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INVERSION OF THE MIXED RIESZ HYPERBOLIC B-POTENTIALS

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Abstract: In this article a theory of fractional powers of a singular hyperbolic operator on arbitrary spaces is discussed. We consider the problem of inversion of the mixed hyperbolic Riesz B-potential operator on weighted Lebesgue spaces. We apply here the method of approximative inverses.

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Key Words: mixed hyperbolic Riesz B–potential, fractional power of a singular hyperbolic operator, Lorentz distance, singular Bessel differential operator, generalized translation, Fourier–Hankel transform, bounded operator, approximative inverse operator

1. Introduction

1.1. Brief History

In this paper, we study the inversion of an integral operator of the mixed hyperbolic Riesz B-potentials which is a fractional power of the operator

$$\frac{\partial^2}{\partial t^2}-\sum_{k=1}^{n}(B_{\gamma_i})_{x_i}, \tag{1}$$

where $\gamma_1>0,...,\gamma_n>0$ and $(B_{\gamma_i})_{x_i}=\dfrac{\partial^2}{\partial x_i^2}+\dfrac{\gamma_i}{x_i}\dfrac{\partial}{\partial x_i}$ is the singular differential

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Bessel operator.

The potential theory comes from mathematical physics. The most well-known areas of its application are electrostatic and gravitational theory, probability theory, scattering theory, biological systems and other. In mathematics, the Newton potential is an operator that acts as the inverse to the negative Laplacian. Thus, the Newton potential can be interpreted as the negative power of the Laplace operator. The Hungarian mathematician Marcell Riesz was the first who consider the fractional negative powers of the Laplace operator, what is now called the *Riesz potentials* (see [1], [2]). Riesz also introduced potentials with Lorentz distances which are the fractional negative powers of the D'Alembert operator. Further study, properties, and applications of classical Riesz potentials could be found in books [3] p. 49, 263; [4] p.117; [5] p. 131, [6] p. 483, 554; [7] p. 215; [8] p. 127, 341, 458, [9] p. 111, [10] p. 254, 316, [11] p. 144.

Along with operators representing fractional powers of ordinary derivatives, the theory of fractional powers of differential operators related to the Bessel operator $B_\nu{=}D^2+\frac{\nu}{x}D$ and $D=\frac{d}{dx}$ develops. The theory of fractional powers of elliptic operators involving Bessel operators instead of all or some second derivatives is well developed. Such operators in the case of negative powers are analogues of the Riesz potentials with Euclidean distance, and for them the term *elliptic B–potentials* is accepted. Elliptic B–potentials and its inverses have been studied in papers [12]–[26] and this list is not complete. It is interesting that the Bessel operator and the hyper–Bessel operator could be considered as generalized fractional derivatives too (see [27], p. 97). In [27], p. 18 a hypergeometric fractional integral was investigated which is the fractional power of the Bessel operator in particular case.

The fractional powers of hyperbolic operators with Bessel operators instead of all or some second derivatives are much less investigated despite the fact that its study opens the widest possibilities for theoretical researches and practical applications not only of singular differential equations but also differential geometry (see [28, 29]). When the power is negative such operators are called *hyperbolic B–potentials*. Some results for hyperbolic B–potentials could be found in [30]–[33].

In this paper we obtain the inverse operator to the *mixed hyperbolic Riesz B–potentials*. This potential is the negative real power of the hyperbolic operator (1). In our research we rely on methods developed in papers [34]–[41]. Namely, we use the methods of such transmutation operators as a generalized translation and the Poisson operator (see [34]–[38]), weighted spaces (see [39]–[41]) and methods of classical Riesz potentials (see [28]–[47]).

1.2. Basic Definitions

In this subsection we give shortly basic notations, terminology and results used in the article.

