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# ON BINARY SOFT TOPOLOGICAL SPACES

Shivanagappa S. Benchalli[1], Prakashgouda G. Patil[2],
Abeda S. Dodamani[3], J. Pradeepkumar[4 §]

[1,2,3] Department of Mathematics
Karnatak University
Dharwad, 580003, INDIA

[4]Department of Mathematics
SKSVMA College of Engineering and Technology
Laxmeshwar, 581126, INDIA

**Abstract:** In the present paper, we introduce binary soft topological spaces which are defined over two initial universe sets with a fixed set of parameters. The notions of binary soft open sets, binary soft closed sets, binary soft closure, binary soft interior, binary soft boundary, binary soft neighborhood of a point are introduced and their basic properties are investigated with the suitable examples. These results are fundamental for further research on binary soft topology and will strengthen the foundations of the theory of binary soft topological spaces.

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## 1. Introduction and Preliminary

In order to solve complicated problems in economics, environmental areas and

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[§]Correspondence author

engineering, Molodtsov [7] introduced the concept of soft sets, since no mathematics tools can successfully deal with various kinds of uncertainties in these problems. In 2005, Pie and Miao [8] improved the results of Maji et al. [5]. Recently, in 2011 Shabir and Munaza Naz [9] initiated the study of soft topological spaces, further many researchers like Aygunoglu [2], Ahmad [3], Maji [6], Hussain [4] continued work on soft topology.

In 2016 Ahu Acikgöz and Nihal Tas [1] introduced the concept of binary soft set theory on two initial universal sets and investigated some properties. In this study we initiate the notion of binary soft topological spaces which are defined over two initial universal sets with a fixed set of parameters. Then we discuss some basic properties of binary soft topological spaces and define binary soft open and closed sets. In this paper, with the help of examples we have shown that a binary soft topological spaces gives collection of parameterized family of binary soft topologies on the two initial universal sets and the converse is not true.

The notions of binary soft open sets, binary soft closed sets, binary soft closure, binary soft interior, binary soft boundary, binary soft neighborhood of a point are introduced and their basic properties are investigated with the suitable examples. Binary soft topological spaces are more wide ranging and generalized than the classical topological spaces and soft topological spaces.

The organization of the paper is as follows: Section 2 briefly reviews some basic concepts about soft sets, binary soft sets and their related properties; Section 3 we present some fundamental concepts in binary soft open sets, binary soft closed sets, binary soft closure, binary soft interior, binary soft boundary, binary soft neighborhood of a point and their basic properties are investigated with the suitable examples. Section 4 is conclusion of the paper.

**Definition 1.** ([9]) Let $X$ be an initial universe and let $E$ be a set of parameters. Let $P(X)$ denote the power set of $X$ and let $A$ be a nonempty subset of $E$. A pair $(F, A)$ is called a soft set over $X$, where $F$ is a mapping given by $F : A \to P(X)$. In other words, a soft set over $X$ is a parameterized family of subsets of the universe $X$. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon$-approximate elements of the soft set $(F, A)$. Clearly, a soft set is not a set.

Let $U_1, U_2$ be two initial universe sets and $E$ be a set of parameters. Let $P(U_1), P(U_2)$ denote the power set of $U_1, U_2$, respectively. Also, let $A, B, C \subseteq E$.

**Definition 2.** ([1]) A pair $(F, A)$ is said to be a binary soft set over $U_1, U_2$, where F is defined as follows: $F : A \to P(U_1) \times P(U_2)$, $F(e) = (X, Y)$ for each $e \in A$ such that $X \subseteq U_1$, $Y \subseteq U_2$.

**Definition 3.** ([1]) A binary soft set $(G, A)$ over $U_1, U_2$ is called a binary absolute soft set, denoted by $\tilde{\tilde{A}}$ if $F(e) = (U_1, U_2)$ for each $e \in A$.

**Definition 4.** ([1]) The union of two binary soft sets of $(F, A)$ and $(G, B)$ over the common $U_1, U_2$ is the binary soft set $(H, C)$, where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} (X_1, Y_1), & \text{if } e \in A - B \\ (X_2, Y_2), & \text{if } e \in B - A \\ (X_1 \cup X_2, Y_1 \cup Y_2), & \text{if } e \in A \cap B \end{cases} \quad (1)$$

such that $(F(e) = (X_1, Y_1)$ for each $e \in A$ and $(G(e) = (X_2, Y_2)$ for each $e \in B$. We denote it $(F, A)\tilde{\tilde{\cup}}(G, A) = (H, C)$.

**Definition 5.** ([1]) The intersection of two binary soft sets $(F, A)$ and $(G, B)$ over a common $U_1, U_2$is the binary soft set (H,C), where $C = A \cap B$, and $H(e) = (X_1 \cap X_2, Y_1 \cap Y_2)$ for each $e \in C$ such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$ for each $e \in B$. We denote it $(F, A)\tilde{\tilde{\cap}}(G, B) = (H, C)$.

**Definition 6.** ([1]) Let $(F, A)$ and $(G, B)$ be two binary soft sets over a common $U_1, U_2$. (F, A) is called a binary soft subset of (G,B) if:
(i) $A \subseteq B$,
(ii) $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ such that $F(e) = (X_1, Y_1)$, $G(e) = (X_2, Y_2)$ for each $e \in A$. We denote it $(F, A) \tilde{\tilde{\subseteq}} (G, B)$.

**Definition 7.** ([1]) A binary soft set $(F, A)$ over $U_1, U_2$ is called a binary null soft set, denoted by $\tilde{\tilde{\phi}}$ if $F(e) = (\phi, \phi)$ for each $e \in A$.

**Definition 8.** ([1]) The difference of two binary soft sets $(F, A)$ and $(G, A)$ over the common $U_1$, $U_2$ is the binary soft set $(H, A)$, where $H(e) = (X_1 - X_2, Y_1 - Y_2)$ for each $e \in A$ such that $(F, A) = (X_1, Y_1)$ and $(G, A) = (X_2, Y_2)$.

## 2. Binary Soft Topological Spaces

Throughout the paper let $U_1$, $U_2$ be two initial universe sets and E be a set of parameters. Let $P(U_1)$, $P(U_1)$ denote the power set of $U_1$, $U_2$, respectively.

**Definition 9.** Let $\tau_\Delta$ be the collection of binary soft sets over $U_1$, $U_2$, then $\tau_\Delta$ is said to be a binary soft topology on $U_1$, $U_2$ if
(i) $\tilde{\tilde{\phi}}$, $\tilde{\tilde{X}} \in \tau_\Delta$
(ii) The union of any member of binary soft sets in $\tau_\Delta$ belongs to $\tau_\Delta$.
(iii) The intersection of any two binary soft sets in $\tau_\Delta$ belongs to $\tau_\Delta$.
Then $(U_1, U_2, \tau_\Delta, E)$ is called a binary soft topological space over $U_1$, $U_2$.

**Definition 10.** Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over $U_1$, $U_2$ then the member of $\tau_\Delta$ are said to be binary soft open sets in $U_1$, $U_2$.

**Definition 11.** Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over $U_1$, $U_2$ then the member of $\tau_\Delta$ are said to be binary soft closed sets in $U_1$, $U_2$ if its relative complement $(F, E)'$ is belongs to $\tau_\Delta$.

