SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS BY USING FOURIER TRANSFORM

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Abstract: In this study, complex differential equations are solved by using the Fourier transform. First, we separate the real and imaginary parts of the equation. Thus, from one unknown equation we obtain a system of two unknown equations. We obtain the Fourier transforms of real and imaginary parts of the solutions using the Fourier transform. Finally, we obtain the real and imaginary parts of the solution by using the inverse Fourier transform.

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1. Introduction

One of the methods used for solving linear differential equations is the method of integral transforms. An integral transform is a mapping from function to function that takes the form

$$g(\alpha) = \int_a^b f(t) K(\alpha, t) dt, \quad (1)$$

where $K(\alpha, t)$ is called the kernel of transform. In (1), $f(t)$ is the input function and $g(t)$ is the output function. Integral transforms have been used in solving many problems in applied mathematics, mathematical physics and engineering sciences. The best known two integral transforms are the Laplace transform and the Fourier transform.

The Fourier Transform, one of the gifts of Jean-Baptiste Joseph Fourier to the world of science, is an integral transform used in many areas of engineering such as it has been very useful for analyzing harmonic signals or signals for which there is known need for local information. Then the Fourier transform analysis has also been very useful in many other areas such as quantum mechanics, wave motion, turbulence, etc., [1, 2, 3, 4, 5]. Furthermore, it has been useful in mathematics such as generalized integrals, integral equations, ordinary differential equations, partial differential equations can be solved by using the Fourier transform. Another example of its applications could be: voice of every human being can be expressed as the sum of sine and cosine. Since the electromagnetic spectrum of the frequency of each voice is different, the frequency of each sine and cosine sum will be different. In this way, a voice record can be found belongs to whom using the Fourier transform. In fact, our ear automatically runs this process instead of us.

In this study, we obtain solutions of complex differential equations from first order constant coefficients. Some equations which have not general solution in real space can have to general solution in complex space. As an example, we can give Laplace equation, [6, 7, 8]. In this study, we have studied equations of the form

$$A.\frac{\partial w}{\partial z} + B.\frac{\partial w}{\partial \bar{z}} + C.w = F(z, \bar{z}), \quad (2)$$

where A, B, C are real constants, $w(z, \bar{z})$ is unknown function.

These kind of equations have been studied using the Laplace transform and the Elzaki transform in [6] and [7]. Moreover, in [8] solutions of complex differential equations of first order with variable coefficients have been studied by using the Adomian decomposition method.

This paper has been organized as follows: In Section 2, basic definitions and theorems are given. In Section 3, we have a formulation to solve of equation (2) and we have given some examples for validity of the method.

2. Basic Definitions and Theorems

Definition 1. The Fourier transform of $f(x)$ is defined as follows:

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-iwx} dt. \quad (3)$$

As a function of w , the integral (1) is written as

$$\mathcal{F}[f(x)] = F(w).$$

Definition 2. If $\mathcal{F}[f(x)] = F(w)$, then $f(x)$ is called the inverse Fourier transform of $F(w)$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwx} dt,$$

and it is denoted by $f(x) = \mathcal{F}^{-1}[F(w)]$.

Theorem 3. ([1, 3]) *The Fourier transform is linear. That is, if $c_1, c_2 \in \mathbb{R}$, then*

$$\mathcal{F}[c_1 f_1(x) + c_2 f_2(x)] = c_1 \mathcal{F}[f_1(x)] + c_2 \mathcal{F}[f_2(x)].$$

Theorem 4. ([1, 3]) *Let $f(x)$ be continuous or partly continuous in the interval $(-\infty, \infty)$, and $f(x), f'(x), \dots, f^{(n-1)}(x) \rightarrow 0$ for $|x| \rightarrow \infty$. If $f, f', \dots, f^{(n-1)}$ are absolutely integrable in the interval $(-\infty, \infty)$, then*

$$\mathcal{F}[f^{(n)}(x)] = (iw)^n F(w).$$

Definition 5. The Dirac delta function can be rigorously thought of as a function on real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}.$$

