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COMPOSITE RUNGE KUTTA METHOD FOR OIL PRODUCTION MODEL BY USING INTUITIONISTIC FUZZY DIFFERENTIAL EQUATION

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Abstract: The aim of this paper is to construct Runge-Kutta (RK) methods to obtain the numerical solution of intuitionistic fuzzy differential equations (IFDEs) and Convergence RK methods for solving intuitionistic differential equations. Then third order RK methods have been compared Arithmetic mean (AM) to Heronian mean (HeM) for solving intuitionistic fuzzy initial value problems (IFIVPs). The absolute error results are compared with AM to HeM which show good accuracy.

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Key Words: intuitionistic Fuzzy differential equations, IVP, RK methods, convergence

1. Introduction

Fuzzy differential equations plays a significant role in the field of biology, engineering, physics as well as among other field of science. For example, in population models, civil engineering, bioinformatics and computational biology, economic model, quantum optics and gravity, modeling hydraulic and etc. The conception of fuzzy set theory is considered to be one of Intuitionistic fuzzy

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set (IFS) introduced by Zadeh in [13]. Atanassov [2] extended the concept of fuzzy set theory and its applications. Melliani et al. [5, 6] discussed differential and partial differential equations under intuitionistic fuzzy environment. Abbasbandy and Allahviranloo [1] have discussed numerical solution of FDE by Runge-Kutta method with intuitionistic treatment. Sneh Lata and Amit Kumar [10] have introduced time dependent intuitionistic fuzzy linear differential equation. Sankar Prasad Mondal and Tapan Kumar Roy [4, 9] illustrated strong and weak solution of intuitionistic fuzzy ordinary differential equation and introduced second order linear differential equations by using the fuzzy boundary value. Atanassov [2], Ma et al. [7] discussed convergence and stability of fuzzy initial value problems. Nirmala and chenthur Pandian [8] proposed for solving IFDE under the differentiability. This paper presents third order RK methods based AM, and HeM, RM for solving intuitionistic fuzzy IVPs. The efficiency of these methods has been illustrated by numerical example. The RK methods compared with AM and HeM. System of fuzzy differential equation (SFDE) is the one of the most important differential equation for uncertainty modeling.

2. Preliminaries

Definition 1. Let a set X be fixed. An IFS A in X is an object having the form $A = \{\langle x, \mu_A(x), \vartheta_A(x)\rangle 0 : x \in X\}$ where the $\mu_A(x) : X \to [0,1]$ and $\vartheta_A(x) : X \to [0,1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set A which is a a subset of X, for every element of $x \in X, 0 < \mu_A(x), \vartheta_A(x) \leq 1$.

Definition 2. An IFN A is defined as follows:

1. an intuitionistic fuzzy subset of real life,
2. normal, i.e. there is any $x_0 \in R$ such that $\mu_A(x) = 1$ (so $\vartheta_{\tilde{A}^i}(x) = 0$),
3. a convex set for the membership function $\mu_A(x)$, that is, $\mu_A(\lambda x_1 + (1-\lambda x_2)) \geq \min(\mu_A(x_1), \mu_A(x_2))$, $\forall x_1, x_2 \in R, \lambda \in [0,1]$,
4. a concave set for the non-membership function $\vartheta_{\tilde{A}^i}(x)$, that is, $\vartheta_A(\lambda x_1 + (1-\lambda)x_2) \leq \min(\vartheta_A(x_1), \vartheta_A(x_2))$, $\forall x_1, x_2 \in R, \lambda \in [0,1]$.

Definition 3. The α, β- cut of an IFN $A = \{x, \mu_A(x), \vartheta_A(x) :$ $x \in X\}$ is defined as follows:
$A = \{x, \mu_A(x), \vartheta_A(x) : x \in X, \mu_A(x) \geq \alpha$ and $\vartheta_A(x) \leq 1 - \alpha\}$,
$\forall x \in [0, 1]$. The $\alpha-$cut representation of IFN A generates the following pair of intervals and is denoted by
$[a]_\alpha = \{[A_L^+(\alpha), A_U^+(\alpha)], [A_L^-(\beta), A_U^-(\beta)]\}$.

Definition 4. An intuitionistic fuzzy set $A = \{x, \mu_A(x), \vartheta_A(x) : x \in X\}$ such that $\mu_A(x)$ and $(1 - \vartheta_A)(x) = 1 - \vartheta_A(x), \forall x \in R$ are fuzzy numbers, is called an intuitionistic fuzzy number. Therefore IFS $A\{x, \mu_A(x), \vartheta_A(x) :$ $x \in X\}$ is a conjecture of two fuzzy number, A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \vartheta_A(x)$.