Suppose that R^{n+1} is the $n+1$–dimensional Euclidean space,

$$R_+^{n+1}=\{(t,x)=(t,x_1,\ldots,x_n)\in R^{n+1},\ \ x_1>0,\ldots,x_n>0\},$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multiindex consisting of positive fixed real numbers γ_i, i=1, ..., n, and $|\gamma|=\gamma_1+\ldots+\gamma_n$. Let Ω be a finite or infinite open set in R^{n+1} symmetric with respect to each hyperplane x_i=0, $i=1,...,n$, $\Omega_+=\Omega\cap R_+^{n+1}$ and $\overline{\Omega}_+=\Omega\cap\overline{R}_+^{n+1}$ where $\overline{R}{}_+^{n+1}=\{(t,x)=(t,x_1,\ldots,x_n)\in R^{n+1},x_1\geq 0,\ldots,x_n\geq 0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all existing derivatives of these functions with respect to x_i for any $i=1,...,n$ are continuous up to x_i=0 and all existing derivative with respect to t are continuous for $t\in R$. The class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all nonnegative integer $k\leq\frac{m-1}{2}$ and for $i=1,...,n$ (see [39] and [40], p. 21). In the following we will denote $C^m_{ev}(\overline{R}{}_+^{n+1})$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m. Let $C^\infty_{ev}(\overline{R}{}_+^{n+1})=C^\infty_{ev}$. Assuming that $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ is the space of all functions $f\in C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. We will use the notation $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)=\mathcal{D}_+(\overline{\Omega}_+)$.

Let $L_p^\gamma(\Omega_+)$, $1\leq p<\infty$ be the space of all measurable in Ω_+ functions such that

$$\int\limits_{\Omega_+}|f(t,x)|^p x^\gamma dtdx<\infty,$$

where and further

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p\geq 1$, the $L_p^\gamma(\Omega_+)$–norm of f is defined by

$$||f||_{L_p^\gamma(\Omega_+)}=\left(\int\limits_{\Omega_+}|f(t,x)|^p x^\gamma dtdx\right)^{1/p}.$$

The weighted measure of Ω_+ is denoted by $\mathrm{mes}_\gamma(\Omega_+)$ and is defined by the formula

$$\mathrm{mes}_\gamma(\Omega_+) = \int\limits_{\Omega_+} x^\gamma dtdx.$$

For every measurable function $f(x)$ defined on R^{n+1}_+ we consider

$$\mu_\gamma(f,\sigma) = \mathrm{mes}_\gamma\{(t,x) \in R^{n+1}_+ : |f(t,x)| > \sigma\}$$

$$= \int\limits_{\{(t,x):\ |f(t,x)|>\sigma\}^+} x^\gamma dtdx,$$

where $\{(t,x):|f(t,x)|>\sigma\}^+=\{(t,x)\in R^{n+1}_+:|f(t,x)|>\sigma\}$. We will call the function $\mu_\gamma=\mu_\gamma(f,\sigma)$ a *weighted distribution function* $|f(t,x)|$.

Let a space $L^\gamma_\infty(\Omega_+)$ be defined as a set of measurable on Ω_+ functions $f(t,x)$ such that

$$||f||_{L^\gamma_\infty(\Omega_+)} = \underset{(t,x)\in\Omega_+}{\mathrm{ess\,sup}_\gamma}|f(t,x)| = \inf_{\sigma\in\Omega_+}\{\mu_\gamma(f,\sigma) = 0\} < \infty.$$

For $1 \leq p \leq \infty$ the $L^\gamma_{p,loc}(\Omega_+)$ is the set of functions u defined almost everywhere in Ω_+ such that $uf \in L^\gamma_p(\Omega_+)$ for any $f \in \mathcal{D}_+(\overline{\Omega}_+)$.

Let us define $\mathcal{D}'_+(\overline{\Omega}_+)$ as a set of continuous linear functionals on $\overline{\Omega}_+$. Each function $u \in L^\gamma_{1,loc}(\Omega_+)$ will be identified with the functional $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ acting according to the formula

$$(u,f)_\gamma = \int\limits_{\Omega_+} u(t,x)\, f(t,x)\, x^\gamma\, dtdx, \qquad f \in \mathcal{D}_+(\overline{\Omega}_+). \tag{2}$$

Functionals $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ acting by the formula (2) will be called *regular weighted functionals*. All other continuous linear functionals $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ will be called *singular weighted functionals*. We will use the notation $D'_+ = \mathcal{D}'_+(\overline{R}_+)$.