Some of the properties of the Dirac delta function are for example, [2, 3]:

$$i) \int_{-\infty}^{\infty} \delta(x) dt = 1,$$

$$ii) \int_{-\infty}^{\infty} f(x) \delta(x - x_0) dt = f(x_0),$$

$$iii) \int_{-\infty}^{\infty} f(x) \delta^{(n)}(x - x_0) dt = (-1)^n f^{(n)}(x_0),$$

$$iv) \int_{-\infty}^{\infty} \frac{\delta(w - w_0) f(w)}{(w - w_0)^n} dw = \frac{1}{n!} \frac{d^n f(w)}{dw^n} (w = w_0),$$

$$v) (w - w_0)^n \delta^{(n)}(w - w_0) = (-1)^n n! \delta(w - w_0).$$

Theorem 6. ([2, 3]) *The Fourier transform of the Dirac Delta function is 1, that is, $\mathcal{F}[\delta(x)] = 1$.*

Theorem 7. ([1, 3]) *The Fourier transforms of some functions are as follows:*

- i) $\mathcal{F}[1] = 2\pi\delta(w)$
- ii) $\mathcal{F}[x^n] = 2\pi i^n \delta^{(n)}(w)$
- iii) $\mathcal{F}[y^{(n)}] = (iw)^n \mathcal{F}[y]$
- iv) $\mathcal{F}[x^n.y] = i^n \frac{d^n \mathcal{F}[y]}{dw^n}$
- v) $\mathcal{F}[e^{iw_0x}] = 2\pi\delta(w - w_0)$
- vi) If $\mathcal{F}[y] = Y(w)$, then $\mathcal{F}[e^{iw_0x}.y] = Y(w - w_0)$
- vii) $\mathcal{F}[e^{ax}] = 2\pi\delta(w + ia)$
- viii) If $\mathcal{F}[y] = Y(w)$, then $\mathcal{F}[e^{w_0t}.y] = Y(w + iw_0)$.

2.1. Complex Derivatives

Let $w = w(z, \bar{z})$ be a complex function, $z = x + iy$, $w(z, \bar{z}) = u(x, y) + i.v(x, y)$. The first and second order derivatives w.r.t. z and $\bar{z}$ of $w(z, \bar{z})$ are defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right), \quad (4)$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right). \quad (5)$$

Theorem 8. ([2]) *The Fourier transforms of partial derivatives of first order of $f(x, y)$ are:*

$$\begin{aligned} i) \mathcal{F} \left[\frac{\partial f}{\partial y} \right] &= \frac{\partial F(k, t)}{\partial y}, \\ ii) \mathcal{F} \left[\frac{\partial f}{\partial x} \right] &= ikF(k, y), \end{aligned}$$

where $F(k, y) = \mathcal{F}[f(x, y)]$.

3. Solution of complex differential equations of first order with constant coefficients

Theorem 9. *Let A, B, C be real constants, $F(z, \bar{z})$ is a polynomial of $z, \bar{z}$, and $w = u + iv$ is a complex function. Then a special solution of*

$$A. \frac{\partial w}{\partial z} + B. \frac{\partial w}{\partial \bar{z}} + C.w = F(z, \bar{z})$$

is given by

$$u = \mathcal{F}^{-1} \left[\frac{2F_1^*(k, y) [ik(A+B) + 2C] - 2.(A-B) \frac{\partial F_2^*(k, y)}{\partial y}}{[ik(A+B) + 2C]^2 + (A-B)^2 D^2} \right],$$

$$v = \mathcal{F}^{-1} \left[\frac{2F_2^*(k, y) [ik(A+B) + 2C] - 2.(B-A) \frac{\partial F_1^*(k, y)}{\partial y}}{[ik(A+B) + 2C]^2 + (A-B)^2 D^2} \right].$$