Definition 5. A triangular intuitionistic Fuzzy Number ($TIFN$) A is an intuitionistic fuzzy set in R with following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$, given as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, a_2 \leq x \leq a_3 \\ 0, \text{ otherwise} \end{cases}, \quad \vartheta_A(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'}, a_1' \leq x \leq a_2 \\ \dfrac{x - a_2}{a_3' - a_2}, a_2 \leq x \leq a_3' \\ 0, \text{ otherwise} \end{cases},$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$ and $TIFN$ is denoted by
$A = (a_1, a_2, a_3, a_1', a_2', a_3')$. Arithmetic operators over $TIFN$ can be found in [8].

Definition 6. For arbitrary $u, v \in E^n$, the quality
$D(u, v) = \sup -0 \leq 1d([u]^\alpha, [v]^\alpha)$ is the distance between u and v, where d is the Hausdroff metric in E^n.

Definition 7. Let mapping $f : I \to W^n$ for some interval I be an intuitionistic fuzzy function. The α-cut of f is given by

$$[f(t)]_{\alpha,\beta} = \{[\underline{f^+}(f; \alpha), \overline{f^+}(t, \alpha)], [\underline{f^-}(f; \alpha), \overline{f^-}(t, \alpha)], \}$$

where

$$\underline{f^+}(t; \alpha) = \min\{f^+(t; \alpha) | t \in I,\ 0 \leq \alpha \leq 1\},$$
$$\overline{f^+}(t; \alpha) = \max\{f^+(t; \alpha) | t \in I,\ 0 \leq \alpha \leq 1\}.$$

$$\underline{f^-}(t;\alpha) = \min\{f^-(t;\beta)|t \in I,\ 0 \leq \beta \leq 1\},$$
$$\overline{f^-}(t;\alpha) = \max\{f^-(t;\beta)|t \in I,\ 0 \leq \beta \leq 11\}.$$

3. Intuitionistic Fuzzy Cauchy Problem

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$\begin{cases} y'(t) = f(t, y(t)),\ t \in [a,b] \\ y(t_0) = y_0 \end{cases}, \tag{1}$$

where y is an intuitionistic fuzzy function of the crisp variable t, $f(t, y(t))$ is an intuitionistic fuzzy function of the crisp variable t and the intuitionistic fuzzy variable y and y is the intuitionistic fuzzy derivative. If an initial value $y(t_0) = y_0$ intuitionistic fuzzy number, we get an intuitionistic fuzzy Cauchy problem of first order $y'(t) = f(t, f(t)), y(t_0) = y_0$. As each intuitionistic fuzzy number is a conjecture two fuzzy numbers, equation (1) can be replaced by an equivalent system as follows:

$$y'(t) = \{[\underline{y'^+}(t;\alpha), \overline{y'^+}(t;\alpha)], [\underline{y'^-}(t;\alpha), \overline{y'^-}(t;\alpha)]\},$$

where

$$\begin{cases} \underline{y'^+}(t;\alpha) & = \underline{f^+}(t, y^+) = \min\{f^+(t,u)|u \in [\underline{y^+}, \overline{y^+}]\} \\ & = F(t, \underline{y^+}, \overline{y^+}), \underline{y^+}(t_0) = \underline{y_0^+} \end{cases} \tag{2}$$

$$\begin{cases} \underline{y'^+}(t;\alpha) & = \overline{f^+}(t, y^+) = \max\{f^+(t,u)|u \in [\underline{y^+}, \overline{y^+}]\} \\ & = G(t, \underline{y^+}, \overline{y^+}), \overline{y^+}(t_0) = \underline{y_0^+} \end{cases} \tag{3}$$

$$\begin{cases} \underline{y'^-}(t;\alpha) & = \underline{f^-}(t, y^+) = \min\{f^-(t,u)|u \in [\underline{y^-}, \overline{y^-}]\} \\ & = H(t, \underline{y^-}, \overline{y^-}), \underline{y^-}(t_0) = \underline{y_0^+} \end{cases} \tag{4}$$

$$\begin{cases} \overline{y'^-}(t;\alpha) & = \overline{f^+}(t, y^+) = \max\{f^+(t,u)|u \in [\underline{y^+}, \overline{y^+}]\} \\ & = I(t, \underline{y^-}, \overline{y^-}), \overline{y^-}(t_0) = \overline{y_0^-} \end{cases}. \tag{5}$$

