International Journal of Applied Mathematics

Volume 30 No. 6 2017, 547-568
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i6.7

NON-SMOOTH DECOMPOSITION AND THE MARCINKIEWICZ INTEGRAL

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Abstract: We develop a theory of non-smooth decomposition in homogeneous Triebel-Lizorkin spaces. As a byproduct, we can recover the decomposition results for Hardy spaces as a special case. The result extends what Frazier and Jawerth obtained in 1990. The result by Frazier and Jawerth covers only the limited range of the parameters but the result in this paper is valid for all admissible parameters for Triebel-Lizorkin spaces. As an application of the main results, we prove that the Marcinkiewicz operator is bounded. What is new in this paper is to reconstruct sequence spaces other than classical ℓ^p spaces.

AMS Subject Classification: 41A17, 42B35
Key Words: atomic decomposition, Triebel-Lizorkin spaces, Marcinkiewicz integral

1. Introduction

It is well known that the Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbf{R}^n)$ for $0 < p \leq 1$, $0 < p \leq q < \infty$ and $s \in \mathbf{R}$ admit the non-smooth atomic decomposition (see [2, Theorem 7.4], [6]). The aim in this paper is to remove this restriction and to study the non-smooth decomposition of $\dot{F}^s_{p,q}(\mathbf{R}^n)$ for $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$.

Received: November 10, 2017

Definition 1. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbf{R}$. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ satisfy $\chi_{B(4)\setminus B(2)} \le \varphi \le \chi_{B(8)\setminus B(1)}$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$ for which the quantity

$$\|f\|_{\dot{F}^s_{p,q}} \equiv \|\{2^{js}\varphi_j(D)f\}_{j\in\mathbf{Z}}\|_{L^p(l^q)}$$

is finite, where $\varphi_j(x) \equiv \varphi(2^{-j}x)$, $\mathcal{P}(\mathbf{R}^n)$ denotes the set of all polynomials on $\mathbf{R}^n$,

$$\psi(D)f(x) \equiv \mathcal{F}^{-1}\psi * f(x) \quad (x \in \mathbf{R}^n)$$

for $\psi \in \mathcal{S}(\mathbf{R}^n)$ and $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\|\{f_j\}_{j\in\mathbf{Z}}\|_{L^p(l^q)}$ stands for the vector-norm of a sequence $\{f_j\}_{j=-\infty}^{\infty}$ of mesurable functions:

$$\|\{f_j\}_{j\in\mathbf{Z}}\|_{L^p(l^q)} \equiv \left(\int_{\mathbf{R}^n}\left(\sum_{j=-\infty}^{\infty}|f_j(x)|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}, \quad 0 < p, q \le \infty. \tag{1.1}$$

The space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ realizes many function spaces: Indeed,

$$\begin{cases} \dot{F}^0_{p,2}(\mathbf{R}^n) = L^p(\mathbf{R}^n) & (1 < p < \infty) \\ \dot{F}^0_{p,2}(\mathbf{R}^n) = H^p(\mathbf{R}^n) & (0 < p \le 1) \end{cases}$$

with equivalence of quasi-norms, where $H^p(\mathbf{R}^n)$ stands for the Hardy space. See [3, Theorem 6.1.2] for the first equivalence and [4, Theorem 2.2.9] for the second equivalence. See [10] for more details on the Triebel-Lizorkin type spaces. Thus, our result will cover the ones for Hardy spaces as well as Lebesgue spaces.

To handle $\dot{F}^s_{p,q}(\mathbf{R}^n)$, it may be convenient to work on the corresponding sequence space $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$: it is simpler to handle sequences than to handle distributions.

Definition 2. For $\nu \in \mathbf{Z}$ and $m = (m_1, m_2, \ldots, m_n) \in \mathbf{Z}^n$, we define

$$Q_{\nu,m} \equiv \prod_{j=1}^{n}\left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right).$$

Denote by $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ the set of such cubes. The elements in $\mathcal{D}(\mathbf{R}^n)$ are called dyadic cubes.

We adopt the definition by Grafakos, [4, Definition 2.3.5].

Definition 3. Let $0 < q \le \infty$ and $s \in \mathbf{R}$. We consider the set of sequences $\{r_Q\}_{Q\in\mathcal{D}} \subset \mathbf{C}$ such that the function

$$g_q^s(\{r_Q\}_{Q\in\mathcal{D}};x) \equiv \left(\sum_{Q\in\mathcal{D}} (|Q|^{-\frac{s}{n}}|r_Q|\chi_Q(x))^q\right)^{\frac{1}{q}} \quad (x \in \mathbf{R}^n)$$

is in $L^p(\mathbf{R}^n)$. Let $0 < p < \infty$. For such sequences $r = \{r_Q\}_{Q\in\mathcal{D}}$ we set

$$\|r\|_{\dot{\mathbf{f}}^s_{p,q}} \equiv \|g_q^s(r)\|_{L^p}.$$

A sequence $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ is said to belong to $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ if $\|\lambda\|_{\dot{\mathbf{f}}^s_{p,q}} < \infty$. Sometimes, we identity $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbf{Z},m\in\mathbf{Z}^n}$ with $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ via $\lambda_{\nu,m} = \lambda_Q$ when $Q = Q_{\nu,m}$.

To obtain our result, we follow the book [4] by Grafakos.

Definition 4. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbf{R}$. A sequence $r = \{r_Q\}_{Q\in\mathcal{D}}$ is called an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_0 if there exists a dyadic cube Q_0 such that

$$g_q^s(\{r_Q\}_{Q\in\mathcal{D}};\cdot) \equiv \left(\sum_{Q\in\mathcal{D}} (|Q|^{-\frac{s}{n}}|r_Q|\chi_Q)^q\right)^{\frac{1}{q}} \le \chi_{Q_0}. \tag{1.2}$$

Our first theorem is as follows:

Theorem 5. *Suppose that we are given parameters p, q, s, u satisfying*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbf{R}, \quad 0 < u \le \min(1, q).$$

1. *For any $t \in \dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$, there exists a decomposition*

$$t = \sum_{j=1}^{\infty} \lambda_j r_j, \tag{1.3}$$

where each r_j is an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}$ with cube Q_j and $\{\lambda_j\}_{j=1}^{\infty}$ satisfies

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j}\right)^{\frac{1}{u}} \right\|_{L^p} \le C\|t\|_{\dot{\mathbf{f}}^s_{p,q}}. \tag{1.4}$$

2. *If a sequence* $\{Q_j\}_{j=1}^\infty$ *of cubes and a sequence* $\{\lambda_j\}_{j=1}^\infty$ *of complex numbers satisfy*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} < \infty, \tag{1.5}$$

then for any ∞*-atoms* r_j *for* $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ *with cube* Q_j*, the series* t *given by* (1.3) *belongs to* $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$.