**Definition 12.** Let $U_1$, $U_2$ be the two initial universe sets and E be a set of parameters and $\tau_\Delta = \{\tilde{\tilde{\phi}}, \tilde{\tilde{X}}\}$. Then $\tau_\Delta$ is called the binary soft indiscrete topology on $U_1$, $U_2$ and $(U_1, U_2, \tau_\Delta, E)$ is said to be a binary soft indiscrete space over $U_1$, $U_2$.

**Definition 13.** Let $U_1$, $U_2$ be the two initial universe sets and E be a set of parameters and let $\tau_\Delta$ be the collection of all binary soft sets which can be defined over $U_1$, $U_2$. The $\tau_\Delta$ is called the binary soft discrete topology on $U_1$, $U_2$ and $(U_1, U_2, \tau_\Delta, E)$ is said to be a binary soft discrete space over $U_1$, $U_2$.

**Example 14.** Consider the following sets:
$U_1 = \{a_1, a_2, a_3, a_4, a_5\}$
$U_2 = \{b_1, b_2, b_3, b_4\}$
$E = \{e_1, e_2, e_3, e_4\}$
and $\tau_\Delta = \{\tilde{\tilde{\phi}}, \tilde{\tilde{X}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$,
where $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ are binary soft sets defined as follows:
$(F_1, E) = \{(e_1, (\{a_1\}, \{b_1\})), (e_2, (\{a_2\}, \{b_2\})), (e_4, (\{a_3\}, \{b_3\}))\}$
$(F_2, E) = \{(e_1, (\{a_4\}, \{b_4\})), (e_2, (\{a_3\}, \{b_1\})), (e_3, (\{a_1, a_2\}, \{b_3\})),$
$(e_4, (\{a_3, a_5\}, \{b_1, b_2\}))\}$

$(F_3, E) = \{(e_1, (\{a_1, a_4\}, \{b_1, b_4\})), (e_2, (\{a_2, a_3\}, \{b_1, b_2\})),$
$(e_3, (\{a_1, a_2\}, \{b_3\})), (e_4, (\{a_3, a_5\}, \{b_1, b_2, b_3\}))\}$
$(F_4, E) = \{(e_4, (\{a_3\}\}).$
Clearly, $\tau_\Delta$ is binary soft topology.
$\tilde{\tilde{\phi}}$, $\tilde{\tilde{X}}$, $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ are binary soft open sets.
$\tilde{\tilde{\phi}}$, $\tilde{\tilde{X}}$, $(F_1, E)', (F_2, E)', (F_3, E)', (F_4, E)'$ are binary soft closed sets.

**Remark 15.** Any binary soft collection of sets needs not to be a binary soft topology. The following example shows this.

**Example 16.** $\tau_\Delta = \{\tilde{\tilde{X}}, \tilde{\tilde{\phi}}, \{(e_1, (\{a_4, a_5\}, \{b_2, b_3\}), (e_2, (\{a_3\}, \{b_4\})\}$
$\{(e_1, (\{a_2, a_3\}, \{b_1, b_4\})), (e_2, (\{a_2\}, \{b_1\})), (e_5, (\{a_1, a_3\}, \{b_2)\}\}$
$\{(e_3, (\{a_1, a_3, \{b_2, b_3\})), (e_4, (\{a_1\}, \{b_1, b_2\}))\}\}.$

**Remark 17.** Let $(U_1, U_2, \tau_\Delta, E)$ and $(U_1, U_2, \tau'_\Delta, E)$ be two binary soft topological spaces over the same universal sets $U_1$,$U_2$, then $(U_1, U_2, \tau_\Delta \tilde{\tilde{\cup}} \tau'_\Delta, E)$ may not be binary soft topological spaces over $U_1$,$U_2$.

**Example 18.** Let
$\tau_\Delta = \{\tilde{\tilde{X}}, \tilde{\tilde{\phi}}, \{(e_1, (\{a_1\}, \{b_1\})), (e_2, (\{a_2\}, \{b_2\})), (e_4, (\{a_3\}, \{b_3\}))\},$
$\{(e_1, (\{a_4\}, \{b_4\})), (e_2, (\{a_3\}, \{b_1\})), (e_3, (\{a_1, a_2\}, \{b_3\})),$
$(e_4, (\{a_3, a_5\}, \{b_1, b_2\}))\}$, $\{(e_1, (\{a_1, a_4\}, \{b_1, b_4\})), (e_2, (\{a_2, a_3\},$
$\{b_1, b_2\})), (e_3, (\{a_1, a_2\}, \{b_3\})), (e_4, (\{a_3, a_5\}, \{b_1, b_2, b_3\}))\},$
$\{(e_4, (\{a_3\}))\}\},$
$\tau'_\Delta = \{\tilde{\tilde{X}}, \tilde{\tilde{\phi}}, \{(e_1, (\{a_2\}, \{b_2\})), (e_5, (\{a_3, a_4)\}, \{b_1, b_3\})),$
$(e_8, (\{a_1, a_3\}, \{b_2\}))\}$, $\{(e_2, (\{a_1, a_2\}, \{b_4\})), (e_5, (\{a_3, a_5)\}, \{b_3\})),$
$(e_7, (\{a_1\}, \{b_2, b_3\}))\}$, $\{(e_1, (\{a_2\}, \{b_2\})), (e_2, (\{a_1, a_2)\}, \{b_4\})),$
$(e_5, (\{a_3, a_4, a_5\}, \{b_1, b_3\})), (e_7(\{a_1\}, \{b_1, b_3\}))\}\},$
clearly, $\tau_\Delta$ and $\tau'_\Delta$ are binary soft topological spaces. Then
$\tau_\Delta \tilde{\tilde{\cup}} \tau'_\Delta = \{\tilde{\tilde{X}}, \tilde{\tilde{\phi}}, \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_2\}, \{b_2\})),$
$(e_4, (\{a_3\}, \{b_3\})), (e_5, (\{a_3, a_4)\}, \{b_1, b_3\})), (e_8, (\{a_1, a_3\}, \{b_2\}))\},$
$\{(e_1, (\{a_1\}, \{b_1\})), (e_2, (\{a_1, a_2\}, \{b_2, b_4\})), (e_4, (\{a_3\}, \{b_3\})),$
$(e_5, (\{a_3, a_5\}, \{b_3\})), (e_7, (\{a_1\}, \{b_2, b_3\}))\}$, $\{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})),$
$(e_2, (\{a_1, a_2\}, \{b_2, b_4\})), (e_4, (\{a_3\}, \{b_3\})), (e_5, (\{a_3, a_4, a_5\}, \{b_1, b_3\})),$
$(e_7, (\{a_1\}, \{b_2, b_3\}))\}$, $\{(e_1, (\{a_2, a_4\}, \{b_2, b_4\})), (e_2, (\{a_3\}, \{b_1\})),$
$(e_3, (\{a_1, a_2\}, \{b_3\})), (e_5, (\{a_3, a_4\}, \{b_1, b_3\})), (e_8, (\{a_1, a_3\}, \{b_2\}))\},$
$\{(e_1, (\{a_4\}, \{b_4\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_4\})), (e_3, (\{a_1, a_2\}, \{b_3\})),$

