**ANALYTICAL SOLUTION OF TIME-FRACTIONAL
NONLINEAR BENJAMIN-BONA-MAHONY EQUATION
BY RESIDUAL POWER SERIES METHOD**

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Abstract: In this paper a new iterative technique, named as residual power series (RPS) method, is applied to find the approximate solution of the non-linear time-fractional Benjamin-Bona-Mahony (BBM) equation. The results obtained by numerical experiments are compared with the analytical solutions to confirm the accuracy and efficiency of the proposed technique.

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Key Words: fractional derivative, Benjamin-Bona-Mahony equation, residual power series method

1. Introduction

Nonlinear partial differential equations play important roles in engineering and applied sciences. It is known that except a limited number of these equations, most of them do not have analytical solution. In recent years, scientists have presented some new methods for solving nonlinear partial differential equations, such as inverse scattering method [1], Adomian's decomposition method [2], Homotopy analysis method [3, 4], Homotopy perturbation method [5, 6], variational iteration method [7, 8], Hirota's bilinear method [9] and Lie group method [10].

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The nonlinear time-fractional BBM equation is a mathematical model of propagation of small-amplitude long waves in nonlinear dispersive media in the form

$$D_t^\alpha u - u_{xxt} + \beta u_x + g(u)_x = 0, \quad x \in R, \quad t \geq 0 \quad (1)$$

with the initial data

$$u(x, 0) = f(x) \rightarrow 0, \quad x \rightarrow \pm\infty, \quad (2)$$

where $u(x, t)$ represents the fluid velocity in the horizontal direction x , α is a positive constant, $\beta \in R$ and $g(u)$ is a C^2 -smooth nonlinear function [11].

In this study, the residual power series (RPS) method is applied in the numerical solution of nonlinear time-fractional BBM equation. The new method supplies the solution in the shape of a convergence series without implementing linearization, perturbation or discretization techniques.

This paper is organized as follows: In Section 2, some preliminaries of fractional calculus are given. In Section 3, a residual power series solution for nonlinear time-fractional BBM is constructed. In Section 4, some numerical results are presented.

2. Preliminaries

In this section, the basic definitions and various features for fractional calculus theory are shown [12, 13, 14, 15].

Definition 1. The Riemann-Liouville fractional integral of order α ($\alpha \geq 0$) is given as ([16, 17]):

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0,$$

$$J^0 f(x) = f(x).$$

Definition 2. The Caputo fractional derivative with order α is defined as ([16]-[17]):

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \int_0^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} f(t) dt,$$

where D^m is the classical differential operator of integer order m , $m-1 < \alpha < m$, $x > 0$.

For the Caputo derivative, we have

$$D^\alpha x^\beta = 0, \quad \beta < \alpha,$$

$$D^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} x^{\beta - \alpha}, \quad \beta \geq \alpha.$$

Definition 3. The Caputo time (partial) fractional derivative of order α of $u(x, t)$ is defined as ([16, 17]):

$$D_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \xi)^{m - \alpha - 1} \frac{\partial^m u(x, \xi)}{\partial t^m} d\xi, & m - 1 < \alpha < m \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

Definition 4. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t - t_0)^{m\alpha} = c_0 + c_1 (t - t_0)^\alpha + c_2 (t - t_0)^{2\alpha} + \dots,$$

$$0 \leq m - 1 < \alpha \leq m, \quad t \geq t_0,$$

is called fractional power series about $t = t_0$, [18].

Definition 5. A power series expansion of the form

$$\sum_{m=0}^{\infty} f_m(x) (t - t_0)^{m\alpha} = f_0(x) + f_1(x) (t - t_0)^\alpha + f_2(x) (t - t_0)^{2\alpha} + \dots,$$

$$0 \leq m - 1 < \alpha \leq m, \quad t \geq t_0,$$

is called multiple fractional power series about $t = t_0$, [18].