In Theorem 5 the case of $s \in \mathbf{R}$, $0 < p = u \leq 1$ and $p \leq q \leq \infty$ is proved in [2, Theorem 7.2]. In this case there is no condition on the position of the cubes since

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} = \left(\sum_{j=1}^{\infty} |\lambda_j|^p |Q_j| \right)^{\frac{1}{p}}. \tag{1.6}$$

We can refine our Theorem 5.

Definition 6. Let $0 < p < \infty$, $0 < q \leq \infty$, $v \in (0, \infty)$ and $s \in \mathbf{R}$. One says that a sequence $r = \{r_Q\}_{Q \in \mathcal{D}}$ is called a v-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_0 if there exists a dyadic cube Q_0 such that

$$\operatorname{supp}(g^s_q(\{r_Q\}_{Q \in \mathcal{D}}; \cdot)) \subset Q_0, \quad \|g^s_q(\{r_Q\}_{Q \in \mathcal{D}}; \cdot)\|_{L^v} \leq |Q_0|^{\frac{1}{v}}.$$

We can refine the latter half of Theorem 5 as follows:

Theorem 7. *In addition to the assumption in Theorem 5, let* $v \in (\max(1, p), \infty)$. *If a sequence* $\{Q_j\}_{j=1}^\infty$ *of cubes and a sequence* $\{\lambda_j\}_{j=1}^\infty$ *of complex numbers satisfy* (1.5)*, then for any* v*-atoms* r_j *with cube* Q_j*, the series* t *given by* (1.3) *belongs to* $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$.

The above results cover the ones in [2, Section 7]. What is new about this paper is the case where $p > \min(q, 1)$. The case when $p > 1$ and $q = 2$ is especially interesting because this yields the decomposition for $L^p(\mathbf{R}^n) = \dot{F}^0_{p,2}(\mathbf{R}^n)$.

We now transform the results to the one of the sequences.

Definition 8 (Atoms for Triebel-Lizorkin spaces)**.** Let $0 < p < \infty, 0 < q \leq \infty$, $s \in \mathbf{R}$, and let $\nu \in \mathbf{Z}$ and $m \in \mathbf{Z}^n$. Suppose that the integers $K, L \in \mathbf{Z}$

satisfy $K \geq 0$ and $L \geq -1$. A function $a \in C^K(\mathbf{R}^n)$ is said to be a *smooth (K, L)-atom centered at $Q_{0,m}$* for $\dot{f}^s_{p,q}(\mathbf{R}^n)$, if it is supported on $3\,Q_{0,m}$ and if it satisfies the differential inequality and the moment condition:

$$\|\partial^\alpha a\|_{L^\infty} \leq 2^{\nu|\alpha|}, \quad |\alpha| \leq K, \tag{1.7}$$

$$\int_{\mathbf{R}^n} x^\beta a(x)\, dx = 0, \quad |\beta| \leq L. \tag{1.8}$$

The case $L = -1$ is excluded in (1.8).

To state our main result, we present the following definition:

Definition 9. Let $0 < p < \infty, 0 < q \leq \infty, s \in \mathbf{R}$. We say that A is a non-smooth atom for $\dot{F}^s_{p,q}(\mathbf{R}^n)$ with cube $\tilde{Q}$ if there exists a cube $\tilde{Q}$ such that $A = \sum_{Q \subset \tilde{Q}} r_Q a_Q$ where $r = \{r_Q\}_{Q \in \mathcal{D}}$ is an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ and each a_Q is a smooth $(K, L)-$atom centered at Q.

The following theorem, which is a conclusion of this note, extends [4, Corollary 2.3.9]. Define $\sigma_p \equiv n\left(\frac{1}{\min(1,p)} - 1\right)$ and $\sigma_{p,q} \equiv \max(\sigma_p, \sigma_q)$.

Theorem 10. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbf{R}$, $0 < u \leq \min(1, q)$, and let*

$$L \geq \max(-1, [\sigma_{p,q} - s]),$$

where $[\cdot]$ denotes the Gauss sign. Then we have the following:

1. *Let $f \in \dot{F}^s_{p,q}(\mathbf{R}^n)$. Then we can write*

$$f = \sum_{j=1}^{\infty} \lambda_j A_j$$

in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$, where $\{A_j\}_{j=1}^{\infty}$ is a sequence of non-smooth atoms and $\{\lambda_j\}_{j=1}^{\infty}$ and $\{Q_j\}_{j=1}^{\infty}$ satisfy $\mathrm{supp} A_j \subset 3Q_j$ *and*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} \leq C\, \|f\|_{\dot{F}^s_{p,q}}. \tag{1.9}$$

2. *Suppose that each A_j is a non-smooth atom with cube Q_j and the complex sequence $\{\lambda_j\}_{j=1}^{\infty}$ satisfies*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} < \infty.$$

Then the sum

$$f \equiv \sum_{j=1}^{\infty} \lambda_j A_j,$$

converges in the topology of $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$ and satisfies

$$\|f\|_{\dot{F}^s_{p,q}} \leq C \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p}.$$

In Theorem 10 the case of $s \in \mathbf{R}$, $0 < p = u \leq 1$ and $p \leq q \leq \infty$ is [2, Theorem 7.4 (ii)]. To conclude this section, we recall the following definition to compare our atoms with the ones in Hardy spaces.

Definition 11 (Atoms in Hardy spaces)**.** Let $0 < p \leq 1 < v \leq \infty$. Fix $L \geq L_0 \equiv [\sigma_p]$. A (non-smooth) (p, v)-*atom centered at a cube* Q is an $L^v(\mathbf{R}^n)$-function A which is supported on Q and satisfies the moment condition of order L, that is,

$$\int_{\mathbf{R}^n} x^\alpha A(x)\, dx = 0$$

for all multi-indexes α with $|\alpha| \leq L$ and $\|A\|_{L^v} \leq |Q|^{\frac{1}{v}}$.