$(e_5, (\{a_3, a_5)\}, \{b_3\})), (e_7, (\{a_1\}, \{b_2, b_3\}))\}$, $\{(e_1, (\{a_2, a_4\}, \{b_2, b_4\}))$, $(e_2, (\{a_1, a_2, a_3\}, \{b_1, b_4\})), (e_3, (\{a_1, a_2\}, \{b_3\})), (e_5, (\{a_3, a_4, a_5)\}$, $\{b_1, b_3\})), (e_7, (\{a_1\}, \{b_2, b_3\}))\}$, $\{(e_1, (\{a_1, a_2, a_4\}, \{b_1, b_2, b_4\}))$, $(e_2, (\{a_1, a_2, a_3\}, \{b_1, b_2, b_4\})), (e_3, (\{a_1, a_2\}, \{b_3\})), (e_4, (\{a_3, a_5)\}$, $\{b_1, b_2, b_3\}))$, $(e_5, (\{a_3, a_4, a_5\}, \{b_1, b_3\}))(e_7, (\{a_1\}, \{b_2, b_3\}))\}$.
Clearly, $\{(e_5, (\{a_3, a_4)\}, \{b_1, b_3\}))\}\tilde{\tilde{\cap}}\{(e_5, (\{a_3, a_5)\}, \{b_3\}))\} =$ $\{(e_5, (\{a_3\}, \{b_3\}))\} \notin \tau_\Delta \tilde{\tilde{\cup}} \tau_\Delta^{'}$.
Thus, $\tau_\Delta \tilde{\tilde{\cup}} \tau_\Delta^{'}$ is not binary soft topology.

**Theorem 19.** *Let $(U_1, U_2, \tau_\Delta, E)$ and $(U_1, U_2, \tau_\Delta^{'}, E)$ be two binary soft topological spaces over the common initial universal sets $U_1$, $U_2$, then $(U_1, U_2, \tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}, E)$ is a binary soft topological space over $U_1$, $U_2$.*

*Proof.* (i) $\tilde{\tilde{\phi}}$, $\tilde{\tilde{X}}$ belongs to $\tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$.
(ii) Let $\{(G_i, E)/i \in I\}$ be a family of binary soft sets in $\tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$.Then $(G_i, E) \in \tau_\Delta$ and $(G_i, E) \in \tau_\Delta^{'}$, for all $i \in I$, So $\tilde{\tilde{\cup}}_{i\in I}(G_i, E) \in \tau_\Delta$ and $\tilde{\tilde{\cup}}_{i\in I}(G_i, E) \in \tau_\Delta^{'}$. Thus $\tilde{\tilde{\cup}}_{i\in I}(G_i, E) \in \tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$.
(iii) Let the two binary soft sets $(H, E), (I, E) \in \tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$. Then $(H, E), (I, E) \in \tau_\Delta$ and $(H, E), (I, E) \in \tau_\Delta^{'}$. Since $(H, E)\tilde{\tilde{\cap}}(G, E) \in \tau_\Delta$ and $(H, E)\tilde{\tilde{\cap}}(G, E) \in \tau_\Delta^{'}$, so $(H, E)\tilde{\tilde{\cap}}(G, E) \in \tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$. Thus $\tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}$ defines the binary soft topology on $U_1, U_2$ and
$(U_1, U_2, \tau_\Delta \tilde{\tilde{\cap}} \tau_\Delta^{'}, E)$ is a binary soft topological space over $U_1$, $U_2$. This completes the proof. □

**Definition 20.** Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over $U_1$, $U_2$ and (G, E) be the binary soft set over common universal sets $U_1$, $U_2$. Then the binary soft closure of (G, E) denoted by $\overline{\overline{(G, E)}}$ is the intersection of all binary soft closed sets of (G, E). Thus, $\overline{\overline{(G, E)}}$ is the smallest binary soft closed sets over $U_1$, $U_2$ which contains (G, E).

**Theorem 21.** *Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over $U_1$, $U_2$ and let (H, E),(I, E) are binary soft sets over $U_1$, $U_2$. Then:*
*(i)* $\overline{\overline{\tilde{\tilde{\phi}}}} = \tilde{\tilde{\phi}}$ *and* $\overline{\overline{\tilde{\tilde{X}}}} = \tilde{\tilde{X}}$,
*(ii)* $(H, E)\tilde{\tilde{\subseteq}}\overline{\overline{(H, E)}}$ *implies* $\overline{\overline{(H, E)}}$ *is a binary soft closed set and* $\overline{\overline{(H, E)}}$ *contains* $(H, E)$,
*(iii)* $(H, E)$ *is a binary soft closed if and only if* $(H, E)\tilde{\tilde{=}}\overline{\overline{(H, E)}}$,

*(iv)* $\overline{\overline{\overline{\overline{(H,E)}}}} = (H,E)$,
*(v)* $(H,E)\tilde{\tilde{\subseteq}}(I,E)$ *implies* $\overline{\overline{(H,E)}}\ \tilde{\tilde{\subseteq}}\ \overline{\overline{(I,E)}}$,
*(vi)* $\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}} = \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}$,
*(vii)* $\overline{\overline{(H,E)\tilde{\tilde{\cap}}(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\cap}}\overline{\overline{(I,E)}}$.

*Proof.* (i) is obvious.

(ii) Let $\{(H_i,E)/i \in I\}$ be the family of all the binary closed sets containing (H, E). Then by definition, we know that
$\overline{\overline{(H,E)}} = \tilde{\tilde{\cap}}_{i\in I}(H_i,E) \to (1)$.
Now, we know that $\{(H_i,E)/i \in I\}$ is a binary soft closed set $\forall i \in I$
$\Rightarrow \tilde{\tilde{\cap}}_{i\in I}(H_i,E)$ is also binary soft closed set. Since arbitrary intersection of binary soft closed sets is binary soft closed.
$\overline{\overline{(H,E)}}$ is a binary soft closed set (from (1))
$\Rightarrow$ Thus $\overline{\overline{(H,E)}}$ is a binary soft closed set.
Now, we prove that $\overline{\overline{(H,E)}}\tilde{\tilde{\supseteq}}(H,E)$.
We know that $\forall i \in I$, $\{(H_i,E)/i \in I\}\tilde{\tilde{\supseteq}}(H,E)$
$\Rightarrow (H,E)\tilde{\tilde{\subseteq}}\tilde{\tilde{\cap}}_{i\in I}(H_i,E)$
$\Rightarrow (H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$ [using (1)]
$\Rightarrow \overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$.
Thus $\overline{\overline{(H,E)}}$ contains (H,E). Hence $\overline{\overline{(H,E)}}$ is a binary soft closed set and $\overline{\overline{(H,E)}}$ contains (H,E).
(iii) Let (H,E) is a binary soft closed set and to prove $\overline{\overline{(H,E)}} = (H,E)$, Suppose $(H,E)$ is binary soft closed set. Now we have $(H,E)\tilde{\tilde{\supset}}(H,E)$.
Therefore $(H,E)$ is a binary soft closed set containing $(H,E) \to (1)$.
But $\overline{\overline{(H,E)}}$ is the smallest binary soft closed set containing $(H,E) \to (2)$.
Therefore from (1) and (2) it follows that $\overline{\overline{(H,E)}}$ is smaller than $(H,E)$ that is $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$.
But from (ii) of this theorem, we have $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$ is always true. Therefore we have $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$ and $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$. Thus $(H,E) = \overline{\overline{(H,E)}}$. Consequently, if $(H,E)$ is binary soft closed set, then $(H,E) = \overline{\overline{(H,E)}}$.