The generalized function δ_γ is defined by the equality by analogy with ([40], p. 12)

$$(\delta_\gamma,\varphi)_\gamma = \varphi(0), \quad \varphi \in \mathcal{D}_+(\overline{\Omega}_+).$$

We will use the generalized convolution product defined by the formula

$$(f*g)_\gamma = \int\limits_{R^{n+1}_+} f(\tau,y)({}^\gamma\mathbf{T}^y_x g)(t-\tau,x)y^\gamma d\tau dy, \tag{3}$$

where ${}^\gamma\mathbf{T}_x^y$ is multidimensional generalized translation

$$({}^\gamma\mathbf{T}_x^y f)(t,x)=({}^{\gamma_1}T_{x_1}^{y_1}...{}^{\gamma_n}T_{x_n}^{y_n} f)(t,x). \tag{4}$$

Each of one–dimensional generalized translations ${}^{\gamma_i}T_{x_i}^{y_i}$ is defined for i=1, ..., n by the next formula (see [34], p. 122, formula (5.19))

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(t,x)=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi \sin^{\gamma_i-1}\varphi_i$$

$$\times f(t,x_1,...,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\varphi_i},x_{i+1},...,x_n)\,d\varphi_i.$$

As the space of basic functions we will use the subspace of rapidly decreasing functions:

$$S_{ev}(R_+^{n+1})=\left\{f\in C_{ev}^\infty:\sup_{(t,x)\in R_+^{n+1}}\left|t^{\alpha_0}x^\alpha D^\beta f(t,x)\right|<\infty\right\},$$

where $\alpha=(\alpha_1,...,\alpha_n)$, $\beta=(\beta_0,\beta_1,...,\beta_n)$, $\alpha_0,\alpha_1,...,\alpha_n,\beta_0,\beta_1,...,\beta_n$ are arbitrary integer nonnegative numbers, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$, $D^\beta=D_t^{\beta_0}D_{x_1}^{\beta_1}...D_{x_n}^{\beta_n}$, $D_t=\frac{\partial}{\partial t}$, $D_{x_j}=\frac{\partial}{\partial x_j}$, $j=1,...,n$. In the same way as $\mathcal{D}'_+$ we introduce the space S'_{ev}. In fact we identify S'_{ev} with a subspace of $\mathcal{D}'_+$ since $\mathcal{D}_+$ which is dense in S_{ev}.

Let $\Psi_{\gamma,V}$ denote the following class of functions

$$\Psi_{\gamma,V}=\{\psi\in S_{ev}(R_+^{n+1}):(D^k\psi)(x)=0,x\in V,|k|=0,1,2,...\}$$

and

$$\Phi_{\gamma,V}=\{\varphi:\mathcal{F}_\gamma\varphi\in\Psi_{\gamma,V}\}.$$

2. Mixed Hyperbolic Riesz B–Potential and its Inversion

2.1. Basic Properties of Mixed Hyperbolic Riesz B–Potential

Let $|x|=\sqrt{x_1^2+...+x_n^2}$. First for $(t,x)\in R_+^{n+1}$, $\lambda\in C$ we define function s^λ by the formula

$$s^\lambda(t,x)=\begin{cases}\frac{(t^2-|x|^2)^\lambda}{N(\alpha,\gamma,n)}, & \text{when } t^2\geq|x|^2 \text{ and } t\geq 0;\\ 0, & \text{when } t^2<|x|^2 \text{ or } t<0,\end{cases} \tag{5}$$

where

$$N(\alpha,\gamma,n)=\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \qquad (6)$$

Regular weighted generalized function corresponding to (5) we will denote by s_+^{λ}.

We introduce the *mixed hyperbolic Riesz B–potential* $I^{\alpha}_{s,\gamma}$ *of order* $\alpha>0$ as a generalized convolution product (3) with a weighted generalized function $s_+^{\frac{\alpha-n-|\gamma|-1}{2}}$ and $f\in S_{ev}$:

$$(I^{\alpha}_{s,\gamma}f)(t,x)=(s_+^{\frac{\alpha-n-|\gamma|-1}{2}}*f)_{\gamma}(t,x). \qquad (7)$$

The precise definition of the constant $N(\alpha,\gamma,n)$ allows to obtain the semigroup property or index low of the potential (7).

