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PERMUTABLE SUBGROUPS OF GROUPS OF ORDER 16

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Abstract: A subgroup H of a group G is said to be permutable subgroup if and only if $HK = KH$ for every subgroup K of G. Certainly, every normal subgroup is permutable. The converse is not true. In this research we will find all permutable subgroups of the groups of order 16. Then, find which subgroup is permutable and not normal.

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Key Words: permutable subgroup, normal subgroup, group centre, factor group

1. Introduction

A subgroup H of a finite group G is permutable (quasinormal) if and only if $HK = KH = \langle H, K\rangle$ for all $K \leq G$.

Permutable subgroups of finite groups have certain properties, such as; any permutable subgroup of finite group is nilpotent module their core [6]. S.E. Stonehewer in [4], proved that a permutable subgroup is locally subnormal. In [1] the authors developed several local approaches for a classes of finite groups which called PT-groups (permutability is a transitive relation). In [5], J. Evan considered permutability within a direct product of finite groups.

Clearly, if H is normal subgroup of a finite group G, then $HK = KH$ for all subgroup K of G. The converse is not always true. That is, the current

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work has been considered to construct the permutable subgroups structure of groups of order 16, which leads to the fact that not all permutable subgroups of finite groups are normal. This will be shown by a counterexample.

2. Notations and preliminaries

For a finite group G, the centre of G will be denoted by $Z(G)$. The order of the group will be denoted by $|G|$. If G is a p-group ($|G|$ is a power of prime p), then $Z(G)$ is a non-trivial subgroup of G. So, for a group of order $16 = 2^4$ the order of the group centre will be $2, 2^2, 2^3$ or 2^4. Therefore, the factor group $G/Z(G)$ will be of order $\dfrac{|G|}{|Z(G)|} = 2^3, 2^2, 2$ or 1.

We deduce the properties of the group for various cases using the Correspondence Theorem, which indicates that, for a normal subgroup N of a group G, the structure of the subgroups of the factor group is exactly the same as the structure of the subgroups containing N, with N collapsed to the identity element, this is shown in the next theorem.

Theorem 1. ([7]) *If N is a normal subgroup of a group G, let $\mathfrak{S}$ be the set of all subgroups A of G such that $N \subseteq A \subseteq G$ and $\mathfrak{F}$ be the set of all subgroups of G/N. Then there is a bijection map $\phi : \mathfrak{S} \to \mathfrak{F}$ such that $\phi(A) = A/N$ for all $A \in \mathfrak{S}$. Furthermore, the normal subgroups in $\mathfrak{S}$ correspond to normal subgroups in $\mathfrak{F}$.*

In order to describe a finite group G of generators set $\{a_1, a_2, \cdots, a_n\}$, the next representation which found in [3] will be used:

$$\begin{aligned} G = \{a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \cdots a_n^{\alpha_n} \mid \quad & a_1^{\beta_1} = a_2^{\beta_2} = a_3^{\beta_3} = \cdots = a_n^{\beta_n} = e, \\ & a_2 a_1 = a_1 a_2 a_{1,2}, \\ & a_3 a_2 = a_2 a_3 a_{2,3}, \\ & \vdots \\ & a_n a_{n-1} = a_{n-1} a_n a_{n-1,n} \} . \end{aligned}$$

The preceding representation shows clearly the elements of G in terms of generators and the orders of these generators. Also it describes how the generators commute, so we can condense a string of the elements into the form used in this presentation.

3. Classifying all groups of order 16

The classification of all 2-generators non-abelian groups of order 2^n, $n \geq 4$ is found in [2]. To classify both cases abelian and non-abelian groups of order 16, we will use David Clausen method [3], which based on considering the different cases for the order of the centre. Since $Z(G)$ is a nontrivial subgroup (G is p-group), then the possibilities of $|Z(G)|$ are 16, 8, 4, and 2. Therefore, $|G/Z(G)| = 1, 2, 4$ or 8.

The next tools will be used to complete our classification.