(iv) Since $\overline{\overline{(H,E)}}$ is binary soft closed set, therefore by (iii) we have $\overline{\overline{\overline{\overline{(H,E)}}}} = (H,E)$.
(v) If $(H,E)\tilde{\tilde{\subseteq}}(I,E)$, then $\overline{\overline{(H,E)}}\ \tilde{\tilde{\subseteq}}\ \overline{\overline{(I,E)}}$.

Suppose $(H,E)\tilde{\tilde{\subseteq}}(I,E)$, we know that $(I,E)\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}$,
and we have $(H,E)\tilde{\tilde{\subseteq}}(I,E)\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}$. Therefore $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}$.
Therefore $\overline{\overline{(I,E)}}$ is binary soft closed set containing (I,E)$\rightarrow$(1).
But $\overline{\overline{(H,E)}}$ is the smallest binary soft closed set containing (H,E)$\Rightarrow$(2).
From (1) and (2) it follows that $\overline{\overline{(H,E)}}$ is smaller than $\overline{\overline{(I,E)}}$, that is $\overline{\overline{(H,E)}}\ \tilde{\tilde{\subseteq}}$ $\overline{\overline{(I,E)}}$. Thus, if $(H,E)\tilde{\tilde{\subseteq}}(I,E)$, then $\overline{\overline{(H,E)}}\ \tilde{\tilde{\subseteq}}\ \overline{\overline{(I,E)}}$.
(vi) We know that $(H,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(I,E)$ and $(I,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(I,E)$.
Therefore, $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}$ and $\overline{\overline{(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}$.
Since $(H,E)\tilde{\tilde{\subseteq}}(I,E)$ implies $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}$
$\Rightarrow \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}\tilde{\tilde{\subseteq}}\{\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}\}\tilde{\tilde{\cup}}\{\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}\}$
$\overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}} \rightarrow$(1).
Also from the binary soft closure property we have $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$ and $(I,E)$ $\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}$, thus $(H,E)\tilde{\tilde{\cup}}(I,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}$
$\Rightarrow \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}$ is the binary soft closed set containing $(H,E)\tilde{\tilde{\cup}}(I,E)$.
But $\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}$ is the smallest binary soft closed set containing $(H,E)\tilde{\tilde{\cup}}(I,E)$ $\rightarrow$(2).
Comparing (1) and (2), we have $\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}}$ is smaller than $\overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}$.
Thus from (1) and (2) we have $\overline{\overline{(H,E)\tilde{\tilde{\cup}}(I,E)}} = \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(I,E)}}$.
(vii) Since $(H,E)\tilde{\tilde{\cap}}(I,E)\tilde{\tilde{\subseteq}}(H,E)$ and $(H,E)\tilde{\tilde{\cap}}(I,E)\tilde{\tilde{\subseteq}}(I,E)$, so by part (v)

$$\overline{\overline{(H,E)\tilde{\tilde{\cap}}(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}} \text{ and } \overline{\overline{(H,E)\tilde{\tilde{\cap}}(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(I,E)}}.$$

Thus $\overline{\overline{(H,E)\tilde{\tilde{\cap}}(I,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\cap}}\overline{\overline{(I,E)}}$. This completes the proof. □

**Definition 22.** Let $(H,A)$ be the binary soft set of a binary topological space $(U_1,U_2,\tau_\Delta,A)$ over $U_1,U_2$. Then we associate point wise binary soft closure of $(F,E)$ over $U_1,U_2$, which is denoted by $(\overline{\overline{H}},A)$ and defined as $(\overline{\overline{H}},A)_{(\alpha)} = \overline{\overline{(H,A)}}_{(\alpha)}$, where $\overline{\overline{(H,A)}}_{(\alpha)}$ is the binary soft closure of $(H,A)_{(\alpha)}$ in $(U_1,U_2,\tau_\Delta,A)$ for each $\alpha \in A$.

**Theorem 23.** *Let $(U_1,U_2,\tau_\alpha,A)$ be a binary soft topological space and (H, A) be a binary soft set over $U_1,U_2$, then $(\overline{\overline{H}},A)\tilde{\tilde{\subseteq}}\overline{\overline{(H,A)}}$.*

*Proof.* For any parameter $\alpha \in E$, $\overline{\overline{(H,A)}}_{(\alpha)}$ is the smallest binary soft closed set in $(U_1, U_2, \tau_\alpha, A)$ which contains $(H, A)_{(\alpha)}$. Moreover, if $\overline{\overline{(H, A)}}_{(\alpha)} = (L, A)$ then $(L, A)$ is also a binary soft closed set in $(U_1, U_2, \tau_\alpha, A)$ containing $(H, A)_{(\alpha)}$. This implies that $(\overline{\overline{H}}, A)_{(\alpha)} = \overline{\overline{(H, A)}}_{(\alpha)} \tilde{\tilde{\subseteq}} (L, A)$. Thus $(\overline{\overline{H}}, A) \tilde{\tilde{\subseteq}} \overline{\overline{(H, A)}}$. *square*

**Theorem 24.** *Let $(U_1, U_2, \tau_\Delta, A)$ be the binary soft topological space and (F,A) be the binary soft set over $(U_1, U_2$, then $(\overline{\overline{F}}, A) \tilde{\tilde{\subseteq}} \overline{\overline{(F, A)}}$.*

*Proof.* Let $(U_1, U_2, \tau_\Delta, A)$ be the binary soft topological space over $U_1, U_2$. If $(\overline{\overline{F}}, A) \tilde{\tilde{\subseteq}} \overline{\overline{(F, A)}}$, then $(\overline{\overline{F}}, A)$ is a binary soft closed set and so $(\overline{\overline{F}}, A)^c \in \tau_\Delta$. Conversely, if $(\overline{\overline{F}}, A)^c \in \tau_\Delta$, then binary soft closed set containing $(F, A)$. By above theorem $(\overline{\overline{F}}, A) \tilde{\tilde{\subseteq}} \overline{\overline{(F, A)}}$ and by the definition of binary soft closure of $(F, A)$, any binary soft closed set over $U_1, U_2$ which contains $(F, A)$ will contains $\overline{\overline{(F, A)}}$. Thus $\overline{\overline{(F, A)}} \tilde{\tilde{\subseteq}} (\overline{\overline{F}}, A)$, hence $(\overline{\overline{F}}, A) = \overline{\overline{(F, A)}}$. □

**Definition 25.** Let $(U_1, U_2, \tau_\Delta, A)$ be a binary soft topological space over $U_1, U_2$. $(H, A)$ be a binary soft set and $e_x \in E$. Then $e_x$ is said to be a binary soft interior point of $H, A)$ if there exists a binary soft open set $(K, A)$ such that $e_x \in (H, A) \tilde{\tilde{\subseteq}} (K, A)$.