We can rewrite formula (7) as

$$(I^{\alpha}_{s,\gamma}f)(t,x)=\int\limits_{R^{n+1}_+} s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^{\gamma}\mathbf{T}^{y}_{x})f(t-\tau,x)y^{\gamma}d\tau dy. \qquad (8)$$

Theorem 1. *Let* $n+|\gamma|-1<\alpha<n+|\gamma|+1$, $1\leq p<\frac{n+|\gamma|+1}{\alpha}$. *For the following estimate*

$$||I^{\alpha}_{s,\gamma}f||_{q,\gamma}\leq M||f||_{p,\gamma}, \qquad f\in S_{ev} \qquad (9)$$

to be valid it is necessary and sufficient that $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$. *The constant* M *does not depend on* f.

Remark. By virtue of (9) there is unique extension of $I^{\alpha}_{s,\gamma}$ to all L^{γ}_{p}, $1<p<\frac{n+|\gamma|+1}{\alpha}$ preserving boundedness when $n+|\gamma|-1<\alpha<n+|\gamma|$. It follows that this extension is introduced by the integral (8) from its absolute convergence.

Theorem 2. *For* $f\in S_{ev}$ *the Fourier–Hankel transform of the mixed hyperbolic Riesz potential* $I^{\alpha}_{s,\gamma}f$ *is defined by the formula*

$$\mathcal{F}_{\gamma}[I^{\alpha}_{s,\gamma}f](\tau,\xi)=q\left|\tau^2-|\xi|^2\right|^{-\frac{\alpha}{2}}\cdot\mathcal{F}_{\gamma}[f(t,x)](\tau,\xi), \qquad (10)$$

where

$$q=\begin{cases}1, & |\xi|^2\geq\tau^2;\\ e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\geq 0;\\ e^{\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0.\end{cases}$$

2.2. Inversion of the Mixed Hyperbolic Riesz B–Potential

For the inversion of the potential (7) we will use approach based on the idea of approximative inverse operators (see [28, 29]). This method gives an inverse operator as a limit of regularized operators. Namely, taking into account the formula (10) we will construct inverse operator for the potential (7) in the form

$$(I^{\alpha}_{s,\gamma})^{-1}f=\lim_{\varepsilon\to 0}\left(\mathcal{F}^{-1}_{\gamma}(q|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|})*f\right)_{\gamma},$$

where the limit is understood in the norm L_p^{γ} or almost everywhere.

Let

$$g_{\alpha,\gamma,\varepsilon}(t,x)=\mathcal{F}^{-1}_{\gamma}(q^{-1}|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|})(t,x).$$

Theorem 3. *The function $g_{\alpha,\gamma,\varepsilon}(t,x)$ belongs to the space $L_{r,\gamma}$, $1<r<\infty$ with additional restriction $\frac{2(n+|\gamma|)-1}{2(n+|\gamma|)-2}<r$ for $n+|\gamma|-1<\alpha<n+|\gamma|$ when $n+|\gamma|+1$ is odd.*

Now we introduce the homogenizing kernel $N_{\gamma}(t,x,\varepsilon)$ which is defined as follows

$$N_{\gamma}(t,x,\varepsilon)=\frac{C(n,\gamma,\varepsilon)}{(t^2+\varepsilon^2)(|x|^2+\varepsilon^2)^{\frac{n+|\gamma|}{2}}},$$

where $x=(x_1,...,x_n)$, $\varepsilon>0$,

$$C(n,\gamma,\varepsilon)=\frac{2^n\varepsilon^2\Gamma\left(\frac{n+1+|\gamma|}{2}\right)}{\pi^{\frac{3}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

We give some properties for the function $N_{\gamma}(t,x,\varepsilon)$ proved in [32] for a more general case.