Definition 6. Suppose that $u(x, t)$ has a multiple fractional power series representation at $t = t_0$ of the form

$$u(x, t) = \sum_{m=0}^{\infty} f_m(x) (t - t_0)^{m\alpha}, \quad x \in I, \quad t_0 \leq t \leq t_0 + R.$$

If $D_t^{m\alpha} u(x, t)$, $m = 0, 1, 2, \dots$ are continuous on $I \times (t_0, t_0 + R)$, then

$$f_m(x) = \frac{D_t^{m\alpha} u(x, t_0)}{\Gamma(m\alpha + 1)}.$$

3. RPS of the nonlinear time-fractional BBM equation

The RPS method proposes the solution for Eqs. (1)-(2) as a fractional power series expansion about the initial point $t = 0$

$$u(x, t) = \sum_{k=0}^{\infty} f_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq 1, \quad x \in I, \quad 0 \leq t < R. \quad (3)$$

To obtain the numerical values from this series, let $u_m(x, t)$ denotes the m -th truncated series of $u(x, t)$. That is,

$$u_m(x, t) = \sum_{k=0}^m f_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq 1, \quad x \in I, \quad 0 \leq t < R. \quad (4)$$

By the initial condition, the 0^{th} residual power series approximate solution of $u(x, t)$ can be written as follows:

$$u_0(x, t) = f_0(x) = u(x, 0) = f(x). \quad (5)$$

Eq.(5) can be written as

$$u_m(x, t) = f(x) + \sum_{k=1}^m f_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad (6)$$

$$0 < \alpha \leq 1, \quad x \in I, \quad 0 \leq t, \quad k = 1, 2, 3, \dots \quad (7)$$

Define the residual function for Eq.(1) as

$$Res(x, t) = D_t^\alpha u - u_{xxt} + u_x + uu_x \quad (8)$$

and the m^{th} residual function can be expressed as

$$Res_m(x, t) = D_t^\alpha(u_m) - (u_m)_{xxt} + (u_m)_x + (u_m)(u_m)_x. \quad (9)$$

From [19, 20, 21, 22], some results such as $Res(x, t) = 0$, $\lim_{m \rightarrow \infty} Res_m(x, t)$ for each $x \in I$ and $t \geq 0$ and $D_t^\alpha Res(x, 0) = D_t^\alpha Res_m(x, 0) = 0$, $r = 0, 1, 2, \dots, m$ are stated.

Substitute the m^{th} truncated series of $u(x, t)$ into Eq. (8), calculate the fractional derivative $D_t^{(m-1)\alpha}$ of $Res(x, t)$, $m = 1, 2, 3, \dots$ at $t = 0$ and solve the following obtained algebraic system

$$D_t^{(m-1)\alpha} Res_m(x, 0) = 0, \quad 0 < \alpha \leq 1, \quad m = 1, 2, 3, \dots, \quad (10)$$

to get the required coefficients $f_k(x)$, $k = 1, 2, 3, \dots, m$ in Eq. (6). To determine $f_1(x)$, the 1st residual function in Eq. (8) can be written as follows:

$$Res_1(x, t) = D_t^\alpha(u_1) - (u_1)_{xxt} + (u_1)_x + (u_1)(u_1)_x, \quad (11)$$

where $u_1(x, t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$. Therefore,

$$\begin{aligned} Res_1(x, t) = & f_1(x) - f_1''(x)\frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)} + f'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \left(f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\left(f'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right). \end{aligned} \quad (12)$$

From Eq.(9), we deduce that $Res_1(x, 0) = 0$, and thus

$$f_1(x) = -f'(x) - f(x)f'(x). \quad (13)$$

The 1st residual power series approximate solution is

$$u_1(x, t) = f(x) + (-f'(x) - f(x)f'(x))\frac{t^\alpha}{\Gamma(1+\alpha)}. \quad (14)$$

To obtain $f_2(x)$, the 2nd residual function in Eq. (8) can be written in the following form

$$Res_2(x, t) = D_t^\alpha(u_2) - (u_2)_{xxt} + (u_2)_x + (u_2)(u_2)_x, \quad (15)$$

where $u_2(x, t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$. Therefore,

$$\begin{aligned} Res_2(x, t) = & f_2(x) + f_2'(x)\frac{t^\alpha}{\Gamma(1+\alpha)} - f_2''(x)\frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)} \\ & - f_2''(x)\frac{2\alpha t^{2\alpha-1}}{\Gamma(1+2\alpha)} + f'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \left(f(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ & \left(f'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right). \end{aligned} \quad (16)$$

The operator D_t^α is applied on both sides of Eq. (15) as follows:

$$D_t^\alpha Res_2(x, t) = f_2(x) + f_1'(x) + f_2'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$\begin{aligned}
& + \left(f_1(x) + f_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \left(f'(x) + f'_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f'_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) \\
& + \left(f(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) \left(f'_1(x) + f'_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \right). \quad (17)
\end{aligned}$$