The system of equations given in (2) and (3) will have unique solution. $[\underline{y^+}, \overline{y^+}] \in B = \overline{c}[0,1] \times \overline{c}[0,1]$ and the system of equations given in (4) and (5) will leave unique solution, $[\underline{y^-}, \overline{y^-}] \in B$, $B = \overline{c}[0,1] \times \overline{c}[0,1]$. Therefore the system given from equation (2) to (5) possesses unique solution,

$$y(t) = \{[\underline{y^+(t)}, \overline{y^+}(t)], [\underline{y^-(t)}, \overline{y^-}(t)]\} \in B \times B,$$

which is an intuitionistic fuzzy function. That is for each t,

$$y(t;\alpha) = \{[\underline{y}^+(t;\alpha), \overline{y^+}(t;\alpha)], [\underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)]\}, \alpha \in [0,1]$$

is an intuitionistic fuzzy number. The parametric form of the system of equation (2) to (5) is given by

$$\begin{aligned}
\underline{y}^+(t;\alpha) &= F(t, \underline{y}^+(t;\alpha), \overline{y^+}(t;\alpha)), \underline{y}^+(t_0;\alpha) = \underline{y_0^+}(\alpha)\\
\overline{y^+}(t;\alpha) &= G(t, \underline{y}^+(t;\alpha), \overline{y^+}(t;\alpha)), \overline{y^+}(t_0,\alpha) = \underline{y_0^+}(\alpha)\\
\underline{y}^-(t;\alpha) &= H(t, \underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)), \underline{y}^-(t_0;\alpha) = \underline{y_0^+}(\alpha)\\
\overline{y^-}(t;\alpha) &= I(t, \underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)), \overline{y^-}(t_0;\alpha) = \overline{y_0^-}(\alpha)
\end{aligned}$$

for $\alpha \in [0,1]$.

4. Runge-Kutta Methods for Solving Intuitionistic fuzzy Differential Equations

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$y' = f(t,y),\ \ y(t_0) = y_0. \tag{6}$$

The construction of Runge-Kutta methods is based on the variation of constants formula

$$y_{n+1} = y_n + \int_0^h f(t_n + \tau, y(t_n, \tau))d\tau. \tag{7}$$

Then the integral in (7) is approximated by a quadrature formula can be written as

$$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i K_i, \tag{8}$$

$$\text{where,}\ \ K_i = f(t_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij}K_j),\ \ i = 1, 2, \cdots, s. \tag{9}$$

The explicit RK method satisfies

$$\sum_{i=1}^{s} b_i(0) = 1,\ \sum_{j=1}^{s} a_{ij}(0) = c_i,\ i = 1, 2, \cdots, s.$$

The parametric form of the system of intuitionistic fuzzy equations is given by

$$\begin{aligned}
\underline{y}^+(t;\alpha) &= F(t, \underline{y}^+(t;\alpha), \overline{y^+}(t;\alpha)), \underline{y}^+(t_0;\alpha) = \underline{y_0^+}(\alpha),\\
\overline{y^+}(t;\alpha) &= G(t, \underline{y}^+(t;\alpha), \overline{y^+}(t;\alpha)), \overline{y^+}(t_0,\alpha) = \underline{y_0^+}(\alpha),\\
\underline{y}^-(t;\alpha) &= H(t, \underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)), \underline{y}^-(t_0;\alpha) = \underline{\underline{y_0^+}}(\alpha),\\
\overline{y^-}(t;\alpha) &= I(t, \underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)), \overline{y^-}(t_0;\alpha) = \overline{\overline{y_0^-}}(\alpha),
\end{aligned}$$

for $\alpha \in [0,1]$. The unique solution of the system of equation,

$$y(t) = \{[y^+(t), \overline{y^+}(t)], [\underline{y}^-(t), \overline{y^-}(t)]\} \in B \times B$$

which is an intuitionistic fuzzy function of t,

$$y(t;\alpha) = \{[\underline{y}^+(t;\alpha)], [\underline{y}^-(t;\alpha), \overline{y^-}(t;\alpha)]\}$$

is an intuitionistic fuzzy number.