Theorem 4. *The homogenizing kernel $N_{\gamma}(t,x,\varepsilon)$ has the properties:*

1. $\mathcal{F}_{\gamma}[N_{\gamma}(t,x,\varepsilon)](\xi)=e^{-\varepsilon\tau-\varepsilon|\xi|}$,
2. $\int\limits_{R^{n+1}_+}N_{\gamma}(t,x,\varepsilon)x^{\gamma}dtdx=\int\limits_{R^{n+1}_+}N_{\gamma}(t,x,1)x^{\gamma}dtdx=1$,
3. $N_{\gamma}(t,x,\varepsilon)\in L_p^{\gamma}$, $1\leq p\leq\infty$.

Theorem 5. *Let $f \in L_p^\gamma$ and*

$$(N_{\gamma,\varepsilon} f)(\tau, y) = \int\limits_{R_+^{n+1}} N_\gamma(t, x, \varepsilon) T_x^y f(\tau - t, x) x^\gamma dt dx.$$

Then $(N_{\gamma,\varepsilon} f)(\tau, y)$ converges to $f(\tau, y)$ almost everywhere for ε tending to zero:

$$\lim_{\varepsilon \to 0} (N_{\gamma,\varepsilon} f)(\tau, y) = f(\tau, y), \qquad a.e.$$

Proof. Let us consider

$$(N_{\gamma,\varepsilon} f)(\tau, y) = \int\limits_{R_+^{n+1}} N_\gamma(t, x, \varepsilon) T_x^y f(\tau - t, x) x^\gamma dt dx$$

$$= C(n, \gamma, \varepsilon) \int\limits_{-\infty}^{+\infty} \frac{dt}{(t^2 + \varepsilon^2)} \int\limits_{R_+^n} \frac{T_x^y f(\tau - t, x) x^\gamma dx}{(|x|^2 + \varepsilon^2)^{\frac{n+|\gamma|}{2}}}.$$

The inner integral is estimated using Theorem 2, pp. 315 from [18] (see also [19]) and this estimate has the form

$$(N_{\gamma,\varepsilon} f)(\tau, y) \leq C(n, \gamma, \varepsilon) \int\limits_{-\infty}^{+\infty} \frac{(M_x f)(\tau - t, \xi) dt}{(t^2 + \varepsilon^2)},$$

where M_x is the maximal function (see [18] pp. 313). Applying now Stein's theorem (see [4], pp. 77, Theorem 2) we obtain the following estimate

$$\sup_{\varepsilon > 0} |(N_{\gamma,\varepsilon} f)(\tau, y)| \leq A (M_C f)(\tau, \xi),$$

where

$$(M_C f)(\tau, \xi) = \sup_{\beta > 0} \sup_{r > 0} \frac{1}{2\beta |B_\xi^+(r)|_\gamma} \int\limits_{|t - \tau| < \beta} dt \int\limits_{B_\xi^+(r)} |f(t, x)| x^\gamma dx$$

is the maximum function with respect to the cylinders in R_+^{n+1} with centers at the point ξ. Now the existence of the limit is proved also as in the book [4], pp. 58. □

Theorem 6. *Let* $n+|\gamma|-1<\alpha<n+1+|\gamma|$, $1<p<\frac{n+1+|\gamma|}{\alpha}$ *with the additional restriction* $p<\frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+|\gamma|+3\alpha(n+|\gamma|)}$ *when* $n+|\gamma|-1<\alpha<n+|\gamma|$ *and* n *is odd. Then for* $f(x)\in L_p^\gamma$ *the following representation is true*

$$((I^\alpha_{s,\gamma})^{-1}_\varepsilon I^\alpha_{s,\gamma}f)(t,x)=(N_{\gamma,\varepsilon}f)(t,x), \tag{11}$$

where $(I^\alpha_{s,\gamma})^{-1}_\varepsilon f{=}\left(\mathcal{F}^{-1}_\gamma(q|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|})*f\right)_\gamma$.