Let $s = 0$, $0 < p < \infty$, $q = 2$ and $1 < v < \infty$. In Theorem 10, the function A_j is a (p, v)-atom modulo a multiplicative constant since

$$\|A_j\|_{L^v} \sim \|A_j\|_{\dot{F}^0_{v,2}} \sim \|g_2^0(r_j)\|_{L^v} \leq |Q_j|^{\frac{1}{v}}.$$

The second equivalence follows from the Littlewood–Paley theory, which indicates $\dot{F}^0_{v2}(\mathbf{R}^n) \sim L^v(\mathbf{R}^n)$.

We organize the remaining part of this paper as follows: Sections 2-4 are devoted to the proof of the above theorems. As an application, we prove the boundedness of the Marcinkiewicz operators. Basically, the key idea is to investigate closely the behavior of these operators for non-smooth atoms. In [5, Theorem 2.1], Liu and Yang proposed a criterion for the case of $0 < p \le 1$ and $q \ge p$. Here, we will remove the restiction $0 < p \le 1$. Our results will be valid for $1 \le p < \infty$ and $1 < q < \infty$ as well as for some extra parameters. Unfortunately, we can not present a general criterion for the operators to be bounded from homogeneous Triebel–Lizokin spaces to Banach spaces. This disadvantage comes from the fact that we need to take care of the position of the support of the atoms.

2. Proof of Theorem 5

We recall the following facts in [4, p. 115–116]. Let $t = \{t_Q\}_{Q\in\mathcal{D}}$ be a sequence, and let $s \in \mathbb{R}$ and $0 < q \le \infty$.

Lemma 12. *Let $R \in \mathcal{D}$. Define*

$$g^s_{q,R}(\{t_Q\}_{Q\in\mathcal{D}}; x) \equiv \left(\sum_{Q\in\mathcal{D},\, R\subset Q} (|Q|^{-\frac{s}{n}} |t_Q| \chi_Q(x))^q \right)^{\frac{1}{q}} \qquad (x \in \mathbf{R}^n)$$

and

$$\mathcal{D}_\nu \equiv \{Q \in \mathcal{D}; l(Q) = 2^{-\nu}\}.$$

1. *If $R_1 \subset R_2$ and $x \in \mathbf{R}^n$, then $g^s_{q,R_2}(t;x) \le g^s_{q,R_1}(t;x)$.*
2. *For any $x \in \mathbf{R}^n$,*
$$\lim_{\nu\to\infty} \sum_{Q\in\mathcal{D}_\nu} \chi_Q(x) g^s_{q,Q}(t;x) = 0. \tag{2.1}$$
3. *For any $x \in \mathbf{R}^n$,*
$$\lim_{\nu\to-\infty} \sum_{Q\in\mathcal{D}_\nu} \chi_Q(x) g^s_{q,Q}(t;x) = g^s_q(t;x). \tag{2.2}$$

Lemma 13. *For $k \in \mathbf{Z}$, we set*

$$\mathcal{A}_k \equiv \{R \in \mathcal{D} : g^s_{q,R}(t;x) > 2^k, \text{ for } x \in R\}.$$

1. [4, p. 116] *If $Q \in \mathcal{D}$ does not belong to any $\mathcal{A}_k, k \in \mathbf{Z}$, then $t_Q = 0$.*
2. [4, p. 115] *For each $k \in \mathbf{Z}$, $\mathcal{A}_{k+1} \subset \mathcal{A}_k$.*

3. [4, p. 115 (2.3.16)]

$$\{x \in \mathbf{R}^n \,:\, g_q^s(t;x) > 2^k\} = \bigcup_{R \in \mathcal{A}_k} R. \tag{2.3}$$

4. [4, p. 115 (2.3.17)] *For all* $k \in \mathbf{Z}$,

$$\left(\sum_{Q \in \mathcal{D} \setminus \mathcal{A}_k} (|Q|^{-\frac{s}{n}} |t_Q| \chi_Q)^q \right)^{\frac{1}{q}} \le 2^k. \tag{2.4}$$

Lemma 14. *Let* $\mathcal{A}_k$ *be as in Lemma 13.*
Let $t = \{t_Q\}_{Q \in \mathcal{D}}$ *be a sequence indexed by* $Q \in \mathcal{D}$. *Assume*

$$g_q^s(t;x) < \infty$$

for a.e. $x \in \mathbf{R}^n$. *We set*

$$\mathcal{B}_k \equiv \{J \in \mathcal{D} \,:\, J \text{ is a maximal dyadic cube in } \mathcal{A}_k \setminus \mathcal{A}_{k+1}\}.$$

For $J \in \mathcal{B}_k$, *we define*

$$v(k,J) \equiv \{v(k,J)_Q\}_{Q \in \mathcal{D}} \equiv \{t_Q \chi_{\mathcal{A}_k \setminus \mathcal{A}_{k+1}}(Q) \chi_{\{S \in \mathcal{D} \,:\, S \subset J\}}(Q)\}_{Q \in \mathcal{D}},$$

$$r(k,J) \equiv 2^{-k-1} v(k,J).$$

1. [4, p. 116 (2.3.18)] and [4, p. 116 (2.3.21)] *We have*

$$t = \sum_{k \in \mathbf{Z}} \sum_{J \in \mathcal{B}_k} v(k,J) = \sum_{k \in \mathbf{Z}} \sum_{J \in \mathcal{B}_k} 2^{k+1} r(k,J). \tag{2.5}$$

2. [4, p. 116 (2.3.19)] *For all* $k \in \mathbf{Z}$ *and* $J \in \mathcal{B}_k$, $g_q^s(v(k,J)) \le 2^{k+1}$.

Remark that in the definition of $t(k,J)$,

$$t(k,J)_Q = v_Q \chi_{\{S \in \mathcal{D} \,:\, S \subset J\}}(Q)$$

if $Q \in \mathcal{A}_k \setminus \mathcal{A}_{k+1}$ is a cube contained in J otherwise $t(k,J)_Q = 0$.