The general form of RK for intuitionistic fuzzy $IVPs$ is

$$\begin{cases}
\underline{\underline{y}}^+(t_{n+1};\alpha) = \underline{\underline{y}}^+(t_n,\alpha) + h\sum_{i=1}^{s} b_i K_i\\
\overline{\overline{y^+}}(t_{n+1};\alpha) = \overline{\overline{y^+}}(t_n;\alpha) + h\sum_{i=1}^{s} b_i L_i\\
\underline{\underline{y}}^-(t_{n+1};\beta) = \underline{\underline{y}}^-(t_n,\beta) + h\sum_{i=1}^{s} b_i M_i\\
\overline{\overline{y^-}}(t_{n+1};\beta) = \overline{\overline{y^-}}(t_n;\beta) + h\sum_{i=1}^{s} b_i N_i
\end{cases}, \tag{10}$$

where

$$\begin{cases}
K_i = F(t_n + c_i h, \underline{\underline{y}}^+(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j K_j)\\
L_i = G(t_n + c_i h, \overline{\overline{y^+}}(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j L_j)\\
K_i = H(t_n + c_i h, \underline{\underline{y}}^+(t_n;\beta) + h\sum_{j=1}^{i-1} a_j M_j)\\
N_i = I(t_n + c_i h, \overline{\overline{y^+}}(t_n;\beta) + h\sum_{j=1}^{i-1} a_j N_j),\ i = 1, 2, \cdots, s
\end{cases}, \tag{11}$$

where $b_i's$ and $c_i's$ are constants.

4.1. RK3 for Intuitionistic Fuzzy IVP

The RK3 formula for intuitionistic fuzzy IVP is given by

$$\begin{cases} \underline{y^+}(t_{n+1};\alpha) & = \underline{y^+}(t_n;\alpha) + \frac{h}{2}\sum_{i=1}^{2}\frac{K_i + K_{i+1}}{2} \\ & = \underline{y^+}(t_n;\alpha) + \frac{h}{2}\sum_{i=1}^{2}(AM) \\ \overline{y^+}(t_{n+1};\alpha) & = \overline{y^+}(t_n;\alpha) + \frac{h}{2}\sum_{i=1}^{2}\frac{L_i + L_{i+1}}{2} \\ & = \overline{y^+}(t_n;\alpha) + \frac{h}{2}\sum_{i=1}^{2}(AM) \\ \overline{y^-}(t_{n+1};\beta) & = \overline{y^+}(t_n;\beta) + \frac{h}{2}\sum_{i=1}^{2}\frac{M_i + M_{i+1}}{2} \\ & = \overline{y^-}(t_n;\beta) + \frac{h}{2}\sum_{i=1}^{2}(AM) \\ \underline{y^-}(t_{n+1};\beta) & = \underline{y^-}(t_n;\beta) + \frac{h}{2}\sum_{i=1}^{2}\frac{N_i + N_{i+1}}{2} \\ & = \underline{y^-}(t_n;\beta) + \frac{h}{2}\sum_{i=1}^{2}(AM) \end{cases}, \tag{12}$$

$$\begin{cases} K_i = F(t_n + c_i h, \underline{y^+}(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j K_j) \\ L_i = G(t_n + c_i h, \overline{y^+}(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j L_j) \\ K_i = H(t_n + c_i h, \underline{y^+}(t_n;\beta) + h\sum_{j=1}^{i-1} a_j M_j) \\ N_i = I(t_n + c_i h, \overline{y^+}(t_n;\beta) + h\sum_{j=1}^{i-1} a_j N_j), \;\; i = 1, 2, \cdots, s \end{cases}. \tag{13}$$

The classical third order RK formula is given by

$$\underline{y^+}(t_{n+1};\alpha) = \underline{y^+}(t_n;\alpha) + \frac{h}{4}[K_1 + 2K_2 + K_3],$$

$$\overline{y^+}(t_{n+1};\alpha) = \overline{y^+}(t_n;\alpha) + \frac{h}{4}[I_1 + 2I_2 + I_3].$$

$$\underline{y^-}(t_{n+1},\alpha) = \underline{y^-}(t_n;\alpha) + \frac{h}{4}[M_1 + 2M_2 + M_3],$$

$$\overline{y^+}(t_{n+1};\alpha) = \overline{y^-}(t_n;\alpha) + \frac{h}{4}[N_1 + 2N_2 + N_3].$$