Proof. The equation is

$$A. \frac{\partial w}{\partial z} + B. \frac{\partial w}{\partial \bar{z}} + Cw = F(z, \bar{z}). \quad (6)$$

We use equalities (4), (5) in equality (6), to obtain:

$$A. \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) + B. \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) + Cw = F_1(x, y) + iF_2(x, y). \quad (7)$$

If in (7) we write $w = u + iv$, the following equality follows:

$$A \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) + B. \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) + 2C.(u + iv) = 2F_1(x, y) + 2iF_2(x, y). \quad (8)$$

If the real and imaginary parts are separated, then the following system of equations is obtained:

$$(A+B) \frac{\partial u}{\partial x} + (A-B) \frac{\partial v}{\partial y} + 2Cu = 2F_1(x, y), \quad (9)$$

$$(A+B) \frac{\partial v}{\partial x} + (B-A) \frac{\partial u}{\partial y} + 2Cv = 2F_2(x, y). \quad (10)$$

If we use the Fourier transform to the above (9), (10) equalities, we get

$$(A+B)ikU(k, y) + (A-B) \frac{\partial V(k, y)}{\partial y} + 2CU(k, y) = 2F_1^*(k, y), \quad (11)$$

$$(A+B)ikV(k, y) + (B-A) \frac{\partial U(k, y)}{\partial y} + 2CV(k, y) = 2F_2^*(k, y), \quad (12)$$

where U, V, F_1^*, F_2^* are the Fourier transforms of u, v, F_1, F_2 , respectively. If (11), (12) is regularized and Cramer's rule is used, then we obtain as follows:

$$[ik(A+B) + 2C]U(k, y) + (A-B) \frac{\partial V(k, y)}{\partial y} = 2F_1^*(k, y),$$

$$(B - A) \frac{\partial U(k, y)}{\partial y} + [ik(A + B) + 2C] V(k, y) = 2F_2^*(k, y),$$

$$\begin{vmatrix} ik(A + B) + 2C & (A - B)D \\ (B - A)D & ik(A + B) + 2C \end{vmatrix} \\ = [ik(A + B) + 2C]^2 + (A - B)^2 D^2,$$

where $D = \frac{\partial}{\partial y}$ is the partial derivative w.r.t. to y ,

$$\begin{aligned} U &= \frac{\begin{vmatrix} 2F_1^*(k, y) & (A - B)D \\ 2F_2^*(k, y) & ik(A + B) + 2C \end{vmatrix}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2} \\ U &= \frac{2F_1^*(k, y) [ik(A + B) + 2C] - 2.(A - B) \frac{\partial F_2^*(k, y)}{\partial y}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2}, \end{aligned} \quad (13)$$

$$\begin{aligned} V &= \frac{\begin{vmatrix} ik(A + B) + 2C & 2F_1^*(k, y) \\ (B - A)D & 2F_2^*(k, y) \end{vmatrix}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2} \\ V &= \frac{2F_2^*(k, y) [ik(A + B) + 2C] - 2.(B - A) \frac{\partial F_1^*(k, y)}{\partial y}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2}. \end{aligned} \quad (14)$$

The following solutions are then obtained by the inverse Fourier transform:

$$u(x, y) = \mathcal{F}^{-1} \left[\frac{2F_1^*(k, y) [ik(A + B) + 2C] - 2.(A - B) \frac{\partial F_2^*(k, y)}{\partial y}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2} \right], \quad (15)$$

$$v(x, y) = \mathcal{F}^{-1} \left[\frac{2F_2^*(k, y) [ik(A + B) + 2C] - 2.(B - A) \frac{\partial F_1^*(k, y)}{\partial y}}{[ik(A + B) + 2C]^2 + (A - B)^2 D^2} \right]. \quad (16)$$