**Definition 26.** Let $(U_1, U_2, \tau_\Delta, A)$ be a binary soft topological space over $U_1, U_2$. $(F, A)$ be a binary soft set and $e_x \in A$. Then $(F, A)$ is said to be a binary soft neighborhood of $e_x$ if there exists a binary soft open set (K, A) such that $e_x \in (F, A) \tilde{\tilde{\subseteq}} (K, A)$.

**Theorem 27.** *Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over $U_1, U_2$. $(F, E)$ be a binary soft set over $U_1, U_2$ and $e_x \in E$. If $e_x$ is a binary soft interior point of $(F, E)$, then $e_x$ is a binary soft interior point of $(F, E)_{(\alpha)}$ in $(U_1, U_2, \tau_\alpha, E)$ for each $\alpha \in E$.*

The above theorem is not true in general.

**Theorem 28.** *Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space over initial parameter sets $(U_1, U_2$, then:*
*(i) Each $e_x \in E$ has a binary soft neighborhood.*
*(ii) The intersection of any two binary soft neighborhood of a binary soft point $e_x$, is again binary soft neighborhood.*
*(iii) Every binary soft superset of a binary soft neighborhood of a point $e_x$ is*

*again a binary soft neighborhood of the point $e_x$.*

*Proof.* (i) For any $e_x \in \tilde{\tilde{X}}$, so $e_x \in \tilde{\tilde{X}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Thus $\tilde{\tilde{X}}$ is a binary soft neighborhood of $e_x$.
(ii) Let $e_x$ be a binary soft topological space $(U_1, U_2, \tau_\Delta, E)$. Let $e_x \in E$ be any binary soft point and let $(F, E)$ and $(G, E)$ be any two binary soft neighborhoods of $e_x$. Now to prove $(F, E) \tilde{\tilde{\cap}} (G, E)$ is also a binary soft neighborhood of $e_x$. Now $(F, E)$ is a binary soft neighborhood of $e_x$ implies there exist a binary soft open set (K, E) such that $e_x \in (K, E) \tilde{\tilde{\subseteq}} (F, E)$. Also $(G, E)$ is binary soft neighborhood of $e_x$ implies there exist a binary soft open set (L, E) such that $e_x \in (L, E) \tilde{\tilde{\subseteq}} (F, E) \to (1)$.
Now $(K, E) \tilde{\tilde{\cap}} (L, E)$ is binary soft open set, also we have from (1)

$$e_x \in [(K, E) \tilde{\tilde{\cap}} (L, E)] \tilde{\tilde{\subseteq}} [(F, E) \tilde{\tilde{\cap}} (G, E)].$$

Thus there exists a binary soft open set $[(K, E) \tilde{\tilde{\cap}} (L, E)]$ such that

$$e_x \in [(K, E) \tilde{\tilde{\cap}} (L, E)] \tilde{\tilde{\subseteq}} [(F, E) \tilde{\tilde{\cap}} (G, E)].$$

From the definition of soft binary neighborhood, it follows that $[(F, E) \tilde{\tilde{\cap}} (G, E)]$ is a binary soft neighborhood of $e_x$. Thus, the intersection of any two binary soft neighborhood is again binary soft neighborhood.
(iii) Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space. Let $e_x \in E$ be any binary soft point, and let (F,E) be a binary soft neighborhood of $e_x$. Let $(G, E)$ be any binary soft superset of $(F, E)$. Now, since $(G, E)$ is also a binary soft neighborhood of $e_x$ therefore there exists a binary soft open set $(H, E)$ such that $e_x \in (H, E) \tilde{\tilde{\subseteq}} (F, E) \to (1)$.
Now, $(F, E)$ is binary soft subset of $(G, E)$ this implies $(G, E) \tilde{\tilde{\supseteq}} (F, E)$ implies $(F, E) \tilde{\tilde{\subseteq}} (G, E) \to (2)$.
From (1) and (2) we have $e_x \in (H, E) \tilde{\tilde{\subseteq}} (F, E) \tilde{\tilde{\subseteq}} (G, E)$, which implies $e_x \in (H, E) \tilde{\tilde{\subseteq}} (G, E)$. Thus there exists a binary soft open set $(H, E)$ such that $e_x \in (H, E) \tilde{\tilde{\subseteq}} (G, E)$. Therefore $(G, E)$ is a binary soft neighborhood of $e_x$. Thus every binary soft superset of a binary neighborhood is again a binary soft neighborhood of that point. □

**Theorem 29.** *Let $(U_1, U_2, \tau_\Delta, E)$ be a binary soft topological space. Let $(F, E)$ be any binary soft subset over $U_1, U_2$. Then the following properties hold true:*

*(i)$(F,E)^{\odot}$ is an binary soft open set contained in $(F,E)$,*
*i.e. $(F,E)^{\odot}$ is an binary soft open and $(F,E)^{\odot}\tilde{\tilde{\subseteq}}(F,E)$.*
*(ii) $(F,E)^{\odot}$ is the largest binary soft open set contained in $(F,E)$.*
*(iii) $(F,E)$ is binary soft open if and only if $(F,E)\tilde{\tilde{=}}(F,E)^{\odot}$.*

*Proof.* (i) By definition of binary soft interior, we have $(F,E)^{\odot} = \tilde{\tilde{\cup}}_{\lambda\in\Lambda}(H,E)_{\lambda}$ where $\{(H,E)_{\lambda} : \lambda \in \Lambda\}$ is the family of all binary soft open sets contained in $(F,E)$.
$(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)\ \forall\lambda \in \Lambda$
$\Rightarrow$ Union of all binary soft open sets
$\Rightarrow$ Binary soft open sets (By definition of binary soft topology).
Also, we have $(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)\ \forall\lambda \in \Lambda$
$\Rightarrow \tilde{\tilde{\cup}}_{\lambda\in\Lambda}(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)$
$\Rightarrow (F,E)^{\odot}\tilde{\tilde{\subseteq}}(F,E)$. Hence $(F,E)^{\odot}$ is a binary soft open set and $(F,E)^{\odot}\tilde{\tilde{\subseteq}}(F,E)$.
(ii) From (i) we have, $(F,E)^{\odot}$ is a binary soft open set contained in $(F,E)$. Let $(H,E)$ be any binary soft open set contained in $(F,E)$. This implies $\{(H,E)_{\lambda} : \lambda \in \Lambda\}=$ The family of all binary soft open sets contained in $(F,E)$. which implies $(H,E)\tilde{\tilde{\subseteq}}\tilde{\tilde{\cup}}_{\lambda\in\Lambda}(H,E)_{\lambda}$
$\Rightarrow (H,E)\tilde{\tilde{\subseteq}}(F,E)^{\odot}$
$\Rightarrow (F,E)^{\odot}\tilde{\tilde{\supseteq}}(H,E)$
$\Rightarrow (F,E)^{\odot}$ is larger than $(H,E)$. $(F,E)^{\odot}$ is larger than every binary soft open set $(H,E)$ contained in $(F,E)$. Thus $(F,E)^{\odot}$ is the largest binary soft open set contained in $(F,E)$.
(iii) Suppose $(F,E)$ is binary soft open. Therefore $(F,E)$ is a binary soft open set contained in $(F,E)$ (i.e. $(F,E)\tilde{\tilde{\subseteq}}(F,E)$). $\rightarrow$(1).
But $(F,E)^{\odot}$ is the largest binary soft open set contained in $(F,E)$. $\rightarrow$ (2).
Therefore from (1) and (2) it follows that $(F,E)^{\odot}$ must be larger than $(F,E)$, that is $(F,E)^{\odot}\tilde{\tilde{\supseteq}}(F,E)$ or $(F,E)$ is smaller than $(F,E)^{\odot}$, that is $(F,E)\tilde{\tilde{\subseteq}}(F,E)^{\odot}$ $\rightarrow$(3).
But $(F,E)^{\odot}\tilde{\tilde{\subseteq}}(F,E)$ is always true. $\rightarrow$ (4).
From (3) and (4) we have $(F,E)\tilde{\tilde{=}}(F,E)^{\odot}$. Note that the right hand side result, that is $(F,E)^{\odot}$ is a binary soft open set. Therefore left hand side of the result that is $(F,E)$ must be a binary soft open set. Consequently, $(F,E)$ is a binary soft open set. If $(F,E)\tilde{\tilde{=}}(F,E)^{\odot}$, then $(F,E)$ is a binary soft open set. Hence $(F,E)$ is binary soft open if and only if $(F,E)\tilde{\tilde{=}}(F,E)^{\odot}$. □