The value of AM can be replaced by HeM:

$$HeM = \frac{2(AM) + (GM)}{3},$$

formula is given by

$$\begin{cases} \underline{y^+}(t_{n+1};\alpha) & = \underline{y^+}(t_n;\alpha) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(K_i + K_{i+1}\sqrt{K_i K_{i+1}}) \\ & = \underline{y^+}(t_n;\alpha) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(HeM) \\ \overline{y^+}(t_{n+1};\alpha) & = \underline{y^+}(t_n;\alpha) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(L_i + L_{i+1}\sqrt{L_i L_{i+1}}) \\ & = \overline{y^+}(t_n;\alpha) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(HeM) \\ \underline{y^-}(t_{n+1};\beta) & = \underline{y^-}(t_n;\beta) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(M_i + M_{i+1}\sqrt{M_i M_{i+1}}) \\ & = \underline{y^-}(t_n;\beta) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(HeM) \\ \overline{y^-}(t_{n+1};\beta) & = \underline{y^-}(t_n;\beta) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(N_i + N_{i+1}\sqrt{N_i N_{i+1}}) \\ & = \underline{y^-}(t_n;\beta) + \dfrac{h}{4}\displaystyle\sum_{i=1}^{2}(HeM) \end{cases},$$

where

$$\begin{cases} K_i = F(t_n + c_i h), \underline{y^+}(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j K_j \\ L_i = G(t_n + c_i h), \overline{y^+}(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j L_j \\ M_i = H(t_n + c_i h), \underline{y^+}(t_n;\beta) + h\sum_{j=1}^{i-1} a_j M_j \\ N_i = I(t_n + c_i h), \overline{y^+}(t_n;\beta) + h\sum_{j=1}^{i-1} a_j N_j \end{cases}. \tag{14}$$

The missing elements in the matrix $A = (a_{ij}) = (a_{ij}), i, j = 1, 2, 3, \cdots$, are defined to be zero. The values a of the parameters a_j for the above of means are listed in [11, 12].

5. Convergence of Fuzzy Exponential Runge-Kutta Methods

The solution is obtained by grid points at

$$a = t_0 \leq t_1 \leq \cdots \leq t_N = b \text{ and } h = \frac{b-a}{N} = t_{n+1} - t_n. \tag{15}$$

We define

$$\begin{cases} F[t_n, y(t_n;\alpha)] = \sum_{i=1}^{s} b_i K_i(t_n, y(t_n;\alpha)) \\ G[t_n, y(t_n;\alpha)] = \sum_{i=1}^{s} b_i L_i(t_n, y(t_n,;\alpha)) \\ H[t_n, y(t_n;\beta)] = \sum_{i=1}^{s} b_i M_i(t_n, y(t_n;\beta)) \\ I[t_n, y(t_n;\beta)] = \sum_{i=1}^{s} b_i N_i(t_n, y(t_n,;\beta)) \end{cases}. \tag{16}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted respectively by

$$[Y(t_n)]_{\alpha,\beta} = [\underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha), \underline{Y^-}(t_n;\beta), \overline{Y^-}(t_n;\beta)]$$

and

$$y(t_n)]_{\alpha,\beta} = [\underline{y^+}(t_n;\alpha), \overline{y^+}(t_n;\alpha), \underline{y^-}(t_n;\beta), \overline{y^-}(t_n;\beta)].$$

We have

$$\begin{aligned} \underline{Y^+}(t_{n+1};\alpha) &\approx \underline{Y^+}(t_n;\alpha) + hF[t_n, \underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha)], \\ \overline{Y^+}(t_{n+1};\alpha) &\approx \overline{Y^+}(t_n;\alpha) + hG[t_n, \underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha)], \\ \underline{Y^-}(t_{n+1};\beta) &\approx \underline{Y^-}(t_n;\beta) + hH[t_n, \underline{Y^+}(t_n;\beta), \overline{Y^-}(t_n;\beta)], \\ \overline{Y^-}(t_{n+1};\beta) &\approx \overline{Y^-}(t_n;\beta) + hI[t_n, \underline{Y^-}(t_n;\beta), \overline{Y^-}(t_n;\beta)], \end{aligned}$$

and

$$\begin{aligned} \underline{y^+}(t_{n+1};\alpha) &= \underline{y^+}(t_n;\alpha) + hF[t_n, \underline{y^+}(t_n;\alpha), \overline{y^+}(t_n;\alpha)], \\ \overline{Y^+}(t_{n+1};\alpha) &= \overline{y^+}(t_n;\alpha) + hG[t_n, \underline{y^+}(t_n;\alpha), \overline{y^+}(t_n;\alpha)], \\ \underline{Y^-}(t_{n+1};\beta) &= \underline{y^-}(t_n;\beta) + hH[t_n, \underline{y^+}(t_n;\beta), \overline{y^-}(t_n;\beta)], \\ \overline{Y^-}(t_{n+1};\beta) &= \overline{y^-}(t_n;\beta) + hI[t_n, \underline{y^-}(t_n;\beta), \overline{y^-}(t_n;\beta)]. \end{aligned}$$

We need the following corollaries to show the convergence of these approximates, that is,$\underline{y^+}(t_n;\alpha), \overline{y^+}(t_n;\alpha), \underline{y^-}(t_n;\alpha)$ and $\overline{y^-}(t_n;\alpha)$ converges to $\underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha), \underline{Y^-}(t_n;\alpha)$ and $\overline{Y^-}(t_n;\alpha)$ respectively whenever $h \longrightarrow 0$.