Now we prove Theorem 5. Let $t \in \dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ be given. By (2.5), we can write

$$t = \sum_{k \in \mathbf{Z}} \sum_{J \in \mathcal{B}_k} 2^{k+1} r(k,J).$$

Let $\iota \equiv (\iota_1, \iota_2) : \mathbf{N} \to \{(k, J) : k \in \mathbf{Z}, J \in \mathcal{B}_k\}$ be a bijection. By letting $\lambda_j \equiv 2^{\iota_1(j)+1}$ and $r_j \equiv r(\iota_1(j), \iota_2(j))$, we can write

$$t = \sum_{j=1}^{\infty} \lambda_j r_j.$$

Therefore we get the desired decomposition (1.3). We will check that r_j is an ∞-atom. Letting

$$k = \iota_1(j), \quad J = Q_j = \iota_2(j),$$

we have

$$g_q^s(r_j) = g_q^s(r(k, J)) = g_q^s(2^{-k-1} t(k, J)) = 2^{-k-1} g_q^s(t(k, J)).$$

Now, suppose that $t(k, J) = \{v_Q\}_{Q \in \mathcal{D}}$. If $v_Q \neq 0$, then

$$g_q^s(t(k, J)) \leq 2^{k+1}.$$

Furthermore if $t(k, J) = 0$, then $g_q^s(t(k, J)) = 0$. Therefore, since $g_r^s(r_j) \leq \chi_J$ holds, it follows that r_j is an ∞-atom with cube J.

Recall that any $J \in \mathcal{B}_k$ is a cube in $\mathcal{A}_k$ and that $\mathcal{B}_k$ is disjoint family. So, we have

$$\sum_{J \in \mathcal{B}_k} \chi_J \leq \chi_{\cup_{Q \in \mathcal{A}_k} Q} = \chi_{\{g_q^s(t) > 2^k\}}. \tag{2.6}$$

Using (2.6), we calculate

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} &= \left\| \left(\sum_{k \in \mathbf{Z}} \sum_{J \in \mathcal{B}_k} (2^{k+1} \chi_J)^u \right)^{\frac{1}{u}} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{k \in \mathbf{Z}} 2^{ku} \chi_{\{g_q^s(t) > 2^k\}} \right)^{\frac{1}{u}} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{k=-\infty}^{[1+\log_2 g_q^s(t)]} 2^{ku} \right)^{\frac{1}{u}} \right\|_{L^p}. \end{aligned}$$

If we calculate the geometric series, then we obtain

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} \leq C \left\| \left(\frac{2^{[1+\log_2 g_q^s(t)]u}}{1 - 2^{-u}} \right)^{\frac{1}{u}} \right\|_{L^p}$$

$$\le C\left\|\left(\frac{2^{(1+\log_2 g_q^s(t))u}}{1-2^{-u}}\right)^{\frac{1}{u}}\right\|_{L^p}$$
$$= C\left\|\left(\frac{2^u}{1-2^{-u}}g_q^s(t)^u\right)^{\frac{1}{u}}\right\|_{L^p}$$
$$= C\left\|g_q^s(t)\right\|_{L^p} = C\|t\|_{\dot{\mathbf{f}}^s_{p,q}},$$

keeping in mind that $u > 0$.

Conversely suppose we are given a sequence $r_j = \{r_{j,Q}\}_{Q\in\mathcal{D}}$. Denote by Q_j the cube for r_j in the definition of atoms. Then setting

$$t = \sum_{j=1}^{\infty}\lambda_j r_j,$$

we have

$$\|t\|_{\dot{\mathbf{f}}^s_{p,q}} = \left\|g_q^s\left(\sum_{j=1}^{\infty}\lambda_j r_j;\cdot\right)\right\|_{L^p}$$
$$= \left\{\left\|\left(g_q^s\left(\sum_{j=1}^{\infty}\lambda_j r_j;\cdot\right)\right)^u\right\|_{L^{p/u}}\right\}^{\frac{1}{u}}$$
$$\le \left\{\left\|\left(\sum_{j=1}^{\infty}|\lambda_j| g_q^s(r_j;\cdot)\right)^u\right\|_{L^{p/u}}\right\}^{\frac{1}{u}}$$
$$\le \left\{\left\|\sum_{j=1}^{\infty}|\lambda_j|^u g_q^s(r_j;\cdot)^u\right\|_{L^{p/u}}\right\}^{\frac{1}{u}}.$$

Here we have used $u \le q$ to obtain the penultimate inequality and $u \le 1$ to obtain the last inequality. If we use $g_r^s(r_j;\cdot) \le \chi_{Q_j}$, then we obtain

$$\|t\|_{\dot{\mathbf{f}}^s_{p,q}} \le \left\{\left\|\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right\|_{L^{p/u}}\right\}^{\frac{1}{u}} = \left\|\left(\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{L^p} < \infty. \tag{2.7}$$

Thus, the proof is complete. □

We make a brief remark of the method of the proof. The proof of Theorem 5 is essentially made up of two tools. The first tool is a method to decompose sequences and the second tool serves to describe the condition of coefficients. The first tool consists of the $\frac{1}{8}$ median and the stopping time argument. In [2, Section 6], Frazier and Jarwerth used them together with L^0, the set of all measurable functions f for which $\{f \neq 0\}$ has finite measure. This method is refined in §6.6.4 by Grafakos [4]. Since our proof heavily hinges on §6.6.4 in [4], we essentially used the technique of the paper in [2] and the textbook [4]. What is different from these sources is the second tool. As is described in (7.4) of [2] and (7.7) of [2], we have

$$\|\lambda_1 + \lambda_2\|_{\dot{\mathbf{f}}^s_{p,q}}^p \leq \|\lambda_1\|_{\dot{\mathbf{f}}^s_{p,q}}^p + \|\lambda_2\|_{\dot{\mathbf{f}}^s_{p,q}}^p \quad (\lambda_1, \lambda_2 \in \dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)) \tag{2.8}$$

and

$$\|f_1 + f_2\|_{\dot{F}^s_{p,q}}^p \leq \|f_1\|_{\dot{F}^s_{p,q}}^p + \|f_2\|_{\dot{F}^s_{p,q}}^p \quad (f_1, f_2 \in \dot{F}^s_{p,q}(\mathbf{R}^n)) \tag{2.9}$$

for $0 < p \leq 1$, $0 < p \leq q \leq \infty$ and $s \in \mathbf{R}$. Frazier and Jawerth used (2.8) and (2.9) to decompose the sum into small units. One of the important facts on the decomposition of Frazier and Jawerth is that the condition on the position of the cubes Q_j does not appear as is hinted in the right-hand side of (1.6). Since (2.8) and (2.9) are no longer available for general case, we need a trick. To accomodate all admissible parameters, we took into account the position of the cubes Q_j.