From Eqs. (9) and (16),

$$f_2(x) = -f'_1(x) - (f(x)f_1(x))'. \quad (18)$$

To derive $f_3(x)$, the 3^{rd} residual function can be written as follows:

$$Res_3(x, t) = D_t^\alpha(u_3) - (u_3)_{xxt} + (u_3)_x + (u_3)(u_3)_x, \quad (19)$$

where $u_3(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$. Therefore,

$$\begin{aligned}
Res_3(x, t) = & f_1(x) + f_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
& - f''_1(x) \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)} - f''_2(x) \frac{2\alpha t^{2\alpha-1}}{\Gamma(1+2\alpha)} - f''_3(x) \frac{3\alpha t^{3\alpha-1}}{\Gamma(1+3\alpha)} \\
& + f'(x) + f'_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f'_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f'_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
& + \left(f(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) \\
& \left(f'(x) + f'_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f'_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f'_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right). \quad (20)
\end{aligned}$$

From Eqs. (9) and (19),

$$f_3(x) = -f'_2(x) - (f(x)f_2(x))' - 2f_1(x)f'_1(x). \quad (21)$$

The same manner is repeated as above and applied to Eq. (9), the following recurrence results is obtained

$$f_4(x) = -f'_3(x) - (f(x)f_3(x))' - 3(f_1(x)f_2(x))',$$

$$f_5(x) = -f'_4(x) - (f(x)f_4(x))' - 4(f_1(x)f_3(x))' - 6(f_2(x)f'_2(x)),$$

and so on.

4. Numerical Results

Consider the following nonlinear time fractional BBM equation [23, 24, 25]:

$$D_t^\alpha u(x, t) - u_{xx}(x, t) + u_x(x, t) + u(x, t)u_x(x, t) = 0,$$

subject to initial condition

$$u(x, 0) = \operatorname{sech}^2\left(\frac{x}{4}\right).$$

Then, the exact solution is given by

$$u(x, t) = \operatorname{sech}^2\left(\frac{x}{4} - \frac{t}{3}\right).$$

Based on the obtained results, we apply the 5th RPS approximate solution. Figures 1-5 show the 5th RPS approximate solution of the function $u(x, t)$ for different values of the fractional derivative α for $0 < x < 1, 0 < t < 1$. In Table 1 we give the exact solution, RPS solution for several values of x and t when α and the absolute error $|u_{exact} - u_{approx}^5|$ compared with the exact solution for several values of x and t when $\alpha = 1$.

Figure 1. The 5th RPS approximate solution of the nonlinear time-fractional BBM equation for $\alpha = 0.7$.

Figure 2. The 5th RPS approximate solution of the nonlinear time-fractional BBM equation for $\alpha = 0.8$.

Figure 3. The 5th RPS approximate solution of the nonlinear time-fractional BBM equation for $\alpha = 0.9$.

Figure 4. The 5th RPS approximate solution of the nonlinear time-fractional BBM equation for $\alpha = 1$.

Figure 5. Exact solution.

x	t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact	Error
0	0	1.0000	1.0000	1.0000	1.0000	1.0000	0
	0.2	0.9636	0.9752	0.9841	0.9902	0.9956	0.0054
	0.4	0.9293	0.9364	0.9496	0.9627	0.9824	0.0198
	0.6	0.9305	0.9084	0.9108	0.9235	0.9610	0.0375
	0.8	0.9906	0.9131	0.8841	0.8827	0.9321	0.0495
	1	1.1301	0.9726	0.8886	0.8542	0.8966	0.0425
0.5	0	0.9845	0.9845	0.9845	0.9845	0.9845	0.0000
	0.2	0.9888	0.9951	0.9985	0.9994	0.9966	0.0028
	0.4	0.9754	0.9763	0.9858	0.9946	0.9999	0.0053
	0.6	1.0136	0.9663	0.9626	0.9734	0.9944	0.0210
	0.8	1.1644	1.0063	0.9522	0.9459	0.9802	0.0343
	1	1.4933	1.1484	0.9890	0.9316	0.9578	0.0262
1	0	0.9400	0.9400	0.9400	0.9400	0.9400	0.0000
	0.2	0.9847	0.9851	0.9822	0.9775	0.9671	0.0104
	0.4	0.9803	0.9869	0.9941	0.9974	0.9865	0.0109
	0.6	0.9919	0.9761	0.9849	0.9974	0.9975	0.0001
	0.8	1.0721	0.9833	0.9680	0.9801	0.9997	0.0196
	1	1.2793	1.0502	0.9673	0.9556	0.9931	0.0375

Table 1: Exact solution, the RPS solution for several values x , t and α and absolute error.

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