Corollary 8. *Let the sequence of numbers* $\{W_n^+\}_{n=0}^N, \{W_n^-\}_{n=0}^N$ *satisfy* $|W_{n+1}| \leq A|W_n| + B,\ 0 \leq n \leq N-1$ *for some given positive constants* A *and* B. *Then* $|W_n| \leq A^n|W_0| + B\dfrac{A^n - 1}{A-1}$.

Corollary 9. *Let the sequence of numbers* $\{W_n\}_{n=0}^N,\ \{V_n\}_{n=0}^N$ *satisfy* $|W_{n+1}| \leq |W_n| + A.\max\{|W_n||V_n|\} + B$, $|V_{n+1}| \leq |V_n| + A.\max\{|W_n||V_n|\} + B$, *for some given positive constants* A *and* B *and denoted,* $U_n = |W_n||VV_n|,\ 0 \leq n \leq N$. *Then* $U_n \leq \overline{A}^n U_0 + \overline{B}\dfrac{\overline{A}^n - 1}{\overline{A} - 1},\ 0\ \leq n \leq N$ *where* $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$. *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *are obtained by substituting* $[y(t)]_r = [u,v]$ *in* (2). *The domain where* F *and* G *are defined is therefore* $K = \{(t,u,v) : 0 \leq t \leq T, -\infty < v < \infty,\ -\infty < u < v\}$.

Theorem 10. *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *belong to* $c^P(K)$ *and let the partial derivatives of* F *and* G *be bounded over* K. *Then for arbitrary fixed* $\alpha, \beta : 0 \leq \alpha, \beta \leq 1$, *the approximate solution of* (6),

$$[\underline{y^+}(t_n;\alpha), \overline{y^+}(t_n;\alpha), \underline{y^-}(t_n;\alpha), \overline{y^-}(t_n;\alpha)]$$

converges to the exact solution $[\underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha), \underline{Y^-}(t_n;\alpha), \overline{Y^-}(t_n;\alpha)]$.

Proof. By using Taylor's theorem,

$$\begin{aligned}
\underline{Y^+}(t_{n+1};\alpha) &= \underline{Y^+}(t_n;\alpha) + hF(t_n, \underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha)) \\
&\quad + \frac{h^{p+1}c_j^p}{(p+1)!}\underline{Y^+}^{(p+1)}(\xi_{n,1}), \\
\overline{Y^+}(t_{n+1};\alpha) &= \overline{Y^+}(t_n;\alpha) + hG(t_n, \underline{Y^+}(t_n;\alpha), \overline{Y^+}(t_n;\alpha)) \\
&\quad + \frac{h^{p+1}c_j^p}{(p+1)!}\overline{Y^+}^{(p+1)}(\xi_{n,2}),
\end{aligned}$$

$$\begin{aligned}
\underline{Y^-}(t_{n+1};\alpha) &= \underline{Y^-}(t_n;\alpha) + hF(t_n, \underline{Y^-}(t_n;\alpha), \overline{Y^-}(t_n;\alpha)) \\
&\quad + \frac{h^{p+1}c_j^p}{(p+1)!}\underline{Y^-}^{(p+1)}(\xi_{n,3}), \\
\overline{Y^-}(t_{n+1};\alpha) &= \overline{Y^-}(t_n;\alpha) + hG(t_n, \underline{Y^-}(t_n;\alpha), \overline{Y^-}(t_n;\alpha)) \\
&\quad + \frac{h^{p+1}c_j^p}{(p+1)!}\overline{Y^-}^{(p+1)}(\xi_{n,4}),
\end{aligned}$$

where $\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \xi_{n,4} \in (t_n, t_{n+1})$. Now, if we denote

$$\begin{aligned} W_n^+ &= \underline{Y^+}(t_n,\alpha) - \underline{y^+}(t_n,\alpha), \\ V_n^+ &= \overline{Y^+}(t_n,\alpha) - \overline{y^+}(t_n,\alpha), \\ W_n^- &= \underline{Y^-}(t_n,\alpha) - \underline{y^-}(t_n,\alpha), \\ V_n^- &= \overline{Y^-}(t_n,\alpha) - \overline{y^-}(t_n,\alpha), \end{aligned}$$

