0$, and $f \in \mathcal{D}(\Lambda)$. The following assertions hold:

- i) If there exist $\rho \in (0, +\infty)$ and a sequence (ε_k) such that $\varepsilon_k \rightarrow +0$ ($k \rightarrow +\infty$) and $\sum_{k=0}^{+\infty} (1 - F_k(\frac{\ln|f_k|}{-\rho+\varepsilon_k})) < +\infty$, then $\sigma(f, \omega) \geq \rho$ a.s.
- ii) If there exist $\rho \in (-\infty, 0]$ and a sequence (ε_k) such that $\varepsilon_k \rightarrow +0$ ($k \rightarrow +\infty$) and $\sum_{k=0}^{+\infty} F_k(\frac{\ln|f_k|}{-\rho+\varepsilon_k}) < +\infty$, then $\sigma(f, \omega) \geq \rho$ a.s.

Proof of Theorem 9. i) We note $1 - F_k(\ln|f_k|/(-\rho + \varepsilon_k)) = P(A_k)$, where

$$A_k := \{\omega : \lambda_k(\omega) \geq \ln|f_k|/(-\rho + \varepsilon_k)\}.$$

Therefore, from condition one has $\sum_{k=0}^{+\infty} P(A_k) < \infty$. Thus, by the first part of Borel-Cantelli Lemma $P(\overline{C}) = 1$, $C := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k$. Since, $\overline{C} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_k$, then for all $\omega \in \overline{C}$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A}_k$ and $-\rho + \varepsilon_k < 0$ for all $k \geq k^*(\omega)$. Here, $(\forall k \geq k^*(\omega)) : \lambda_k(\omega) < \frac{\ln|f_k|}{-\rho+\varepsilon_k}$. Using $\frac{-\ln|f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$, we get

$$\sigma(f, \omega) = \varliminf_{k \rightarrow +\infty} \frac{-\ln|f_k|}{\lambda_k(\omega)} \geq \varliminf_{k \rightarrow +\infty} (\rho - \varepsilon_k) = \rho \quad \text{a.s.} \quad (12)$$

ii) By the condition $\sum_{k=0}^{+\infty} P(A_k) < +\infty$, where

$$A_k := \{\omega : \lambda_k(\omega) < \ln|f_k|/(-\rho + \varepsilon_k)\}.$$

Since, by the first part of Borel-Cantelli Lemma

$$P(\overline{C}) = 1, \quad C := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k.$$

Where, as above for every $\omega \in \overline{C} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_k$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A}_k$ and $-\rho + \varepsilon_k > 0$ for all $k \geq k^*(\omega)$, such that, $(\forall k \geq k^*(\omega)) : \lambda_k(\omega) \geq \frac{\ln|f_k|}{-\rho+\varepsilon_k}$. Hence, $\frac{-\ln|f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$ and, therefore, we have again the “chain” of relations (12).

The proof of Theorem 9 is complete. □

3. Some Corollaries

Corollary 10. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x)$, $k \geq 0$. If $\lim_{k \rightarrow +\infty} F_k(+0) < 1$ and $f_k \rightarrow 0$ ($k \rightarrow +\infty$), then $\sigma(f, \omega) = 0$ a.s.*

Proof of Corollary 10. By Remark 8, $\sigma(f, \omega) \leq 0$ a.s. It remains to prove that $\sigma(f, \omega) \geq 0$ a.s. Indeed, $\lambda_k(\omega) \geq 0$, therefore $F_k(0) = P\{\omega: \lambda_k(\omega) < 0\} = 0$. Hence, $\sum_{k=k_0}^{+\infty} F_k(\ln |f_k|/\varepsilon_k) < +\infty$ because $\ln |f_k|/\varepsilon_k < 0$ ($k \geq k_0$). Thus, by Theorem 9 ii), $\sigma(f, \omega) \geq 0$ a.s. $\square$

Corollary 10 implies immediately the statement of Corollary 11.

Corollary 11. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x)$, $k \geq 0$. If there exists a positive random variable $a(\omega)$ such that $(\forall x \geq 0)(\forall k \in \mathbb{Z}_+)$: $F_k(x) \leq F_a(x) := P\{\omega: a(\omega) < x\}$ and $F_a(+0) < 1$ and $f_k \rightarrow 0$ ($k \rightarrow +\infty$), then $\sigma(f, \omega) = 0$ a.s.*

Corollary 12. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of random variables with distribution functions $F_k(x)$, $k \geq 0$. If $f_k \rightarrow 0$ ($k \rightarrow +\infty$) and there exist a positive random variable $b(\omega)$ and $\rho > 0$ such that $(\forall x \geq 0)(\forall k \in \mathbb{Z}_+)$: $F_k(x) \geq F_b(x) := P\{\omega: b(\omega) < x\}$, $\int_0^{+\infty} n_\mu(t\rho) dF_b(t) < +\infty$, where $n_\mu(t) = \sum_{\mu_k \leq t} 1$ is the counting function of a sequence $\mu_k = -\ln |f_k|$, then $\sigma(f, \omega) \geq \rho$ a.s.*

Proof of Corollary 12. We remark that

$$\begin{aligned} \sum_{k=k_0}^n \left(1 - F_k\left(\frac{\ln |f_k|}{-\rho + \varepsilon_k}\right)\right) &\leq \int_{\mu_{k_0}}^{\mu_n} (1 - F_k(t/\rho)) dn_\mu(t) \\ &\leq \int_{\mu_{k_0}}^{\mu_n} (1 - F_b(t/\rho)) dn_\mu(t) + O(1) \\ &= \int_{\mu_{k_0}/\rho}^{\mu_n/\rho} n_\mu(t\rho) dF_b(t) + O(1), \end{aligned}$$

($n \rightarrow +\infty$), because $-\ln |f_k| > 0$ ($k \geq k_0$) and $\rho - \varepsilon_k < \rho$ for all $k \geq 0$. Therefore, the series $\sum_{k=k_0}^{+\infty} \left(1 - F_k\left(\frac{\ln |f_k|}{-\rho + \varepsilon_k}\right)\right)$ converges. Hence by Theorem 9 ii) we complete the proof. $\square$

Corollary 13. Let $\Lambda = (\lambda_k(\omega))$ be a increasing (a.s.) sequence of pairwise independent random variables and $f \in \mathcal{D}(\Lambda)$. If $F_0(+0) < 1$, where F_0 is distribution function of $\lambda_0(\omega)$, and $f_k \rightarrow 0$ ($k \rightarrow +\infty$), then $\sigma(f, \omega) = 0$ a.s.

Proof of Corollary 13. We remark that $F_{k+1}(x) \leq F_k(x)$, because $\lambda_k(\omega) \leq \lambda_{k+1}(\omega)$ ($k \geq 0$) a.s. Therefore, by Corollary 11 we obtain the conclusion of Corollary 13. $\square$

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