ON $F$-FACE MAGIC MEAN LABELING OF SOME DUPLICATED GRAPHS

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Abstract: In this paper we introduce the $F$-face magic mean labeling of planar graphs, an assignment of labels to the edges which induces an assignment of labels to the faces of a graph such that the mean weight of each face is constant. We discuss about the $F$-face magic mean labelings of some duplicated graphs.

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1. Introduction

By a graph, we mean a finite, connected, undirected planar graph without loops or multiple edges. By a planar graph, we mean that it can be drawn in a plane such that no two edges intersect.

Duplication of an edge $e = uv$ by a vertex $v'$ in a graph $G$ is a new graph $G'$ where $V(G') = V(G) \cup \{v'\}$ and $E(G') = E(G) \cup \{uv', v'v\}$. Vertex duplication of a path $P_n$, denoted by $\hat{P}_n$, is formed by duplicating all the vertices of $P_n$, $n \geq 2$. Vertex duplication of a cycle $C_n$, denoted by $\hat{C}_n$, is formed by duplicating all the vertices of $C_n$, $n \geq 3$, where $n \equiv 0 (mod 2)$.

The middle graph $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in $M(G)$ if and only if either they are adjacent vertices
or edges of $G$ or one is a vertex of $G$ and the other is an edge incident with it.

The Butterfly graph $B_n$, $n \geq 2$ is obtained from a cycle $C_4$ having edges $v_1v_2, v_2v_3, v_3v_4$ and $v_4v_1$ by duplicating the vertex $v_1$ by $v_1^1, v_1^2$ and $v_1^2$ by $v_1^3, \ldots, v_1^{n-1}$, the edges $v_2v_3$ by a vertex $v_2^1, v_2^2$ by $v_2^3, \ldots, v_2^{n-1}$ by $v_2^n$ and the edge $v_3v_4$ by a vertex $v_3^1, v_3^2$ by $v_3^3, \ldots, v_3^{n-1}$ by $v_3^n$.

A function $g$ is called a mean labeling of graph $G$ if $g : V(G) \rightarrow \{0, 1, 2, \ldots, \}$ is injective and the induced edge function $g^* : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ defined as follows is bijective

$$g^*(e = uv) = \begin{cases} \frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\ \frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd.} \end{cases}$$

The graph which admits mean labeling is called a mean graph [5].

A graph $G$ is magic if the edges of $G$ can be labeled by the numbers $1, 2, 3, \ldots, |E(G)|$ so that the sum of the labels of all the edges incident with any vertex is the same [2].

Motivated by these works, we introduce $F$-face magic mean graph as follows:

An injection $\phi : E(G) \rightarrow \{1, 2, \ldots, e\}$ is called a $F$-face magic mean labeling of $G$ if the induced face labeling

$$\phi^*(f_i) = \left[ \frac{\text{sum of the labels of the edges in the boundary of } f_i}{\deg(f_i)} \right]$$

$$= \left[ \frac{\sum_{e_j \text{ is in } f_i} \phi(e_j)}{\deg(f_i)} \right]$$

$$= k, \text{ a constant,}$$

for each face $f_i$, including the exterior face of $G$.

In this paper we prove that $F$-face magic mean labeling exists for certain families of graphs obtained by duplicating the vertices or edges such as $\hat{P}_n, \hat{C}_{2n}$, middle graph of $C_n$, butterfly graph $B_n$.

2. Main Results

**Theorem 1.** $\hat{P}_n$, vertex duplication of a path $P_n$, admits a $F$-face magic mean labeling for all $n \geq 2$ with face magic mean constant $\frac{3n-2}{2}$ while $n \equiv 0(mod \ 2)$ and $\frac{3(n-1)}{2}$ while $n \equiv 1(mod \ 2)$. 

Proof. Let \( \{v_i : 1 \leq i \leq n\} \) be the vertices of \( P_n \) and \( u_i(1 \leq i \leq n) \) be the respective duplicating vertex of \( v_i \) for \( 1 \leq i \leq n \). Then,

\[
E(\hat{P}_n) = \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_{i-1} : 2 \leq i \leq n\} \\
\cup \{u_iv_{i+1} : 2 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n - 1\} \text{ and }
\[
F(\hat{P}_n) = \{f_i = (v_iv_{i+1}, v_{i+1}v_{i+2}, v_{i+2}u_{i+1}, u_{i+1}v_i) : 1 \leq i \leq n - 2\} \\
\cup \left\{f_0 = \left(\bigcup_{i=1}^{n-1} u_iv_{i+1}, \bigcup_{i=2}^{n} u_iv_{i-1}, v_1v_2, v_{n-1}v_1\right)\right\}.
\]

In \( \hat{P}_n \), \( |E(\hat{P}_n)| = 3(n-1) \) and \( |F(\hat{P}_n)| = n - 1 \).