3. Proof of Theorem 7

We use the following lemma:

Lemma 15. *Let $0 < p < \infty, \max(1, q) < p < \infty$. Then for any sequence $\{A_j\}_{j=1}^\infty$ of non-negative measurable functions, each of which is supported on a cube Q_j, and any sequence $\{\lambda_j\}_{j=1}^\infty$ of non-negative real numbers, we have*

$$\left\| \sum_{j=1}^\infty \lambda_j A_j \right\|_{L^p} \leq C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \left(\frac{1}{|Q_j|} \int_{Q_j} A_j(y)^q \, dy \right)^{\frac{1}{q}} \right\|_{L^p}.$$

Proof. Lemma 15 rephrases [7, Lemma 2.5] with $0 < p \leq 1$ and [8, Theorem 1.3.1] with $1 < p < \infty$. □

The proof of Theorem 7 is now easy. Just reexamine the proof of Theorem 5. Then we notice that everything remains unchanged up to (2.7). Instead of using $g_r^s(r_j; \cdot) \leq \chi_{Q_j}$ we use Lemma 15 to have (2.7).

4. Proof of Theorem 10

We use the following decomposition results for $\dot{F}^s_{p,q}(\mathbf{R}^n)$: We invoke the following result in [11, Theorem 13.8].

Theorem 16. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$, and let K be an integer satisfying*

$$K \geq [1+s]_+ \equiv \max(0, [1+s]).$$

Furthermore, suppose that $L \in \mathbf{Z}$ satisfies

$$L \geq \max(-1, [\sigma_{p,q} - s]). \tag{4.1}$$

1. *Let $\kappa = \{\kappa_{\nu,m}\}_{\nu \in \mathbf{Z},\, m \in \mathbf{Z}^n} \in \dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ and each $a_{\nu,m}$ is a smooth L-atom centered at $Q_{\nu,m}$ for each ν, m. Then*

$$f \equiv \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbf{Z}^n} \kappa_{\nu,m} a_{\nu,m}$$

converges in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$ and

$$\|f\|_{\dot{F}^s_{p,q}} \leq C \|\kappa\|_{\dot{\mathbf{f}}^s_{p,q}}. \tag{4.2}$$

2. *Any $f \in \dot{F}^s_{p,q}(\mathbf{R}^n)$ admits a decomposition:*

$$f = \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbf{Z}^n} \kappa_{\nu,m} a_{\nu,m}. \tag{4.3}$$

Here, the convergence takes place in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$, each $a_{\nu,m}$ is a smooth L-atom centered at $Q_{\nu,m}$ and the coefficient $\kappa = \{\kappa_{\nu,m}\}_{\nu \in \mathbf{N}_0,\, m \in \mathbf{Z}^n}$ satisfies

$$\|\kappa\|_{\dot{\mathbf{f}}^s_{p,q}} \leq C \|f\|_{\dot{F}^s_{p,q}}. \tag{4.4}$$

We now turn to the proof of Theorem 10. First we prove the latter half of Theorem 10.

Let

$$f \in \mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$$

be such that

$$f = \sum_{j=1}^{\infty} \lambda_j A_j.$$

Let $A_j = \sum_{\mu \in \mathbf{Z}} \sum_{Q \in \mathcal{D}_\mu} r_{j,Q} a_Q$ as in the definition of non-smooth atoms. We set

$$f^J \equiv \sum_{j=1}^{J} \lambda_j A_j = \sum_{j=1}^{J} \lambda_j \left(\sum_{\mu \in \mathbf{Z}} \sum_{Q \in \mathcal{D}_\mu} r_{j,Q} a_Q \right)$$
$$= \sum_{\mu \in \mathbf{Z}} \sum_{Q \in \mathcal{D}_\mu} \left(\sum_{j=1}^{J} \lambda_j r_{j,Q} \right) a_Q.$$

We set

$$\kappa^J \equiv \left\{ \sum_{j=1}^{J} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}}.$$

Let $0 < u \le \min(1, q)$. Then we have

$$\|\kappa^J\|_{\dot{\mathbf{f}}^s_{p,q}} = \left\| g^s_q \left(\left\{ \sum_{j=1}^{J} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}} \right) \right\|_{L^p}$$
$$= \left(\left\| g^s_q \left(\left\{ \sum_{j=1}^{J} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}} \right)^u \right\|_{L^{p/u}} \right)^{\frac{1}{u}}$$
$$\le \left(\left\| \sum_{j=1}^{J} g^s_q \left(\{\lambda_j r_{j,Q}\}_{Q \in \mathcal{D}} \right)^u \right\|_{L^{p/u}} \right)^{\frac{1}{u}}$$

$$= \left(\left\| \sum_{j=1}^{J} |\lambda_j|^u g^s_q \left(\{r_{j,Q}\}_{Q \in \mathcal{D}} \right)^u \right\|_{L^{p/u}} \right)^{\frac{1}{u}}$$
$$\le \left(\left\| \sum_{j=1}^{J} |\lambda_j|^u \chi_{Q_j} \right\|_{L^{p/u}} \right)^{\frac{1}{u}}.$$

Thus, by Theorem 16, we have

$$\|f^J\|_{\dot{F}^s_{p,q}} \le C\|\kappa^J\|_{\dot{\mathbf{f}}^s_{p,q}} \le C\left(\left\|\sum_{j=1}^{J}|\lambda_j|^u\chi_{Q_j}\right\|_{L^{p/u}}\right)^{\frac{1}{u}}.$$

By the Fatou property of $\dot{F}^s_{p,q}(\mathbf{R}^n)$ or by the classical Fatou lemma, we conclude

$$\|f\|_{\dot{F}^s_{p,q}} \le C\left(\left\|\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right\|_{L^{p/u}}\right)^{\frac{1}{u}}.$$

Also, by letting $J' < J$,

$$\lim_{J\to\infty, J'\to\infty}\|f^J - f^{J'}\|_{\dot{F}^s_{p,q}} \le C\lim_{J\to\infty, J'\to\infty}\left\|\left(\sum_{j=J'}^{J}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{L^p} = 0,$$

$$\left(\sum_{j=J'}^{J}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}} \in L^p(\mathbf{R}^n).$$

Thus $\{f^J\}_{J=1}^\infty$ is a Cauchy sequence in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$. Therefore $\lim_{J\to\infty} f^J = f \in \mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$.