Proof. Let us consider the operator $(I^\alpha_{s,\gamma})^{-1}_\varepsilon I^\alpha_{s,\gamma}$. The following Young inequality for generalized convolution is well known (see, for example, [48]):

$$||(f*g)_\gamma||_{r,\gamma}\le||f||_{p,\gamma}||g||_{q,\gamma},\quad 1\le p,q,r\le\infty,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1. \tag{12}$$

Using (12) we obtain that the operator is bounded from $L_{p,\gamma}$ to $L_{\updownarrow,\gamma}$ where $\frac{1}{\updownarrow}=\frac{1}{r}+\frac{1}{q}-1$, $r=\frac{(n+1+|\gamma|)p}{n+1+|\gamma|-\alpha p}$, q satisfies the conditions of Theorem 2. The operator $N_{\gamma,\varepsilon}$ the operator is bounded in $L_{p,\gamma}$. Therefore, it suffices to verify equality on a weighted Lizorkin space $\Phi_{V,\gamma}$ which is dense in $L_{p,\gamma}$ (see [49], [50]), where $V=\{\tau^2-|\xi|^2=0\}\cup\{\tau=0\}\cup\{\xi_i=0,i{=}1,...,n\}$, $(\tau,\xi)\in R^{n+1}_+$. Thus

$$\mathcal{F}_\gamma\left[\left((I^\alpha_{s,\gamma})^{-1}_\varepsilon\right)I^\alpha_{s,\gamma}f\right](\tau,\xi){=}q^{-1}|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|}\mathcal{F}_\gamma[I^\alpha_{s,\gamma}f](\tau,\xi)$$

$$=e^{-\varepsilon|\tau|-\varepsilon|\xi|}\mathcal{F}_\gamma[f](\tau,\xi)=\mathcal{F}_\gamma\left[\left(\mathcal{F}^{-1}_\gamma e^{-\varepsilon|\tau|-\varepsilon|\xi|}*f\right)_\gamma\right](\tau,\xi)$$

and

$$\mathcal{F}_\gamma\left[\left((I^\alpha_{s,\gamma})^{-1}_\varepsilon\right)I^\alpha_{s,\gamma}f\right](\tau,\xi){=}\mathcal{F}_\gamma\left[\left(\mathcal{F}^{-1}_\gamma e^{-\varepsilon|\tau|-\varepsilon|\xi|}*f\right)_\gamma\right](\tau,\xi). \tag{13}$$

Applying the inverse Fourier-Hankel transform to (13), we get (11). □

Theorem 7. *Let* $n+|\gamma|-1<\alpha<n+1+|\gamma|$, $1<p<\frac{n+1+|\gamma|}{\alpha}$ *with the additional restriction* $p<\frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+1+|\gamma|+2\alpha(n+|\gamma|)}$ *when* $n+|\gamma|-1<\alpha<n+|\gamma|$ *and* n *is odd. Then*

$$((I^\alpha_{s,\gamma})^{-1}I^\alpha_{s,\gamma}f)(t,x)=f(t,x),\qquad f(t,x)\in L_p^\gamma,$$

where $(I^\alpha_{s,\gamma})^{-1}f=\lim\limits_{\varepsilon\to0}(I^\alpha_{s,\gamma})^{-1}_\varepsilon f$.

This theorem follows from Theorem 6 and Theorem 5.

References

[1] M. Riesz, Intégrale de Riemann-Liouville et solution invariantive du probléme de Cauchy pour l'équation de sondes, *Comptes Rendus du Congres International des Mathematiciens*, **2** (1936), 44-45.

[2] M. Riesz, L'integrale de Riemann-Liouville et le probleme de Cauchy, *Acta Mathematica*, **81** (1949), 1-223.

[3] L. Schwartz, *Théorie des distributions*, Hermann, 1966 (1st Ed. 1950-1951).

[4] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1970.

[5] S. Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*, American Mathematical Soc., 1984.

[6] S.G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Science, Amsterdam, 1987.

[7] B. Rubin, *Fractional Integrals and Potentials*, Chapman and Hall/CRC, 1996.

[8] A.A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[9] M.D. Ortigueira, *Fractional Calculus for Scientists and Engineers*, Springer Netherlands, Series: Lecture Notes in Electrical Engineering, **84**, 2011.

[10] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, 2012.