Define \( \phi : E(\hat{P}_n) \to \{1, 2, \ldots, 3n-3\} \) as follows:

\[
\phi(u_1v_2) = 1 \\
\phi(u_nv_{n-1}) = 3n - 3 \\
\phi(u_iv_{i-1}) = n - i + 1, \ 2 \leq i \leq n - 1 \\
\phi(u_iv_{i+1}) = 3n - i - 2, \ 2 \leq i \leq n - 1 \\
\phi(v_iv_{i+1}) = n + i - 1, \ 1 \leq i \leq n - 1.
\]

The induced face labeling \( \phi^* \) is obtained as follows:

\[
\phi^*(f_i) = \left\{ \begin{array}{ll}
\left\lfloor \frac{3n-2}{2} \right\rfloor & \text{if } n \equiv 0(\text{mod } 2) \\
\left\lfloor \frac{3(n-1)}{2} \right\rfloor & \text{if } n \equiv 1(\text{mod } 2)
\end{array} \right.,
\]

and \( \phi^*(f_0) = \left\{ \begin{array}{ll}
\left\lfloor \frac{3n-2}{2} \right\rfloor & \text{if } n \equiv 0(\text{mod } 2) \\
\left\lfloor \frac{3(n-1)}{2} \right\rfloor & \text{if } n \equiv 1(\text{mod } 2)
\end{array} \right. = n.
\]

Therefore, \( \phi \) is a \( F \)-face magic mean labeling of \( \hat{P}_n \). Thus \( \hat{P}_n \) is a \( F \)-face magic mean graph with face magic mean constant \( \left\lfloor \frac{3n-2}{2} \right\rfloor \), where \( n \geq 2 \).}

The \( F \)-face magic mean labeling of \( \hat{P}_8 \) and \( \hat{P}_5 \) are shown in Figure 1.
Theorem 2. $\widehat{C}_n (n \geq 3)$ admits a $F$-face magic mean labeling with face magic mean constant $\frac{3n}{2}$ if and only if $n$ is even.

Proof. While $n$ is odd, $\widehat{C}_n$ is a non-planar graph. Consider $n$ is even. Let $\{v_i : 1 \leq i \leq n\}$ be the vertices of $C_n$ and $u_i$ be the respective duplicating vertex of $v_i$, for $1 \leq i \leq n$. Then,

$$E(\widehat{C}_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i-1} : 2 \leq i \leq n\}$$

$$\cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n v_1, v_n u_1, v_n v_1\}$$

and

$$F(\widehat{C}_n) = \{f_i = (v_{i-1} v_i, v_i v_{i+1}, u_i v_{i+1}, u_i v_{i-1}) : 2 \leq i \leq n-1\}$$

$$\cup \{f_1 = (v_1 v_2, v_n v_1, u_1 v_n, u_1 v_2), f_n = (v_{n-1} v_n, v_n v_1, v_1 u_n, v_{n-1} u_n)\}$$

$$\cup \left\{ f_I = \left( \bigcup_{i \equiv 2 (\text{mod} \ 2)}^{n} v_i v_{i-1} u_i, \bigcup_{i \equiv 2 (\text{mod} \ 2)}^{n} u_i v_{i-1} u_n v_1 \right) \right\}.$$
\[
f_0 = \left\{ \bigcup_{i \equiv 1 \mod 2}^{n-1} u_i v_{i+1}, \bigcup_{i \equiv 1 \mod 2}^{n-1} u_i v_{i-1}, v_n u_1 \right\}.
\]

In \( \widehat{C}_n \), \(|E(\widehat{C}_n)| = 3n\) and \(|F(\widehat{C}_n)| = n + 2\).

Define \( \phi : E(\widehat{C}_n) \rightarrow \{1, 2, \ldots, 3n\} \) as follows:

\[
\begin{align*}
\phi(u_i v_{i+1}) &= \begin{cases} 
  i, & 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\
  n - 1 + i, & 1 \leq i \leq n - 1 \text{ and } i \text{ is even},
\end{cases} \\
\phi(v_i v_{i+1}) &= \begin{cases} 
  3n - i, & 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\
  i, & 1 \leq i \leq n - 1 \text{ and } i \text{ is even},
\end{cases} \\
\phi(u_i v_{i-1}) &= \begin{cases} 
  3n + 3 - i, & 2 \leq i \leq n \text{ and } i \text{ is odd} \\
  2n + 2 - 1, & 2 \leq i \leq n \text{ and } i \text{ is even},
\end{cases}
\end{align*}
\]

\[
\phi(u_n v_1) = 2n - 1, \\
\phi(v_n u_1) = 2n + 2 \text{ and} \\
\phi(v_n v_1) = n.
\]