Definition 2. Denote the subgroups of order 8 in a group G containing $Z(G)$ by G_i where i indexes the subgroup. Also, unless specified z denotes an element of $Z(G)$, and g_i denotes an element of G_i not in $Z(G)$.

Remark 1. ([3]) The centre of G_i contains the centre of G, i.e. $Z(G) \subseteq Z(G_i)$, also The intersection of two G_i must be a group of order 4.

Note that, if H and K be two subgroups of G. Then the subset $HK = \{hk \mid h \in H,\ k \in K\}$ has order $|HK| = \dfrac{|H||K|}{|H \cap K|}$.

Theorem 3. ([3]) *If $|Z(G)| = 2$ and G_i contains only one subgroup of order 4 with the centre, then G_i is either isomorphic to $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$.*

Theorem 4. ([3]) *If $G/Z(G) \cong D_4$, then one of the subgroups G_i is cyclic.*

Theorem 5. ([3]) *If G has two G_i that are abelian (could also be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$), then the centre has order at least 4.*

For a non-abelian group G of order 16, the group centre $Z(G)$ is of order 8,4,2 or 1. (the case that $|Z(G)| = 1$ will omitted, since G is a p-group). So, we have the following cases:

Case(1): $|Z(G)| = 8$.

Let G be a non-abelian group of order 16. If $|Z(G)| = 8$, then $|G/Z(G)| = 2$. Implies that $G/Z(G)$ is cyclic. Which indicates that G is abelian, therefore $Z(G) = G$, which is a contradiction. Hence, no non-abelian group of order 16 have a centre of order 8.

Case(2): $|Z(G)| = 4$.

Let G be a non-abelian group of order 16. If $|Z(G)| = 4$, then $|G/Z(G)| = 4$. Thus

$$G/Z(G) \cong \mathbb{Z}_4 \text{ or } G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

If $G/Z(G) \cong \mathbb{Z}_4$ which is cyclic. Thus G is abelian, which is a contradiction. Hence $G/Z(G) \ncong \mathbb{Z}_4$.

If $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has three subgroups of order 2, and by the Correspondence Theorem, there are three G_i of order 8, where the intersection of any two G_i is the centre, and it is a subgroup of order 4. Certainly, for $|Z(G)| = 4$ the only possibilities of G_i are that all G_i are abelian, since the centre of G_i would have at least four elements $|Z(Gi)| \geq 4$. So, we have 3 abelian subgroups G_i of order 8, where each G_i has at least one subgroup of order 4. This implies that each G_i isomorphic to one of the following: $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$. Since the centre is of order 4, it is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. This will be considered by the next two cases:

(a) If $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and each G_i contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since all Gi are abelian, and $\mathbb{Z}_8$ contains no subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the only possible isomorphism classes for the three G_i are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$. For the subgroups G_i , $i = 1, 2, 3$, see the following possibilities:

1. $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for $i = 1, 2, 3$, which gives no non-abelian group G of order 16 with this classification.
2. $G_1, G_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $G \cong D_4 \times \mathbb{Z}_2$.
3. $G_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$.
4. Finally, If $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, $i = 1, 2, 3$, then $G \cong Q_8 \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$.

(b) Each G_i contains a subgroup isomorphic to $\mathbb{Z}_4$. Since all the G_i are abelian, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has no subgroup isomorphic to $\mathbb{Z}_4$, so the only possible isomorphism classes are $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$. See the next four possibilities:

1. $G_1, G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Then, the group is Pauli matrices $(SU(2))$.
2. $G_1, G_2 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $G_3 \cong \mathbb{Z}_8$. Then, there are no groups with this property.

3. $G_1 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $G_2, G_3 \cong \mathbb{Z}_8$. Then, the group G is Isanowa or Modular Group of order 16 (M_{16}).
4. $G_1, G_2, G_3 \cong \mathbb{Z}_8$. Then, there are no groups with this property.

Case(3): $|Z(G)| = 2$.

Let G be a non-abelian group of order 16, if $|Z(G)| = 2$, then $|G/Z(G)| = 8$. So we have:

1. $G/Z(G)$ is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or Q_8. Implies that, there are no group G with this property.
2. Let $G/Z(G) \cong D_4$. In this case we can indicate that $G_1 \cong \mathbb{Z}_8$, and G_2 and G_3 are isomorphic to Q_8 or D_4.