[11] S.R. Umarov, *Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols*, Springer International Publishing, Series: Developments in Mathematics, **41**, 2015.

[12] V.S. Guliev, Sobolev theorems for B-Riesz potentials, *Dokl. RAN*, **358**, No 4 (1998), 450–451 (In Russian).

[13] V.S. Guliev, Some properties of the anisotropic Riesz-Bessel potential, *Analysis Mathematica*, **26**, No 2 (2000), 20 pp.

[14] D.D. Gasanov, V.S. Guliev, A.X. Narimanov, Certain inequality for the anizotropic B-maximal functions and the anisotropic B-Riesz potentials, *Trydi mst. Math. and Mech. Akad. Nauk Azerb. Rep.*, **4** (1996), 18-24.

[15] A.D. Gadjiev, V.S. Guliyev, A. Serbetci, E.V. Guliyev, The Stein-Weiss type inequalities for the B-Riesz potentials, *J. Math. Inequal.*, **5**, No 1 (2011), 87-106.

[16] E.V. Guliyev, Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, **26**, No 1 (2006), 71-80.

[17] V.S. Guliyev and J.J. Hasanov, Sobolev-Morrey type inequality for Riesz potentials, associated with the Laplace-Bessel differential operator, *Fract. Calc. Appl. Anal.*, **9**, No 1 (2006), 17-32.

[18] V.S. Guliyev, M. Assal, On maximal function on the Laguerre hypergroup, *Fract. Calc. Appl. Anal.*, **9**, No 3 (2006), 1-12.

[19] V.S. Guliev, On maximal function and fractional integral, associated with the Bessel differential operator, *Math. Inequal. Appl.*, **6**, No 2 (2003), 317-330.

[20] L.N. Lyakhov, Inversion of the B-Riesz potentials, *Dokl. Akad. Nauk SSSR*, **321**, No 3 (1991), 466-469 (In Russian).

[21] L.N. Lyakhov, Spaces of Riezs B-potentials, *Dokl. Akad. Nauk SSSR*, **334**, No 3 (1994), 278-280 (In Russian).

[22] L.N. Lyakhov, Symbol of the integral operator of Riezs B-potential with single characteristic, *Dokl. Akad. Nauk*, **351**, No 2 (1996), 164-168 (In Russian).

[23] L.N. Lyakhov, E.L. Shishkina, Generalized Riesz B-potentials of the mixed type, *Dokl. Akad. Nauk*, **73**, No 1 (2006), 42-45.

[24] L.N. Lyakhov, E.L. Shishkina, General B-hypersingular integrals with homogeneous characteristic, *Dokl. Akad. Nauk*, **75**, No 1 (2007), 39-43.

[25] L.N. Lyakhov, E.L. Shishkina, Inversion of general Riesz B-potentials with homogeneous characteristic in weight classes of functions, *Doklady Akademii Nauk*, **426**, No 4 (2009), 443-447 (in Russian).

[26] E.L. Shishkina, Inversion of integral of B-potential type with density from Φ_γ, *Journal of Mathematical Sciences*, **160**, No 1 (2009), 95-102.

[27] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Res. Notes Math. **301**, Longman Scientific & Technical, Harlow, Co-publ. John Wiley, New York, 1994.

[28] V.A. Nogin and E.V. Sukhinin, Inversion and characterization of hyperbolic potentials in L_p-spaces, *Deponierted in VINITI*, **2512**, No 92 (1992), 1-50 (in Russian).

[29] V.A. Nogin and E.V. Sukhinin, Inversion and characterization of hyperbolic potentials in L_p-spaces, *Dokl. Acad. Nauk*, **329**, No 5 (1993), 550-552 (in Russian).

[30] M.Z. Sarikaya, H. Yildirim, Ö. Akin, On generalized Riesz type potential with Lorentz distance, *Lobachevskii Journal of Mathematics*, **28** (2008), 24-31.

[31] E.L. Shishkina, On the boundedness of hyperbolic Riesz B-potential, *Lithuanian Mathematical Journal*, **56**, No 4 (2016), 540-551.