The induced face labeling \( \phi^* \) is obtained as follows:

\[
\begin{align*}
\phi^*(f_i) &= \frac{3n}{2}, 1 \leq i \leq n, \\
\phi^*(f_1) &= \frac{3n}{2} \text{ and} \\
\phi^*(f_0) &= \frac{3n}{2}.
\end{align*}
\]

Therefore, \( \widehat{C}_n \) is a \( F \)-face magic mean graph with face magic mean constant \( \frac{3n}{2} \). \( \square \)

The \( F \)-face magic mean labeling of \( \widehat{C}_8 \) having face magic mean constant 12 is shown in Figure 2.
Theorem 3. The middle graph $M(C_n)$ of a cycle $C_n$ is a $F$-face magic mean graph, for all $n \geq 3$.

Proof. Let the vertex set, edge set and face set of $G = M(C_n)$ be

\begin{align*}
V(G) &= \{v_i : 1 \leq i \leq n\} \cup \{u_i(e_i) : 1 \leq i \leq n\}, \\
E(G) &= \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_i v_i : 1 \leq i \leq n\} \\
&\quad \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n v_1\} \quad \text{and}
\end{align*}

\begin{align*}
F(G) &= \{f_i = (u_i v_i, u_i v_{i+1}, v_i v_{i+1}) : 1 \leq i \leq n-1\} \\
&\quad \cup \left\{ f_n = (u_n v_n, u_n v_1, v_n v_1), f_I = \left( \bigcup_{i=1}^{n-1} v_i v_{i+1}, v_n v_1 \right) \right\} .
\end{align*}

In $G$, $|E(G)| = 3n$ and $|F(G)| = n + 2$.

Case (i) $n \equiv 0(\text{mod 2})$.

Define $\phi : E(G) \to \{1, 2, \ldots, 3n\}$ as follows:

\begin{align*}
\phi(u_i v_i) &= i, \quad 1 \leq i \leq n, \\
\phi(v_i v_{i+1}) &= \begin{cases} 
\frac{3n}{2} + i, & 1 \leq i \leq \frac{n}{2} \\
\frac{n}{2} + i, & \frac{n}{2} + 1 \leq i \leq n - 1,
\end{cases}
\end{align*}

Figure 2: $F$-face magic mean labeling of $\tilde{C}_8$
\[ \phi(v_n v_1) = \frac{3n}{2}, \]
\[ \phi(u_i v_{i+1}) = \begin{cases} 
3n - 2i + 2, & 1 \leq i \leq \frac{n}{2} \\
4n - 2i + 1, & \frac{n}{2} + 1 \leq i \leq n - 1 \end{cases} \] and
\[ \phi(u_n v_1) = 2n + 1. \]

The induced face labeling on \( f_i \) is obtained as follows:
\[ \phi^*(f_i) = \begin{cases} 
\left\lfloor \frac{9n+4}{6} \right\rfloor, & 1 \leq i \leq \frac{n}{2} \\
\left\lfloor \frac{9n+2}{6} \right\rfloor, & \frac{n}{2} + 1 \leq i \leq n \end{cases} \]
\[ \phi^*(f_I) = \left\lfloor \frac{3n+1}{2} \right\rfloor \] and
\[ \phi^*(f_0) = \left\lfloor \frac{3n+1}{2} \right\rfloor. \]

Thus \( \phi \) is a face magic mean labeling with face magic mean constant \( \frac{3n}{2} \).

**Case (ii) \( n \equiv 1(\mod 2) \).**

Define \( \phi : E(G) \to \{1, 2, \ldots, 3n\} \) as follows:
\[ \phi(v_i v_{i+1}) = \begin{cases} 
3n - i - 2, & 1 \leq i \leq \frac{n+1}{2} \\
\frac{n-1}{2} + i, & \frac{n+1}{2} \leq i \leq n - 1 \end{cases} \]
\[ \phi(v_n v_1) = \frac{3n - 1}{2}, \]
\[ \phi(u_i v_i) = i, \text{ for } 1 \leq i \leq n, \]
\[ \phi(u_i v_{i+1}) = \begin{cases} 
3n - 2i + 2, & 1 \leq i \leq \frac{n+1}{2} \\
4n - 2i + 2, & \frac{n+3}{2} \leq i \leq n - 1 \end{cases} \] and
\[ \phi(u_n v_1) = 2n + 2. \]

The induced face labeling \( \phi^* \) is obtained as follows:
\[ \phi^*(f_i) = \frac{3n+1}{2}, \text{ for } 1 \leq i \leq n, \]
\[ \phi^*(f_I) = \frac{3n+1}{2} \] and \( \phi^*(f_0) = \frac{3n+1}{2}. \)

Thus \( \phi \) is a \( F \)-face magic mean labeling with face magic mean constant \( \frac{3n+1}{2} \).