Let $G_1 \cong \mathbb{Z}_8$. Using Theorem 3 and Theorem 4, gives the next scenarios:

1. If $G_2, G_3 \cong Q_8$, then G is dicyclic group of degree 4 (Dic_4).
2. If $G_2 \cong Q_8, G_3 \cong D_4$, then G is semidihedral group of degree 2 (SD_2).
3. If $G_2, G_3 \cong D_4$, then G is dihedral group of degree 8 (D_8).

4. Permutable subgroups structure of groups of order 16

In this section we will use GAP (Groups, Algorithms and Programming) to find all subgroup structure of groups of order 16. The next GAP's code will be used to find our results:

Algorithm 4.1 GAP code used to find all subgroups of group of order 16

```
gap> Y:=AllGroups(16);;n:=Size(Y);;
gap> for i in [1..n] do;
> Print(StructureDescription(Y[i]));
> S:=AllSubgroups(Y[i]);k:=Size(S);
> for j in [1..k] do;
> Print("\n S=",StructureDescription(S[j]),"\n Size(S)",Size(S[j]),"\n Normal:
",IsNormal(Y[i],S[j]),"\n Permutable:  ",IsPermutable(Y[i],S[j]),"\n\n");
> od;od;
```

The data in Table 1 are deduced from the results obtained by Algorithm 4.1.

Groups	Number of permutable subgroups	Permutable subgroup
$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$ or Small Group(16, 3)	11	Trivial group $\langle b\rangle \cong \mathbb{Z}_2$ (It is the commutator subgroup) $\langle a^2\rangle \cong \mathbb{Z}_2$ $\langle a^2 b\rangle \cong \mathbb{Z}_2$ $\langle a^2, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (It is the centre of the group) $\langle b, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\langle b, a^2 c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\langle a, b\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ $\langle ac, b\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ $\langle a^2, b, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ The whole group
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	11	Trivial group $\{e, x^2\} \cong \mathbb{Z}_2$ (It is the commutator subgroup) $\{e, x^2y^2\} \cong \mathbb{Z}_2$ $\{e, y^2\} \cong \mathbb{Z}_2$ $\{e, x^2, y^2, x^2y^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (It is the centre of the group) $\langle x\rangle \cong \mathbb{Z}_4$ $\langle xy^2\rangle \cong \mathbb{Z}_4$ $\langle x, y^2\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ $\langle x^2, xy\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ $\langle x^2, y\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ The whole group
M_{16}	11	Trivial group $\{e, a^4\} \cong \mathbb{Z}_2$ (It is the commutator subgroup) $\{e, x\} \cong \mathbb{Z}_2$ (not normal) $\{e, a^4x\} \cong \mathbb{Z}_2$(not normal) $\{e, a^2, a^4, a^6\} \cong \mathbb{Z}_4$ (It is the centre of the group) $\{e, a^2x, a^4, a^6x\} \cong \mathbb{Z}_4$ $\{e, x, a^4, a^4x\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\{e, ax, a^6, a^7x, a^4, a^5x, a^2, a^3x\} \cong \mathbb{Z}_8$ $\langle a\rangle \cong \mathbb{Z}_8$ $\{e, a^2, a^4, a^6, x, a^2x, a^4x, a^6x\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ The whole group

Table 1: Permutable subgroups of non-abelian groups of order 16.

5. Example of permutable subgroup which is not normal for group of order 16

Let $G = M_{16}$, the group G is defined as follows:

$$M_{16} = \left\langle a, x \mid a^8 = x^2 = e\ ,\ xax^{-1} = a^5 \right\rangle .$$

The elements of G are

$$\left\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, x, ax, a^2x, a^3x, a^4x, a^5x, a^6x, a^7x\right\} . \tag{1}$$

The following table represents all subgroups of M_{16}:

Remark 2. All the preceding subgroups of M_{16} in Table 2 are permutable and normal subgroups except the subgroups $H_1 = \{e, x\}$ and $H_2 = \{e, a^4x\}$, which are permutable but not normal.