**Theorem 30.** *Let $(U_1, U_2, \tau_{\Delta}, E)$ be the binary soft topological space.*

*Let $(F, E)$, $(G, E)$ be any two binary soft subset over $U_1, U_2$, then the following properties hold true:*
*(i)* $\tilde{\tilde{X}}^{\odot} \tilde{\tilde{=}} \tilde{\tilde{X}}$.
*(ii)* $\tilde{\tilde{\phi}}^{\odot} \tilde{\tilde{=}} \tilde{\tilde{\phi}}$.
*(iii) If* $(F, E) \tilde{\tilde{\subseteq}} (G, E)$ *then* $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (G, E)^{\odot}$.
*(iv)* $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{=}} (F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$.
*(v)* $[(F, E)^{\odot}]^{\odot} = (F, E)^{\odot}$.
*(vi)* $(F, E)^{\odot} \tilde{\tilde{\cup}} (G, E)^{\odot} \tilde{\tilde{\subseteq}} [(F, E) \tilde{\tilde{\cup}} (G, E)]^{\odot}$.

*Proof.* (i) We know that $\tilde{\tilde{X}}$ is binary soft open set. This implies $\tilde{\tilde{X}}^{\odot} \tilde{\tilde{=}} \tilde{\tilde{X}}$. (Since $(F, E)$ is open if and only if $(F, E) \tilde{\tilde{=}} (F, E)^{\odot}$). Therefore $\tilde{\tilde{X}}^{\odot} \tilde{\tilde{=}} \tilde{\tilde{X}}$.
(ii) The result follows from (i).
(iii) Suppose $(F, E) \tilde{\tilde{\subseteq}} (G, E)$, then we know that $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (F, E)$ and $(F, E) \tilde{\tilde{\subseteq}} (G, E)$, therefore $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (G, E)$. Therefore $(F, E)^{\odot}$ is a binary soft open set contained in (G, E)$\rightarrow$ (1).
But $(G, E)^{\odot}$ is the largest binary open set contained in (G, E)$\rightarrow$ (2).
From (1) and (2) we have $(G, E)^{\odot}$ is the larger than $(F, E)^{\odot}$, that is $(F, E)^{\odot}$ is smaller than $(G, E)^{\odot}$, then $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (G, E)^{\odot}$. Thus $(F, E) \tilde{\tilde{\subseteq}} (G, E)$ which implies $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (G, E)^{\odot}$.
(iv) Let $(U_1, U_2, \tau_{\Delta}, E)$ be the binary soft topological space, to prove

$$[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{=}} (F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}.$$

We know that $(F, E) \tilde{\tilde{\cap}} (G, E) \tilde{\tilde{\subseteq}} (F, E)$ and $(F, E) \tilde{\tilde{\cap}} (G, E) \tilde{\tilde{\subseteq}} (G, E)$, this implies $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{\subseteq}} (F, E)^{\odot}$ and $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{\subseteq}} (G, E)^{\odot}$ (by (iii)).
This implies $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{\cap}} [(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{\subseteq}} (F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$.
Hence $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot} \tilde{\tilde{\subseteq}} (F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$.
Also we have, $(F, E)^{\odot} \tilde{\tilde{\subseteq}} (F, E)$ and $(G, E)^{\odot} \tilde{\tilde{\subseteq}} (G, E)$
$(F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot} \tilde{\tilde{\subseteq}} (F, E) \tilde{\tilde{\cap}} (G, E)$ which implies $(F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$ is a binary soft open set contained in $(F, E) \tilde{\tilde{\cap}} (G, E) \rightarrow$ (2).
But $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot}$ is the largest binary soft open set contained in $[(F, E) \tilde{\tilde{\cap}} (G, E)] \rightarrow$ (3).
Therefore from (2) and (3) it follows that $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot}$ is larger than $(F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$,
that is $(F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot}$ is smaller than $[(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot}$ this leads to
$(F, E)^{\odot} \tilde{\tilde{\cap}} (G, E)^{\odot} \tilde{\tilde{\subseteq}} [(F, E) \tilde{\tilde{\cap}} (G, E)]^{\odot}$. $\rightarrow$ (4).

From (1) and (4) it follows that $[(F,E)\tilde{\tilde{\cap}}(G,E)]^{\odot}\tilde{\tilde{=}}(F,E)^{\odot}\tilde{\tilde{\cap}}(G,E)^{\odot}$.
(v) We know that $[(F,E)^{\odot}]^{\odot}$ is binary soft open set. Let us assume $(F,E)^{\odot}\tilde{\tilde{=}}(H,E)$. Therefore $(H,E)$ is a binary soft open set, which implies $(H,E)\tilde{\tilde{=}}(H,E)^{\odot}$. Therefore $(F,E)^{\odot}\tilde{\tilde{=}}[(F,E)^{\odot}]^{\odot}$, hence the result.
(vi) Since $(F,E)\tilde{\tilde{\subseteq}}(F,E)\tilde{\tilde{\cup}}(G,E)$ and $(G,E)\tilde{\tilde{\subseteq}}(F,E)\tilde{\tilde{\cup}}(G,E)$. So by (iii),

$$(F,E)^{\odot}\tilde{\tilde{\subseteq}}[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\odot}$$

and

$$(F,E)^{\odot}\tilde{\tilde{\subseteq}}[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\odot}.$$

So that $(F,E)^{\odot}\tilde{\tilde{\cup}}(G,E)^{\odot}\tilde{\tilde{\subseteq}}[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\odot}$, since $(F,E)^{\odot}\tilde{\tilde{\cup}}(G,E)^{\odot}$ is binary soft open set. Which completes the proof. □