Next we prove the first half of Theorem 10. Let $f \in \dot{F}^s_{p,q}(\mathbf{R}^n)$. Decompose f according to Theorem 16, so that (4.3) and (4.4) hold. If $Q = Q_{\nu,m}$, we write $\lambda_Q \equiv \kappa_{\nu,m}$ and $a_Q \equiv a_{\nu,m}$. We let

$$\mathcal{B} \equiv \{(k, J) \,:\, k \in \mathbf{Z}, J \in \mathcal{B}_k\}.$$

Let

$$N : j \in \mathbf{N} \mapsto (k_j, J_j) \in \mathcal{B}$$

be an enumeration. Let $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$. Since

$$g^s_q(\lambda; x) < \infty$$

for almost all $x \in \mathbf{R}^n$, we have a decomposition:

$$\lambda = \sum_{k\in\mathbf{Z}}\sum_{J\in\mathcal{B}_k} 2^{k+1} r(k, J) = \sum_{j=1}^{\infty} 2^{k_j+1} r(k_j, J_j),$$

where each $r(k, J) = \{r(k, J)_Q\}_{Q\in\mathcal{D}}$ is an ∞-atom supported on J. According to the proof of Theorem 5,

$$\left\|\left(\sum_{j=1}^{\infty} 2^{(k_j+1)u}\chi_{J_j}\right)^{\frac{1}{u}}\right\|_{L^p} = \left\|\left(\sum_{k\in\mathbf{Z}}\sum_{J\in\mathcal{B}_k} 2^{(k+1)u}\chi_J\right)^{\frac{1}{u}}\right\|_{L^p} \le C\|\lambda\|_{\dot{\mathbf{f}}^s_{p,q}}. \quad (4.5)$$

If we combine (4.4) and (4.5), then we obtain

$$\|f\|_{\dot{F}^s_{p,q}} \geq C \left\| \left(\sum_{j=1}^{\infty} 2^{(k_j+1)u} \chi_{J_j} \right)^{\frac{1}{u}} \right\|_{L^p}. \tag{4.6}$$

Letting

$$A_{Q_j} = 2^{k_j} r(k_j, J_j) a_Q,$$

we claim that

$$\sum_{j=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} A_{Q,j} = \lim_{T \to \infty} \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=1}^{T} A_{Q,j} \tag{4.7}$$

in $\dot{F}^s_{p,q}(\mathbf{R}^n)$. In fact, for $T \in \mathbf{N}$, we have

$$\left\| \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=1}^{\infty} A_{Q,j} - \sum_{j=1}^{T} \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} A_{Q,j} \right\|_{\dot{F}^s_{p,q}}$$

$$= \left\| \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=1}^{\infty} A_{Q,j} - \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=1}^{T} A_{Q,j} \right\|_{\dot{F}^s_{p,q}}$$

$$= \left\| \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=T+1}^{\infty} A_{Q,j} \right\|_{\dot{F}^s_{p,q}} \leq C \left\| \sum_{j=T+1}^{\infty} 2^{k_j+1} r(k_j, J_j) \right\|_{\dot{\mathbf{f}}^s_{p,q}}$$

thanks to Theorem 16. We define $\lambda_j \equiv 2^{k_j+1}$. Since

$$g^s_q(r(k_j, J_j)) \leq \chi_{J_j},$$

we have

$$\left\| \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \sum_{j=1}^{\infty} A_{Q,j} - \sum_{j=1}^{T} \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} A_{Q,j} \right\|_{\dot{F}^s_{p,q}}$$

$$\leq C \left\| g^s_q \left(\sum_{j=T+1}^{\infty} 2^{k_j+1} r(k_j, J_j) \right) \right\|_{L^p}$$

$$= C \left[\left\| g^s_q \left(\sum_{j=T+1}^{\infty} 2^{k_j+1} r(k_j, J_j) \right)^u \right\|_{L^{\frac{p}{u}}} \right]^{\frac{1}{u}}$$

$$\le C\left[\left\|\sum_{j=T+1}^{\infty}2^{(k_j+1)u}g_q^s(r(k_j,J_j))^u\right\|_{L^{\frac{p}{u}}}\right]^{\frac{1}{u}}$$
$$\le C\left\|\left(\sum_{j=T+1}^{\infty}|\lambda_j|^u\chi_{J_j}\right)^{\frac{1}{u}}\right\|_{L^p}.$$

Letting $T\to\infty$, we obtain (4.7). Thus, we conclude from (4.7) that

$$\begin{aligned} f &= \sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\lambda_Q a_Q \\ &= \sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\sum_{j=1}^{\infty}2^{k_j+1}r(k_j,J_j)_Q a_Q \\ &= \sum_{j=1}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}r(k_j,J_j)_Q a_Q. \end{aligned}$$

If we denote

$$A_j\equiv\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}r(k_j,J_j)_Q a_Q \quad (j\in\mathbf{N}),$$

then we have

$$f=\sum_{j=1}^{\infty}\lambda_j A_j.$$

Thus, we obtain the desired decomposition.

5. Applications to the boundedness of the Marcinkiewicz operators

Let $0<\rho<n$ and $1<q<\infty$. The Marcinkiewicz operator is defined by

$$\mu_{\Omega,\rho,q}f(x)\equiv\left(\int_0^{\infty}\left|\frac{1}{t^\rho}\int_{B(t)}f(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\right|^q\frac{dt}{t}\right)^{\frac{1}{q}}, \tag{5.1}$$

where we write $B(r)=\{|x|<r\}\subset\mathbf{R}^n$ for $r>0$ here and below.