D_{16}	7	Trivial group
		$\{e, a^4\} \cong \mathbb{Z}_2$ (It is the centre of the group)
		$\{e, a^2, a^4, a^6\} \cong \mathbb{Z}_4$ (It is the commutator subgroup)
		$\langle a \rangle \cong \mathbb{Z}_8$
		$\langle a^2, x \rangle \cong D_8$
		$\langle a^2, ax \rangle \cong D_8$
		The whole group
SD_{16}	7	Trivial group
		$\{e, a^4\} \cong \mathbb{Z}_2$ (It is the centre of the group)
		$\{e, a^2, a^4, a^6\} \cong \mathbb{Z}_4$ (It is the commutator subgroup)
		$\langle a \rangle \cong \mathbb{Z}_8$
		$\langle a^2, x \rangle \cong D_8$
		$\langle a^2, ax \rangle \cong Q_8$
		The whole group
Q_{16}	7	Trivial group
		$\langle z = a^4 = b^2 = c^2 \rangle \cong \mathbb{Z}_2$ (It is the centre of the group)
		$\langle a^2 \rangle \cong \mathbb{Z}_4$ (It is the commutator subgroup)
		$\langle a \rangle \cong \mathbb{Z}_8$
		$\langle a^2, b \rangle \cong Q_8$
		$\langle a^2, ab \rangle \cong Q_8$
		The whole group
$D_8 \times \mathbb{Z}_2$	19	Trivial group
		$\langle y \rangle \cong \mathbb{Z}_2$
		$\langle a^2 y \rangle \cong \mathbb{Z}_2$
		$\langle a^2 \rangle \cong \mathbb{Z}_2$ (It is the commutator subgroup)
		$\langle a^2, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (It is the centre of the group)
		$\langle a^2, x \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, ax \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, xy \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, axy \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a \rangle \cong \mathbb{Z}_4$
		$\langle ay \rangle \cong \mathbb{Z}_4$
		$\langle a^2, x, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, ax, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a, x \rangle \cong D_8$
		$\langle a, xy \rangle \cong D_8$
		$\langle ay, x \rangle \cong D_8$
		$\langle ay, xy \rangle \cong D_8$
		$\langle a, y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$
		The whole group
$Q_8 \times \mathbb{Z}_2$	7	All subgroups are permutable and normal
Central product of D_8 and $\mathbb{Z}_4$ $SU(2)$	17	Trivial group
		$\langle a^2 \rangle \cong \mathbb{Z}_2$ (It is the commutator subgroup)
		$\langle y \rangle \cong \mathbb{Z}_4$ (It is the centre of the group)
		$\langle a \rangle \cong \mathbb{Z}_4$
		$\langle xy \rangle \cong \mathbb{Z}_4$
		$\langle axy \rangle \cong \mathbb{Z}_4$
		$\langle a^2, x \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, ax \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a^2, ay \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\langle a, xy \rangle \cong Q_8$
		$\langle a, y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$
		$\langle x, y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$
		$\langle ax, y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$
		$\langle a, x \rangle \cong D_8$
		$\langle xy, ay \rangle \cong D_8$
		$\langle axy, ay \rangle \cong D_8$
		The whole group

Table 1: (Continuation) Permutable subgroups of non-abelian groups of order 16.

Next, we will prove that H_1 and H_2 are permutable subgroup of M_{16} but not normal.

List of subgroups	Isomorphism class	Order of subgroups	Index of subgroups
$K_1 = \{e\}$	trivial group	1	16
$K_2 = \{e, a^4\}$	$\mathbb{Z}_2$	2	8
$H_1 = \{e, x\}$, $H_2 = \{e, a^4x\}$	$\mathbb{Z}_2$	2	8
$K_3 = \{e, a^2, a^4, a^6\}$	$\mathbb{Z}_4$	4	4
$K_4 = \{e, a^2x, a^4, a^6x\}$	$\mathbb{Z}_4$	4	4
$K_5 = \{e, x, a^4, a^4x\}$	Klein four-group	4	4
$K_6 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$ $K_7 = \{e, ax, a^6, a^7x, a^4, a^5x, a^2, a^3x\}$	$\mathbb{Z}_8$	8	2
$K_8 = \{e, a^2, a^4, a^6, x, a^2x, a^4x, a^6x\}$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8	2
M_{16}	M_{16}	16	1

Table 2: The subgroups of M_{16}.