Hence \( M(C_n) \) is a \( F \)-face magic mean graph for all \( n \geq 3 \). \( \square \)
Figure 3 illustrates the $F$-face magic mean labeling of $M(C_8)$ and $M(C_7)$.

**Figure 3**

**Theorem 4.** Butterfly graph $B_n$, $n \geq 2$ is a $F$-face magic mean graph having face magic mean constant $3n + 1$.

**Proof.** Let $n \geq 2$.

Let $V(B_n) = \{v_i : 1 \leq i \leq 4\} \cup \{v_1^j : 1 \leq j \leq n - 1\} \cup \{v_2^j : 1 \leq j \leq n\} \cup \{v_3^j : 1 \leq j \leq n\}$. Then,

$$E(B_n) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\} \cup \{v_2v_1^j : 1 \leq j \leq n - 1\} \cup \{v_2v_2^j : 1 \leq j \leq n\} \cup \{v_3v_3^j : 1 \leq j \leq n\} \cup \{v_3v_4^j : 1 \leq j \leq n\} \cup \{v_4v_3^j : 1 \leq j \leq n - 1\} \quad \text{and}$$

$$F(B_n) = \{f_1 = (v_1v_2, v_2v_3, v_3v_4, v_4v_1), f_2 = (v_2v_2', v_2'v_3, v_3v_2),$$

$$f_3 = (v_3v_3', v_3'v_4, v_4v_3), f_1' = (v_1v_2, v_2v_1', v_1'v_4, v_4v_1)\}$$

$$\cup \{f_j^i = (v_1^jv_2, v_2^jv_1^{j-1}, v_1^{j-1}v_4, v_4v_1^j) : 2 \leq j \leq n - 1\}$$
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∪ \{f^j_2 = (v^{j+1}_2, v_3, v^j_2v_2, v_2v^{j+1}_2) : 1 \leq j \leq n - 1\}
∪ \{f^j_3 = (v^{j+1}_3v_4, v_4v^j_3, v_3v^{j+1}_3) : 1 \leq j \leq n - 1\}
∪ \{f_0 = (v_2v^n_2, v_2v^n_3, v_3v^n_3, v_4v^{n-1}_4, v_1v^{n-1}_2)\}.

In butterfly graph $B_n$, $|E(B_n)| = 6n + 2$ and $F(B_n)| = 3n + 1$.
Define $\phi : E(B_n) \rightarrow \{1, 2, \ldots, 6n + 2\}$ as follows:

$\phi(v_1v_2) = 4n + 2,$
$\phi(v_2v_3) = 3n + 2,$
$\phi(v_3v_4) = 3n + 1,$
$\phi(v_4v_1) = 2n + 1,$
$\phi(v_2v^j_2) = j, 1 \leq j \leq n,$
$\phi(v_3v^j_2) = 6n + 3 - j, 1 \leq j \leq n,$
$\phi(v_3v^j_3) = n + j, 1 \leq j \leq n,$
$\phi(v_4v^j_3) = 5n + 3 - j, 1 \leq j \leq n,$
$\phi(v_2v^j_1) = 4n + 2 - j, 1 \leq j \leq n - 1$ and
$\phi(v_4v^j_1) = 2n + j, 1 \leq j \leq n - 1.$

The induced face labeling $\phi^*$ is obtained as follows:

$\phi^*(f_1) = \left\lfloor \frac{12n + 6}{4} \right\rfloor,$
$\phi^*(f_2) = \left\lfloor \frac{9n + 4}{3} \right\rfloor,$
$\phi^*(f'_1) = \left\lfloor \frac{12n + 5}{4} \right\rfloor,$
$\phi^*(f'_1) = 3n + 1, 2 \leq j \leq n - 1.$
$\phi^*(f^j_2) = \left\lfloor \frac{12n + 6}{4} \right\rfloor, 2 \leq j \leq n - 1,$
$\phi^*(f^j_3) = \left\lfloor \frac{12n + 6}{4} \right\rfloor, 2 \leq j \leq n - 1$ and
$\phi^*(f_0) = \left\lfloor \frac{18n + 8}{6} \right\rfloor.$

Thus $\phi$ is a $F$-face magic mean labeling of $B_n$.
Hence $B_n$ is a $F$-face magic mean graph having face magic mean constant $3n + 1$. □

Figure 4 illustrates the $F$-face magic mean labeling of butterfly graph $B_4$. 
References


*Figure 4*: $F$-face magic mean labeling of $B_4$. 

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Note: The diagram in the document is labeled as $v^1$, $v^2$, and $v^3$, but the text references $v_1$, $v_2$, and $v_3$. The labels in the diagram are consistent with the text references.