then the above two expressions converted. Hence we can write

$$\begin{aligned} |W_{n+1}^+| &\le |W_n^+| + 2Lh\max(|W_n^+|,|V_n^+|) + \frac{h^{p+1}}{(p+1)!}M_1, \\ |V_{n+1}^+| &\le |V_n^+| + 2Lh\max(|W_n^+|,|V_n^+|) + \frac{h^{p+1}}{(p+1)!}M_1, \\ |W_{n+1}^-| &\le |W_n^-| + 2Lh\max(|W_n^-|,|V_n^-|) + \frac{h^{p+1}}{(p+1)!}M_2, \\ |V_{n+1}^-| &\le |V_n^-| + 2Lh\max(|W_n^-|,|V_n^-|) + \frac{h^{p+1}}{(p+1)!}M_2, \end{aligned}$$

where

$$\begin{aligned} M_1 &= \max\{\max|\underline{Y^+}^{(p+1)}(t,\alpha)|, \max|\overline{Y^+}^{(p+1)}(t,\alpha)|\}, \\ M_2 &= \max\{\max|\underline{Y^-}(p+1)(t,\alpha)|, \max|\overline{Y^-}^{(p+1)}(t,\alpha)|\}, \end{aligned}$$

for $t \in [0,T]$, and $L > 0$ is a bound from the partial derivative of F and G. Therefore we can write

$$\begin{aligned} |W_n^+| &\le (1+4Lh)^n|U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M_1)\frac{(1+4Lh)^n-1}{4Lh}, \\ |V_n^+| &\le (1+4Lh)^n|U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M_1)\frac{(1+4Lh)^n-1}{4Lh}, \\ |W_n^-| &\le (1+4Lh)^n|U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M_2)\frac{(1+4Lh)^n-1}{4Lh}, \\ |V_n^-| &\le (1+4Lh)^n|U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M_2)\frac{(1+4Lh)^n-1}{4Lh}, \end{aligned}$$

where $|U_0^+| = |W_n^+| + |V_0^+|$ and $|U_0^-| = |W_n^-| + |V_0^-|$.

In particular,

$$|W_n^+| \le (1+4Lh)^N |U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M)\frac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

$$|V_n^+| \le (1+4Lh)^N |U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M)\frac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

$$|W_n^-| \le (1+4Lh)^N |U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M)\frac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

$$|V_n^-| \le (1+4Lh)^N |U_0^+| + (\frac{2h^{p+1}}{(p+1)!}M)\frac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

where $M = \max\{M_1, M_2\}$ for $t \in [0, T]$.

Since $W_0^+ = V_0^+ = W_0^- = V_0^- = 0$, we have

$$|W_n^+| \le M\frac{(e^{4LT}-1)}{2L(p+1)!}h^p,$$

$$|V_n^+| \le M\frac{(e^{4LT}-1)}{2L(p+1)!}h^p,$$

$$|W_n^-| \le M\frac{(e^{4LT}-1)}{2L(p+1)!}h^p,$$

$$|V_n^-| \le M\frac{(e^{4LT}-1)}{2L(p+1)!}h^p.$$

Thus if $h \longrightarrow 0$ we get $W_n \longrightarrow 0$, and $V_n \longrightarrow 0$, which complete the proof. □

Numerical Examples

Oil Production Model:

In the world rate of increase oil production y in million metric tons per year was assumed to be proportional to y itself. Then what is the amount of oil after five years initially (1061, 1091, 1131; 1066, 1091, 1111) million metric ton oil (the constant of proportionality is 0.084).

Solution: $\frac{dy}{dt} = ky$ with $k = 0.084$ and $y_0 = (1061, 1091, 1131; 1066, 1091, 1111)$ million.

α, β	Absolute Error for $IFRK3\ AM$ at $t = 1$			
	$\underline{y^+}(t;\alpha)$	$\overline{y^+}(t;\alpha)$	$\underline{y^-}(t;\beta)$	$\overline{y^-}(t;\beta)$
0.00	2.392e-009	2.551e-009	2.403e-009	2.504e-009
0.20	2.406e-009	2.532e-009	2.414e-009	2.496e-009
0.40	2.419e-009	2.515e-009	2.427e-009	2.487e-009
0.60	2.432e-009	2.496e-009	2.437e-009	2.479e-009
0.80	2.447e-009	2.479e-009	2.449e-009	2.469e-009
1.00	2.460e-009	2.460e-009	2.460e-009	2.460e-009

Table 1: (Arithmetic Mean)