Proof. For $H_1 = \{e, x\}$. Consider Equation (1), as $a \in M_{16}$ and $a^{-1} = a^7$. Then,

$$\begin{aligned} axa^{-1} &= axa^7 \\ &= aa^5xa^6 \\ &= a^6a^5xa^5 \\ &= a^3a^5xa^4 \\ &= exa^4 \\ &= xa^4 \\ &= a^4x \notin H_1. \end{aligned}$$

Hence, H_1 is not normal subgroup of M_{16}.

To show that H_1 is permutable, we have

$$\begin{aligned} &K_1H_1 = H_1K_1 = \{e, x\} = H_1 \\ &K_2H_1 = H_1K_2 = \{e, x, a^4, a^4x\} = K_5 \\ &K_3H_1 = H_1K_3 = \{e, a^2, a^4, a^6, x, a^2x, a^4x, a^6x\} = K_8 \\ &K_4H_1 = H_1K_4 = \{e, a^2, a^4, a^6, x, a^2x, , a^4x, a^6x\} = K_8 \\ &K_5H_1 = H_1K_5 = \{e, x, a^4, a^4x\} = K_5 \end{aligned}$$

$$K_6H_1 = H_1K_6 = M_{16}$$
$$K_7H_1 = H_1K_7 = M_{16}$$
$$K_8H_1 = H_1K_8 = \{e, a^2, a^4, a^6x, a^2x, a^4x, a^6x\} = K_8$$
$$H_1H_2 = H_2H_1 = K_5.$$

Therefore, H_1 is permutable and not normal subgroup of M_{16}.

For $H_2 = \{e, a^4x\}$. Consider Equation (1), as $a \in M_{16}$ and $a^{-1} = a^7$, then

$$\begin{aligned} aa^4xa^{-1} &= a^5xa^7 \\ &= a^5xaa^6 \\ &= a^5a^5xa^6 \\ &= a^2xaa^5 \\ &= a^2a^5xa^5 \\ &= a^7xaa^4 \\ &= \vdots \\ &= a^3a^5x \\ &= x \notin H_2. \end{aligned}$$

Hence, H_2 is not normal in M_{16}. To show that H_2 is permutable, we have:

$$K_1H_2 = H_2K_1 = \{e, a^4x\} = H_2$$
$$K_2H_2 = H_2K_2 = \{e, x, a^4, a^4x\} = K_5$$
$$K_3H_2 = H_2K_3 = \{e, a^2, a^4, a^6, x, a^2x, , a^4x, a^6x\} = K_8$$
$$K_4H_2 = H_2K_4 = \{e, a^2, a^4, a^6, x, a^2x, , a^4x, a^6x\} = K_8$$
$$K_5H_2 = H_2K_5 = \{e, x, a^4, a^4x\} = K_5$$
$$K_6H_2 = H_2K_6 = M_{16}$$
$$K_7H_2 = H_2K_7 = \{e, x, a^4, a^4x\} = M_{16}$$
$$K_8H_2 = H_2K_8 = \{e, a^2, a^4, a^6, x, a^2x, a^4x, a^6x\} = K_8.$$

So, H_2 is permutable and not normal subgroup of M_{16}. □

The previous description shows that M_{16} has two permutable subgroups H_1 and H_2, and both subgroups are not normal.

6. Conclusions

This research interest in the classification of groups of order 16. Basically, the classification used to determine which of the resulted subgroups is: normal, permutable or permutable and not normal.

The results obtained in this research asserts that, permutability does not coincide with normality. The obtained example in Section 5, that is the modular group M_{16}, is a group of order 16 that has two permutable subgroups which are not normal.

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