We suppose

$$\int_{S^{n-1}}\Omega(\omega)\,d\sigma(\omega)=0,\quad \Omega\in C^1(S^{n-1}),$$

where $S^{n-1} = \{|x| = 1\}$. According to [9, Theorem 1], we have

$$\|\mu_{\Omega,\rho,q} f\|_{L^p} \le C \|f\|_{\dot{F}^0_{p,q}}$$

if $1 < p < \infty$. We remark that if A is a non-smooth atom supported in $3Q_0$, letting $r = \{r_Q\}_{Q \in \mathcal{Q}}$, we have

$$\|A\|_{\dot{F}^s_{\tilde{p},q}} \le C \|g^s_q(r,\cdot)\|_{L^{\tilde{p}}(\mathbf{R}^n)} \le C \|\chi_{3Q_0}\|_{L^{\tilde{p}}(\mathbf{R}^n)} \le C|3Q_0|^{\frac{1}{\tilde{p}}}. \tag{5.2}$$

Theorem 17. *The estimate* $\|\mu_{\Omega,\rho,q} f\|_{L^p} \le C\|f\|_{\dot{F}^0_{p,q}}$ *for all* $f \in \dot{F}^0_{p,q}$ *if*

$$\frac{nq}{nq+1} < p < \infty, \quad 1 < q < \infty.$$

The rest of this paper is devoted to the proof of Theorem 17. Let $f \in \dot{F}^0_{p,q}(\mathbf{R}^n)$. Let

$$f = \sum_{j=1}^{\infty} \lambda_j A_j$$

be a decomposition as in Theorem 10 (1). Then we have

$$\mu_{\Omega,\rho,q} f(x) \le \sum_{j=1}^{\infty} |\lambda_j| \mu_{\Omega,\rho,q} A_j(x)$$

$$= \sum_{j=1}^{\infty} |\lambda_j| \chi_{3nQ_j}(x) \mu_{\Omega,\rho,q} A_j(x) + \sum_{j=1}^{\infty} |\lambda_j| \chi_{\mathbf{R}^n \setminus 3nQ_j}(x) \mu_{\Omega,\rho,q} A_j(x).$$

Lemma 18. *Let* $x \in \mathbf{R}^n \setminus 3nQ_j$.

1. *Let* $0 < t < |x - c(Q_j)| - \frac{3\sqrt{n}}{2}\ell(Q_j)$. *Then*

$$\frac{1}{t^\rho} \int_{B(t)} A_j(x-y) \frac{\Omega(y/|y|)}{|y|^{n-\rho}} dy = 0.$$

2. *Let* $|x - c(Q_j)| - \frac{3\sqrt{n}}{2}\ell(Q_j) \le t \le |x - c(Q_j)| + \frac{3\sqrt{n}}{2}\ell(Q_j)$. *Then*

$$\left| \frac{1}{t^\rho} \int_{B(t)} A_j(x-y) \frac{\Omega(y/|y|)}{|y|^{n-\rho}} dy \right| \le C \frac{\ell(Q_j)^n}{|x - c(Q_j)|^n}.$$

3. *Let* $t > |x - c(Q_j)| + \frac{3\sqrt{n}}{2}\ell(Q_j)$. *Then*

$$\left| \frac{1}{t^\rho} \int_{B(t)} A_j(x-y) \frac{\Omega(y/|y|)}{|y|^{n-\rho}} dy \right| \le C \frac{\ell(Q_j)^{n+1}}{t^\rho |x - c(Q_j)|^{n-\rho+1}}.$$

Proof. 1. This is clear from the condition on the support.

2. We observe

$$\left|\int_{B(t)} A_j(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\right| \le C\int_{\mathrm{supp}(A_j(x-\cdot))}\left|A_j(x-y)\right|\frac{dy}{|y|^{n-\rho}}.$$

Note that $|y| \ge |x-c(Q_j)| - \frac{3\sqrt{n}}{2}\ell(Q_j)$. Hence,

$$\frac{1}{|y|^{n-\rho}} \le \frac{1}{(|x-c(Q_j)| - \frac{3}{2}\sqrt{n}\ell(Q_j))^{n-\rho}}.$$

Now by $x \in \mathbf{R}^n \setminus 3nQ_j$,

$$\begin{aligned}&|x-c(Q_j)| - \frac{3}{2}\sqrt{n}\ell(Q_j)\\ &= \frac{1}{2}|x-c(Q_j)| + \frac{1}{2}|x-c(Q_j)| - \frac{3}{2}\sqrt{n}\ell(Q_j) \ge \frac{1}{2}|x-c(Q_j)|.\end{aligned}$$

Hence $\dfrac{1}{|y|^{n-\rho}} \le \dfrac{2^{\rho-n}}{|x-c(Q_j)|^{n-\rho}}$. Also,

$$\frac{1}{t^\rho} \le \frac{1}{(|x-c(Q_j)| - \frac{3}{2}\sqrt{n}\ell(Q_j))^{\rho}} \le \frac{2^\rho}{|x-c(Q_j)|^\rho}.$$

Therefore,

$$\left|\frac{1}{t^\rho}\int_{B(t)} A_j(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\right| \le C\frac{\ell(Q_j)^n}{|x-c(Q_j)|^n}.$$

3. Let $x^{(j)} = x - c(Q_j)$. We use the moment condition to have:

$$\begin{aligned}&\frac{1}{t^\rho}\int_{B(t)} A_j(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\\ &= \frac{1}{t^\rho}\int_{\mathrm{supp}(A_j(x-\cdot))} A_j(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\\ &= \frac{1}{t^\rho}\int_{\mathrm{supp}(A_j(x-\cdot))} A_j(x-y)\left(\frac{\Omega(\frac{y}{|y|})}{|y|^{n-\rho}} - \frac{\Omega(\frac{x^{(j)}}{|x^{(j)}|})}{|x^{(j)}|^{n-\rho}}\right)dy.\end{aligned}$$

Note that if $A_j(x-y) \ne 0$, then $|x-y-c(Q_j)| \le \frac{3\sqrt{n}}{2}\ell(Q_j)$. Hence

$$|x-c(Q_j)| - \frac{3\sqrt{n}}{2}\ell(Q_j) \le |y| \le |x-c(Q_j)| + \frac{3\sqrt{n}}{2}\ell(Q_j).$$

Thus, we have

$$\left|\frac{\Omega(\frac{y}{|y|})}{|y|^{n-\rho}} - \frac{\Omega(\frac{x-c(Q_j)}{|x-c(Q_j)|})}{|x-c(Q_j)|^{n-\rho}}\right| \leq C\frac{\ell(Q_j)}{|x-c(Q_j)|^{n-\rho+1}}.$$

Thus, we obtain the desired result.