□

Example 1. Solve the following problem

$$\frac{\partial w}{\partial z} + 2 \frac{\partial w}{\partial \bar{z}} = 3z^2 + 2.$$

Solution 1. $A = 1, B = 2, C = 0, F_1(x, y) = 3x^2 - 3y^2 + 2, F_2(x, y) = 6xy.$

The coefficients of the equation are $A = 1, B = 2, C = 0$ and $F(z, \bar{z}) = 3z^2 + 2$,

$$\begin{aligned} F_1^*(k, y) &= \mathcal{F}[F_1(x, y)] = \mathcal{F}(3x^2 - 3y^2 + 2) \\ &= 3.2\pi i^2 \cdot \delta''(k) + (2 - 3y^2) 2\pi \delta(k), \end{aligned}$$

$$F_2^*(k, y) = \mathcal{F}[F_2(x, y)] = \mathcal{F}(6xy) = 12\pi y i \delta'(k).$$

From (11), we can write $u(x, y) = \text{Re}(w(z, \bar{z}))$ as follows

$$\begin{aligned} u(x, y) &= \mathcal{F}^{-1} \left\{ \frac{3ik [-12\pi \delta''(k) + (4 - 6y^2) 2\pi \delta(k)] + 24\pi i \delta'(k)}{D^2 - 9k^2} \right\}, \\ \mathcal{F}^{-1} \left[\frac{1}{-9k^2(1 - \frac{D^2}{9k^2})} (3ik [-12\pi \delta''(k) + (4 - 6y^2) 2\pi \delta(k)] + 24\pi i \delta'(k)) \right] \\ &= \mathcal{F}^{-1} \left[\frac{1}{-9k^2} (-36\pi i k \delta''(k) + 24\pi i k \delta(k) \right. \\ &\quad \left. - 36\pi y^2 i k \delta(k) + 24\pi i \delta'(k) - \frac{72\pi i k \delta(k)}{9k^2}) \right] \\ &= \int_{-\infty}^{\infty} \frac{2i \delta''(k)}{k} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{4i \delta(k)}{-3k} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{2iy^2 \delta(k)}{k} e^{ikx} \\ &\quad + \int_{-\infty}^{\infty} \frac{4i \delta'(k)}{-3k^2} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{4i \delta(k)}{9k^3} e^{ikx} dk \\ &= \int_{-\infty}^{\infty} \frac{4i \delta(k)}{k^3} e^{ikx} dk - \int_{-\infty}^{\infty} \frac{4i \delta(k)}{3k} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{2iy^2 \delta(k)}{k} e^{ikx} \\ &\quad + \int_{-\infty}^{\infty} \frac{4i \delta(k)}{3k^3} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{4i \delta(k)}{9k^3} e^{ikx} dk \\ &= 4i \frac{(ix)^3}{6} - \frac{4i}{3} ix + 2iy^2 ix + \frac{4i}{3} \frac{(ix)^3}{6} + \frac{4i}{9} \frac{(ix)^3}{6} \\ &= \frac{26x^3}{27} - 2xy^2 + \frac{4x}{3}. \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} v(x, y) &= \mathcal{F}^{-1} \left\{ \frac{-72k\pi y \delta'(k) + 24\pi y \delta(k)}{D^2 - 9k^2} \right\} \\ &= \mathcal{F}^{-1} \left[\frac{1}{-9k^2(1 - \frac{D^2}{9k^2})} (-72k\pi y \delta'(k) + 24\pi y \delta(k)) \right] \end{aligned}$$

$$\begin{aligned}
&= \mathcal{F}^{-1} \left[\frac{1}{-9k^2} (-72k\pi y \delta'(k) + 24\pi y \delta(k)) \right] \\
&= \mathcal{F}^{-1} \left(\frac{8\pi y \delta'(k)}{k} - \frac{8\pi y \delta(k)}{3k^2} \right) \\
&= \int_{-\infty}^{\infty} \left(\frac{4y \delta'(k)}{k} - \frac{4y \delta(k)}{3k^2} \right) e^{ikx} dk \\
&= \int_{-\infty}^{\infty} \left(-\frac{16y \delta(k)}{3k^2} \right) e^{ikx} dk = -\frac{16y}{3} \frac{(ix)^2}{2} = \frac{8x^2 y}{3}.
\end{aligned} \tag{18}$$