α, β	Absolute Error for $IFRK3\ HeM$ at $t = 1$			
	$\underline{y^+}(t;\alpha)$	$\overline{y^+}(t;\alpha)$	$\underline{y^-}(t;\beta)$	$\overline{y^-}(t;\beta)$
0.00	2.216e-009	2.363e-009	2.225e-009	2.320e-009
0.20	2.228e-009	2.345e-009	2.237e-009	2.311e-009
0.40	2.241e-009	2.328e-009	2.247e-009	2.302e-009
0.60	2.254e-009	2.311e-009	2.257e-009	2.296e-009
0.80	2.265e-009	2.296e-009	2.267e-009	2.285e-009
1.00	2.278e-009	2.278e-009	2.278e-009	2.278e-009

Table 2: (Heronian Mean)

The exact solution is given by

$$\begin{cases} \underline{\underline{y^+}}(t;\alpha) & = (1061 + 30\alpha)e^{kt} \\ \overline{\overline{y^+}}(t;\alpha) & = (1131 - 40\alpha)e^{kt} \\ \underline{\underline{y^-}}(t;\beta) & = (1091 - 25\alpha)e^{kt} \\ \overline{\overline{y^-}}(t;\beta) & = (1091 + 30\alpha)e^{kt} \end{cases}.$$

The Error results for this example at $t = 1$ and $(\alpha, \beta) = 1$ are shown in Table 1 and Table 2. The solution graphs are given in Figure 1 and Figure 2.

6. Conclusion

In this paper, third order Runge-Kutta methods have been constructed to solve the intuitionistic fuzzy IVPs. The effectiveness of these methods has been illustrated via examples of intuitionistic fuzzy IVPs. The error results have been

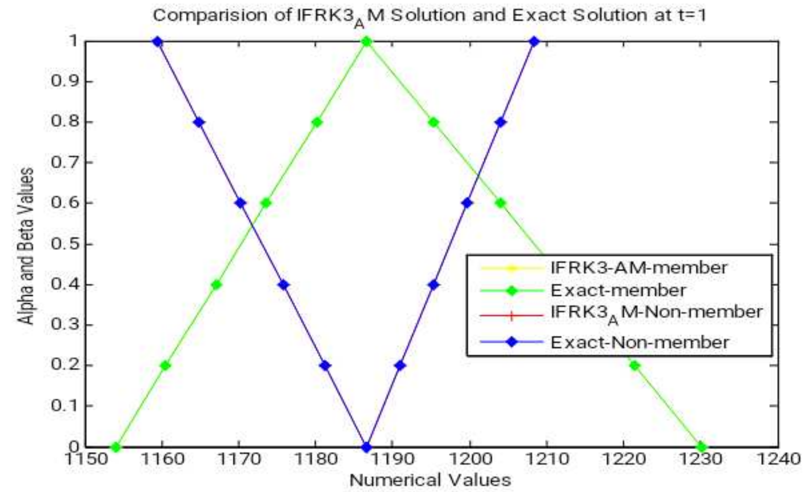

Figure 1: (Arithmetic Mean)

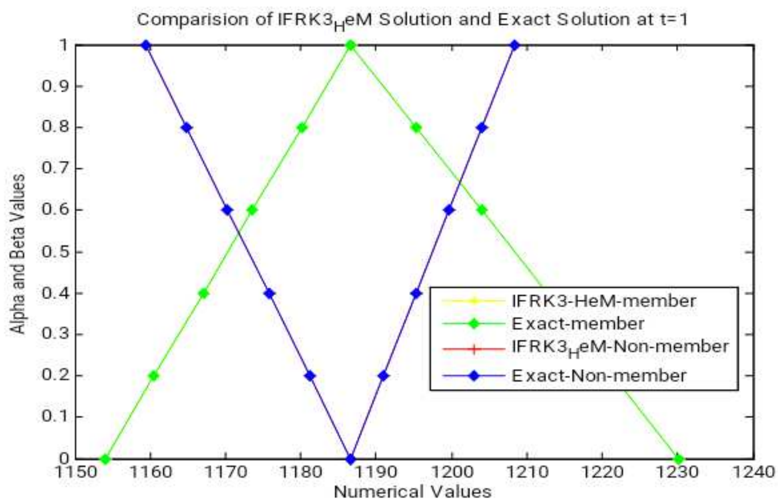

Figure 2: (Heronian Mean)

compared with the AM to HeM. From these tables, we compared with HeM better than good accuracy AM that the absolute error is negligible small. Hence, we have observed that the RK method is suitable for solving intuitionistic fuzzy IVPs.

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