□

By this lemma, for $x \in \mathbf{R}^n \setminus 3nQ_j$ we have

$$\mu_{\Omega,\rho,q}A_j(x) \leq C\frac{\ell(Q_j)^{n+\frac{1}{q}}}{|x-c(Q_j)|^{n+\frac{1}{q}}} \leq C\left(M\chi_{Q_j}(x)\right)^{1+\frac{1}{nq}}.$$

Indeed, by letting $x^{(j)} = x - c(Q_j), a_j = |x^{(j)}| - \frac{3}{2}\sqrt{n}\ell(Q_j)$ and $b_j = |x^{(j)}| + \frac{3}{2}\sqrt{n}\ell(Q_j)$,

$$\begin{aligned}
&\mu_{\Omega,\rho,q}A_j(x) \\
&= \left(\int_{a_j}^{\infty}\left|\frac{1}{t^\rho}\int_{B(t)}A_j(x-y)\frac{\Omega(y/|y|)}{|y|^{n-\rho}}dy\right|^q\frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leq C\left(\int_{a_j}^{b_j}\frac{\ell(Q_j)^{nq}}{|x^{(j)}|^{nq}}\frac{dt}{t} + \int_{b_j}^{\infty}\frac{\ell(Q_j)^{(n+1)q}dt}{t^{q\rho+1}|x^{(j)}|^{(n-\rho+1)q}}\right)^{\frac{1}{q}} \\
&\leq C\left(\frac{\ell(Q_j)^{nq}}{|x^{(j)}|^{nq}}\log\frac{b_j}{a_j} + \frac{\ell(Q_j)^{(n+1)q}}{|x^{(j)}|^{(n+1)q}}\right)^{\frac{1}{q}} \\
&\leq C\left(\frac{\ell(Q_j)^{nq+1}}{|x^{(j)}|^{nq+1}} + \frac{\ell(Q_j)^{(n+1)q}}{|x^{(j)}|^{(n+1)q}}\right)^{\frac{1}{q}} \\
&\leq C\left(\frac{\ell(Q_j)^{nq+1}}{|x^{(j)}|^{nq+1}}\right)^{\frac{1}{q}} = C\frac{\ell(Q_j)^{n+\frac{1}{q}}}{|x^{(j)}|^{n+\frac{1}{q}}}.
\end{aligned}$$

The next lemma is the last step to prove Theorem 17.

Lemma 19.

$$\left\|\sum_{j=1}^{\infty}|\lambda_j|\,\chi_{3nQ_j}\right\|_{L^p} \leq C\left\|\sum_{j=1}^{\infty}|\lambda_j|\,\chi_{Q_j}\right\|_{L^p}.$$

Proof. We have used the Fefferman-Stein vector-valued inequality (see [1]). Indeed, since

$$\chi_{3nQ_j}(x) \leq \frac{1}{|3nQ_j|}\int_{3nQ_j} dy = \frac{(3n)^n}{|3nQ_j|}\int_{Q_j} dy \leq CM\chi_{Q_j}(x),$$

by taking $\alpha > \max(1, \frac{1}{p})$, we obtain

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} |\lambda_j|\, \chi_{3nQ_j} \right\|_{L^p} &\leq C \left\| \sum_{j=1}^{\infty} |\lambda_j|\, M\chi_{Q_j} \right\|_{L^p} \\
&= C \left(\left\| \left(\sum_{j=1}^{\infty} |\lambda_j| \left(M\chi_{Q_j} \right)^{\alpha} \right)^{\frac{1}{\alpha}} \right\|_{L^{p\alpha}} \right)^{\alpha} \\
&= C \left(\left\| \left(\sum_{j=1}^{\infty} M \left(|\lambda_j|^{\frac{1}{\alpha}}\, \chi_{Q_j} \right)^{\alpha} \right)^{\frac{1}{\alpha}} \right\|_{L^{p\alpha}} \right)^{\alpha} \\
&\leq C' \left\| \sum_{j=1}^{\infty} |\lambda_j| \left(\chi_{Q_j} \right)^{\alpha} \right\|_{L^p} = C' \left\| \sum_{j=1}^{\infty} |\lambda_j|\, \chi_{Q_j} \right\|_{L^p}.
\end{aligned}$$

□

We now conclude the proof of Theorem 17. Let $p_0 = p + 1$. We know that $\|\mu_{\Omega,\rho,q} A_j\|_{L^{p_0}} \leq C\|A_j\|_{\dot{F}^0_{p_0,q}} \leq C|3Q_j|^{\frac{1}{p_0}}$. Thus, we can use Lemma 15 to have

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{3nQ_j} \mu_{\Omega,\rho,q} A_j \right\|_{L^p} \\
&\leq C \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{3nQ_j} \left(\frac{1}{|3nQ_j|} \int_{3nQ_j} \mu_{\Omega,\rho,q} A_j(y)^{p_0}\, dy \right)^{\frac{1}{p_0}} \right\|_{L^p} \\
&\leq C \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{3nQ_j} \right\|_{L^p} \leq C \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^p}.
\end{aligned}$$

Here we have used Lemma 19.

Meanwhile, by the Fefferman-Stein vector-valued inequality (see [1])

$$\left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{\mathbf{R}^n \setminus 3nQ_j} \mu_{\Omega,\rho,q} A_j \right\|_{L^p} \leq C \left\| \sum_{j=1}^{\infty} |\lambda_j| (M\chi_{Q_j})^{1+\frac{1}{nq}} \right\|_{L^p}$$

$$\leq C \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^p} \leq C \|f\|_{\dot{F}^0_{p,q}}.$$

Thus, Theorem 17 is proved.

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