**Example 31.** The following example shows that the equality does not hold in 30 of (vi).
By using Example 14, let us consider,
$(F,E) = \{(e_1,(\{a_1\},\{b_1\}),(e_2,(\{a_3\},\{b_1\}),$
$(e_3,(\{a_1,a_2\},\{b_3\}),\ (e_4,(\{a_5\},\{b_2\})\}$
$(G,E) = \{(e_1,(\{a_4\},\{b_4\}),(e_2,(\{a_3\},\{b_1\}),(e_3,(\{a_2\},\{b_3\}),$
$(e_4,(\{a_3\},\{b_1\})\}$
$(F,E)\tilde{\tilde{\cup}}(G,E)\tilde{\tilde{=}}\{(e_1,(\{a_4\},\{b_4\}),(e_2,(\{a_3\},\{b_1\}),(e_3,(\{a_1,a_2\},\{b_3\}),$
$(e_4,(\{a_3,a_5\},\{b_1,b_2\})\}$
$[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\odot}\tilde{\tilde{=}}\{(e_1,(\{a_4\},\{b_4\}),(e_2,(\{a_3\},\{b_1\}),(e_3,(\{a_1,a_2\},\{b_3\}),$
$(e_4,(\{a_3,a_5\},\{b_1,b_2\})\}$
$(F,E)^{\odot} = \tilde{\tilde{\phi}}$ which implies $\tilde{\tilde{\phi}}\tilde{\tilde{\subseteq}}[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\odot}$. Thus equality does not holds.

**Definition 32.** Let $(U_1,U_2,\tau_\Delta,E)$ be a binary soft topological space over initial parameters $U_1,U_2$, then the boundary of binary soft set (H, E) is denoted by $\underline{\underline{(H,E)}}$ or $bd(H,E)$ and is defined as $\underline{\underline{(H,E)}} = \overline{\overline{(H,E)}}\tilde{\tilde{\cap}}\overline{\overline{(H,E)'}}$. Obviously $\underline{\underline{(H,E)}}$ is a smallest binary soft closed set over $U_1,U_2$ containing (H, E).

**Example 33.** $U_1 = \{c_1,c_2,c_3\}$ - set of computers;
$U_2 = \{m_1,m_2\}$ - set of mobiles; , $E = \{e_1 = expensive, e_2 = outlook, e_3 = function\}$;
$\tau_\Delta = \{\tilde{\tilde{X}},\tilde{\tilde{\phi}},\{(e_1,(\{c_1,c_3\},\{m_1\}),(e_2,(\{c_1,c_3\},\{m_2\}),$
$(e_3,(\{c_1,c_3\},\{m_1,m_2\})\},\ \{(e_1,(\{c_3\},\{m_1\}),(e_2,(\{c_3\},\{m_2\}),$
$(e_3,(\{c_3\},\{m_1,m_2\})\},\ \{(e_1,(\{c_2,c_3\},\{m_1\}),(e_2,(\{c_2,c_3\},\{m_2\}),$

$(e_3, (\{c_2, c_3\}, \{m_1, m_2\})\}, \{(e_1, (\{c_1, c_2, c_3\}, \{m_1\}),$
$(e_2, (\{c_1, c_2, c_3\}, \{m_2\}), (e_3, (\{c_1, c_2, c_3\}, \{m_1, m_2\})\}\}$
Now let $(H, E) = \{(e_1, (\{c_2, c_3\}, \{m_2\}), (e_2, (\{c_3\}, \{m_1\}),$
$(e_3, (\{c_1\}, \{m_1\})\}.$
Therefore,
$\underline{\underline{(H, E)}} \tilde{\tilde{=}} \{(e_1, (\{c_3\}, \{m_2\}), (e_2, (\{c_3\}, \{m_2\}), (e_3, (\{c_3\}, \{m_1, m_2\})\}.$

**Remark 34.** Let $(U_1, U_2, \tau_\Delta, E)$ be the binary soft topological space over $U_1, U_2$ and let $(F, E)$ be any binary soft set then we always have

$$(F, E)^{\odot} \tilde{\tilde{\subseteq}} (F, E) \tilde{\tilde{\subseteq}} \overline{\overline{(F, E)}}.$$

**Remark 35.** $\underline{\underline{(H, E)}} \tilde{\tilde{=}} \underline{\underline{[\tilde{\tilde{X}} - (H, E)]}}$
$bd(H, E) = bd[\tilde{\tilde{X}} - (H, E)]$
To prove; $bd(H, E) = bd\overline{\overline{(H, E)}} \tilde{\tilde{\cap}} \overline{\overline{[\tilde{\tilde{X}} - (H, E)]}} \rightarrow (1)$
$bd[\tilde{\tilde{X}} - (H, E)] = \overline{\overline{[\tilde{\tilde{X}} - (H, E)]}} \tilde{\tilde{\cap}} \overline{\overline{[\tilde{\tilde{X}} - (\tilde{\tilde{X}} - (H, E))]}}$
$\Rightarrow \overline{\overline{[\tilde{\tilde{X}} - (H, E)]}} \tilde{\tilde{\cap}} \overline{\overline{(H, E)}}$
$\Rightarrow \overline{\overline{(H, E)}} \tilde{\tilde{\cap}} \overline{\overline{[\tilde{\tilde{X}} - (H, E)]}} = bd(H, E)$. Thus $bd(H, E) = bd[\tilde{\tilde{X}} - (H, E)]$.

**Theorem 36.** *Let $(U_1, U_2, \tau_\Delta, E)$ be the binary soft topological space. Let $(H, E)$ be any binary soft subset of $\tilde{\tilde{X}}$. Then the following properties are true:*
*(i)* $bd(H, E) = \overline{\overline{(H, E)}} - (H, E)^{\odot}$.
*(ii)* $(H, E)^{\odot} = (H, E) - bd(H, E)$.
*(iii)* $\tilde{\tilde{X}} \tilde{\tilde{=}} (H, E)^{\odot} \tilde{\tilde{\cup}} bd(H, E) \tilde{\tilde{\cup}} [\tilde{\tilde{X}} - (H, E)]^{\odot}$.

*Proof.* (i) We know that $bd(H, E) = \underline{\underline{(H, E)}} \tilde{\tilde{=}} \overline{\overline{(H, E)}} \tilde{\tilde{\cap}} \overline{\overline{(H, E)'}}$
$\Rightarrow \overline{\overline{(H, E)}} - [\tilde{\tilde{X}} - \overline{\overline{(\tilde{\tilde{X}} - (H, E)]}}$
$= \overline{\overline{(H, E)}} - [\tilde{\tilde{X}} - (\tilde{\tilde{X}} - (H, E))]^{\odot}$
$= \overline{\overline{(H, E)}} - (H, E)^{\odot}$. Thus, $bd(H, E) = \overline{\overline{(H, E)}} - (H, E)^{\odot}$.
(ii) Consider $(H, E) - bd(H, E) = (H, E) - [\overline{\overline{(H, E)}} - \overline{\overline{(\tilde{\tilde{X}} - (H, E))]}}$
$= (H, E) \tilde{\tilde{\cap}} [\overline{\overline{\tilde{\tilde{X}}}} - (\overline{\overline{(H, E)}} \tilde{\tilde{\cap}} \overline{\overline{(\tilde{\tilde{X}} - (H, E))]}}$