[32] E.L. Shishkina, About properties of the averaging kernel in the Lebesgue weighted class, *Scientific bulletins of the Belgorod State University. Series: Mathematics. Physics*, **42**, No 6 (227) (2016), 12-19 (in Russian).

[33] E.L. Shishkina, Integral representation of the kernel of the operator approximating the inverse operator for the hyperbolic Riesz B-potential, *Bulletin of Tambov University. Series: Natural and Technical Sciences*, **21**, No 2 (2016), 448-454 (in Russian).

[34] B.M. Levitan, Expansion in Fourier series and integrals with Bessel functions, *Uspekhi Mat. Nauk*, **6**, No 2 (42) (1951), 102-143 (in Russian).

[35] S.M. Sitnik, Buschman-Erdélyi transmutations: classification, main properties and applications, *Belgorod State University Scientific bulletin. Mathematics, Physics*, **11**, No 208:39 (2015), 60-76.

[36] S.M. Sitnik, On solution to the problem of unitary generalization to the Sonine-Poisson transmutations, *Belgorod State University Scientific bulletin. Mathematics, Physics*, **5**, No 76:18 (2010), 135-153.

[37] S.M. Sitnik, Buschman-Erdélyi transmutations, classification and applications, In: *Analytic Methods Of Analysis and Differential Equations: AMADE 2012* (Edited by M.V. Dubatovskaya, S.V. Rogosin), Cambridge Scientific Publishers, Cottenham, Cambridge (2013), 171-201.

[38] S.M. Sitnik, A survey of Buschman–Erdélyi transmutations, *Chelyabinskiy Fiziko-Matematicheskiy Zhurnal*, **1**, No 4 (2016), 63-93.

[39] Ya.I. Zhitomirskii, Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type, *Mat. Sb. (N. S.)*, **36**, No 78:2 (1955), 299-310.

[40] I.A. Kipriyanov, *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997.

[41] A.B. Muravnik, Functional differential parabolic equations: integral transformations and qualitative properties of solutions of the Cauchy problem, *Journal of Mathematical Sciences*, **216**, No 3 (2016), 345-496.

[42] I.A. Kipriyanov, L.A. Ivanov, Riezs potentials on the Lorentz spaces, *Mat. Sb.*, **130**, No 172:4(8) (1986), 465-474.

[43] S.G. Samko, S.M. Umarkhadzhiev, Description of a space of Riesz potentials in terms of higher derivatives, *Soviet Math. (Izv. VUZ)*, **24**, No 11 (1980), 95-98.

[44] S.M. Umarkhadzhiev, Boundedness of the Riesz potential operator in weighted grand Lebesgue spaces, *Vladikavkaz. Mat. Zh.*, **16**, No 2 (2014), 62-68.

[45] A.N. Karapetyants, V.A. Nogin, Estimates for twisted convolution operators with singularities of kernels on a sphere and at the origin, *Differ. Uravn.*, **42**, No 5 (2006), 674-683.

[46] A.N. Karapetyants, V.A. Nogin, Inversion of potential-type operators with kernels possessing singularities on the sphere, *Journal of Contemporary Math. Anal.*, **34**, No 1 (1999), 55-69.

[47] S.N. Askhabov, Nonlinear integral equations with kernels of potential type on a segment, In: *Proc. of the Seventh International Conference on Differential and Functional-Differential Equations (Moscow, August 2229, 2014). Part 3, CMFD, 60, PFUR, M.* (2016), 5-22.

[48] I.A. Kipriyanov and M.I. Klyuchantsev, On singular integrals generated by generalised translation, II. *Sibirskij Matematiceskij Zurnal*, **11**, No 5 (1970), 1060-1083.

[49] L.N. Lyakhov, On one class of hypersingular integrals, *Dokl. Acad. Nauk USSR*, **315**, No 2 (1990), 291296.

[50] L.N. Lyakhov, Description of Riesz B-potential spaces $U_\alpha^\gamma(L_p^\gamma)$ using B-derevative of order $2[\alpha/2]$, *Dokl. Acad. Nauk USSR*, **341**, No 2 (1995), 161-165.