From (17) and (18), we get the following solution:

$$\begin{aligned}
w &= u(x, y) + iv(x, y) = \frac{26x^3}{27} - 2xy^2 + \frac{4x}{3} + i \frac{8x^2 y}{3} \\
&= \frac{26}{27} \left(\frac{z + \bar{z}}{2} \right)^3 - 2 \left(\frac{z + \bar{z}}{2} \right) \left(\frac{z - \bar{z}}{2i} \right)^2 + \frac{4}{3} \left(\frac{z + \bar{z}}{2} \right) \\
&\quad + \frac{8i}{3} \left(\frac{z + \bar{z}}{2} \right)^2 \left(\frac{z - \bar{z}}{2i} \right).
\end{aligned}$$

Example 2. Solve the following problem

$$2 \frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}} = 4z + 1.$$

Solution 2. The coefficients of the equation are $A = 2, B = -1, C = 0$ and $F(z, \bar{z}) = 4z + 1$.

$$F_1^*(k, y) = \mathcal{F}[F_1(x, y)] = \mathcal{F}(4x + 1) = 4.2\pi i \delta'(k) + 2\pi \delta(k),$$

$$F_2^*(k, y) = \mathcal{F}[F_2(x, y)] = \mathcal{F}(4y) = 8\pi y \delta(k).$$

From (11), we can write $u(x, y) = \text{Re}(w(z, \bar{z}))$ as follows:

$$\begin{aligned}
u(x, y) &= \mathcal{F}^{-1} \left\{ \frac{[16\pi i \delta'(k) + 4\pi \delta(k)] ik - 48\pi \delta(k)}{9D^2 - k^2} \right\} \\
&= \mathcal{F}^{-1} \left[\frac{1}{-k^2(1 - \frac{9D^2}{k^2})} (-16\pi k \delta'(k) + 4\pi i k \delta(k) - 48\pi \delta(k)) \right] \\
&= \mathcal{F}^{-1} \left[\frac{1}{-k^2} (-16\pi k \delta'(k) + 4\pi i k \delta(k) - 48\pi \delta(k)) \right]
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{8\delta'(k)}{k} e^{ikx} dk - \int_{-\infty}^{\infty} \frac{2i\delta(k)}{k} e^{ikx} dk + \int_{-\infty}^{\infty} \frac{24\delta(k)}{k^2} e^{ikx} dk \\
&= 8 \left(-\frac{1}{2} \right) (ix)^2 - 2i(ix) + 24 \frac{(ix)^2}{2} = 2x - 8x^2,
\end{aligned}$$

$$\begin{aligned}
v(x, y) &= \mathcal{F}^{-1} \left\{ \frac{16\pi y i k \delta(k)}{9D^2 - k^2} \right\} \\
&= \mathcal{F}^{-1} \left[\frac{1}{-k^2(1 - \frac{9D^2}{k^2})} (16\pi y i k \delta(k)) \right] \\
&= \mathcal{F}^{-1} \left[\frac{1}{-k^2} (16\pi y i k \delta(k)) \right] = \mathcal{F}^{-1} \left[\frac{1}{-k} (16\pi y i \delta(k)) \right] \\
&= \int_{-\infty}^{\infty} -\frac{8iy\delta(k)}{k} e^{ikx} dk = -8iy(ix) = 8xy,
\end{aligned}$$

$$w = u + iv = 2x - 8x^2 + 8ixy = -4z\bar{z} - 4\bar{z}^2 + z + \bar{z}.$$

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