$= (H,E)\tilde{\tilde{\cap}}[(\tilde{\tilde{X}} - \overline{\overline{(H,E)}})\tilde{\tilde{\cup}}(\tilde{\tilde{X}} - \overline{\overline{[(\tilde{\tilde{X}} - (H,E))]}})]$
$= (H,E)\tilde{\tilde{\cap}}[(\tilde{\tilde{X}} - \overline{\overline{(H,E)}})\tilde{\tilde{\cup}}(\tilde{\tilde{X}} - [(\tilde{\tilde{X}} - (H,E)^{\odot})])]$
$= (H,E)\tilde{\tilde{\cap}}[(\tilde{\tilde{X}} - \overline{\overline{(H,E)}})\tilde{\tilde{\cup}}[(H,E)\tilde{\tilde{\cap}}(H,E)^{\odot}]$.
Therefore, $(H,E)^{\odot} = (H,E) - bd(H,E)$.
(iii) Consider, RHS $(H,E)^{\odot}\tilde{\tilde{\cup}}bd(H,E)\tilde{\tilde{\cup}}[\tilde{\tilde{X}} - (H,E)]^{\odot}$
$= (H,E)^{\odot}\tilde{\tilde{\cup}}bd(H,E)\tilde{\tilde{\cup}}(\tilde{\tilde{X}} - \overline{\overline{(H,E)}})$
$= \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(\tilde{\tilde{X}} - (H,E))}} = \tilde{\tilde{X}}$. Further we know that $bd(H,E) = \overline{\overline{(H,E)}} - (H,E)^{\odot}$, this implies $bd(H,E)$ and $(H,E)^{\odot}$ are disjoint $\to$ (1)
we replace by $(H,E)$ with $(\tilde{\tilde{X}} - (H,E)^{\odot})$ are binary soft disjoint sets. Hence $\tilde{\tilde{X}} = \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(\tilde{\tilde{X}} - (H,E)}}$. Which is a binary soft disjoint union. □

**Theorem 37.** *Let $(U_1, U_2, \tau_\Delta, E)$ be the binary soft topological space. Let $(H,E)$ be any binary soft subset of $U_1, U_2$. Then $(F,E)$ is binary soft closed if and only if $(F,E) \supseteq bd(F,E)$.*

*Proof.* Suppose $(F,E)$ is binary soft closed set, then

$$bd(H,E) = \overline{\overline{(F,E)}}\tilde{\tilde{\cap}}\overline{\overline{[\tilde{\tilde{X}} - (F,E)]}}\tilde{\tilde{\subseteq}}(F,E)\tilde{\tilde{\cap}}\overline{\overline{[\tilde{\tilde{X}} - (F,E)]}}\tilde{\tilde{\subseteq}}(F,E).$$

Therefore $(F,E)\tilde{\tilde{\supseteq}}bd(F,E)$. Hence $(F,E)$ is binary soft closed if and only if $(F,E)\tilde{\tilde{\supseteq}}bd(F,E)$. $\to$(1).
Conversely, suppose $(F,E)\tilde{\tilde{\supseteq}}bd(F,E)$ that is $bd(F,E)\tilde{\tilde{\subseteq}}(F,E)$ which implies $(F,E)\tilde{\tilde{\cup}}$
$bd(F,E) = \overline{\overline{(F,E)}} = (F,E)$. Therefore $(F,E)$ is binary soft closed. Thus $(F,E)\tilde{\tilde{\supseteq}}bd(F,E)$ implies (F, E) is binary soft closed. $\to$ (2).
From (1) and (2) it is clear that $(F,E)$ is binary soft closed if and only if $(F,E)\tilde{\tilde{\supseteq}}bd(F,E)$. □

**Theorem 38.** *Let $(U_1, U_2, \tau_\Delta, E)$ be the binary soft topological space. Let $(H,E)$ be any binary soft subset of $U_1, U_2$. Then $(F,E)$ is binary soft open if and only if $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}$.*

*Proof.* Suppose $(F,E)$ is binary soft open which implies $[\tilde{\tilde{X}} - (F,E)]$ is binary soft closed.
$\Rightarrow \overline{\overline{[\tilde{\tilde{X}} - (F,E)]}}\tilde{\tilde{=}}[\tilde{\tilde{X}} - (F,E)]$.

Now consider $(F,E)\tilde{\tilde{\cap}}bd(F,E)$
$\Rightarrow (F,E)\tilde{\tilde{\cap}}[\overline{\overline{(F,E)}}\tilde{\tilde{\cap}}\overline{\overline{(\tilde{\tilde{X}}-(F,E))}}]$
$\Rightarrow (F,E)\tilde{\tilde{\cap}}[(\tilde{\tilde{X}}-(F,E))\tilde{\tilde{\cap}}\overline{\overline{(F,E)}}]=\tilde{\tilde{\phi}}$. Therefore $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}$. Thus (F, E) is open implies $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}\rightarrow (1)$.
Conversely, suppose $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}$
$\Rightarrow (F,E)\tilde{\tilde{\cap}}[\overline{\overline{(F,E)}}\tilde{\tilde{\cap}}\overline{\overline{(\tilde{\tilde{X}}-(F,E))}}]\tilde{\tilde{=}}\phi$
$(F,E)\tilde{\tilde{\cap}}\overline{\overline{(F,E)}}\tilde{\tilde{\cap}}[\overline{\overline{(\tilde{\tilde{X}}-(F,E))}}]\tilde{\tilde{=}}\phi$
$\Rightarrow (F,E)\tilde{\tilde{\cap}}[\overline{\overline{(\tilde{\tilde{X}}-(F,E))}}]\tilde{\tilde{=}}\phi$
$\Rightarrow (F,E)\tilde{\tilde{\subseteq}}\tilde{\tilde{X}}-[\overline{\overline{(\tilde{\tilde{X}}-(F,E))}}]\tilde{\tilde{=}}\phi$
$\Rightarrow (F,E)\tilde{\tilde{\subseteq}}\tilde{\tilde{X}}-[\tilde{\tilde{X}}-(F,E)^{\odot}]$
$\Rightarrow (F,E)\tilde{\tilde{\subseteq}}(F,E)^{\odot}$. But $(F,E)^{\odot}\tilde{\tilde{\subseteq}}(F,E)$ is always true. Therefore $(F,E)=(F,E)^{\odot}$.
Therefore $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}$ implies $(F,E)$ is binary soft open.
From (i) and (ii) we have that $(F,E)$ is binary soft open if and only if

$$(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\phi}}.$$ □

## 3. Conclusion

The soft set theory is very important tool to study the concepts of classical and non classical logic. Recently the binary soft set theory has been introduced by Ahu Acikgöz and Nihal Tas. In this paper, we introduce binary soft topological spaces which are defined over two initial universe sets with a fixed set of parameters. Many basic results like binary soft open sets, binary soft closed sets, binary soft closure, binary soft interior, binary soft boundary, binary soft neighborhood of a point are introduced and their basic properties are investigated with the suitable examples. These results are important for further research on binary soft topology.

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