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TIME-VARYING LYAPUNOV FUNCTIONS AND LYAPUNOV STABILITY OF NONAUTONOMOUS FRACTIONAL ORDER SYSTEMS

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Abstract: We present a new inequality which involves the Caputo fractional derivative of the product of two continuously differentiable functions, and establish its various properties. The inequality and its properties enable us to construct potential time-varying Lyapunov functions for the Lyapunov stability analysis of fractional order systems. We use time-varying Lyapunov functions to analyse the stability of nonautonomous fractional order systems. By considering time-varying quadratic Lyapunov function, we establish new stability conditions for certain class of nonautonomous fractional order systems where the fractional order lies between 0 and 1.

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1. Introduction

It is well-known that the Lyapunov method (Lyapunov direct method) [10] is a genuine and powerful method for the stability analysis of nonautonomous

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differential systems (systems involving integer order derivatives). It allows us to draw the conclusions about asymptotic behaviour of solutions to such differential systems without actually having the knowledge of its explicit solutions. Many versions of Lyapunov theorems for integer order differential systems can be found in the book [10]. Based on these theorems, one can ensure the stability, asymptotic stability, exponential stability etc., of the solutions to such systems.

On the other hand, fractional order systems [17], which generalize the differential systems of integer order to the non-integer order, are the center of activity in many studies and have wide range of applications [17, 11, 15, 16, 21]. Recently, in [13, 14, 6], the authors proposed fractional order extension of Lyapunov direct method for nonautonomous fractional order systems. It is worth to mention that the fractional Lyapunov direct method is a powerful method not only for the stability analysis of fractional order systems but also for establishing sufficient conditions for ensuring the stability as well as obtaining the bounds of solutions to such systems.

However, the fractional Lyapunov direct method has a limitation and a difficulty, when it comes to the stability analysis of fractional order systems. First, the limitation lies in the fact that it is usually very difficult to find a Lyapunov function for a given fractional order system. Second, the difficulty lies in the calculation of fractional derivative of Lyapunov function if it exists for a given fractional order system. That means even-if we have identified some candidate functions (in terms of independent variable 'time' and dependent variables 'state variables') for a given system, then the calculation of its fractional derivative along the solutions to the given system is not easy. This is due to the fact that the properties of fractional derivative operators (e.g., Leibniz rule [18, 19], chain rule [20]) are not easy.

In recent years, great efforts have been made by many researchers to construct various types of Lyapunov functions by establishing new inequalities which involve fractional derivatives [4, 8, 22, 5]. For example, quadratic Lyapunov function [4], general quadratic Lyapunov function [8], Volterra-type Lyapunov function [22], convex Lyapunov function [5] have been established for the Lyapunov stability analysis of fractional order systems.

It may be noted that all these works have been focused on time-invariant or autonomous Lyapunov functions (functions depend only on the state variables of fractional order system) which are continuously differentiable. However, in general, it is very essential to construct continuously differentiable time-varying or nonautonomous Lyapunov functions (functions in terms of independent variable 'time' and the dependent variables 'state variables') for the stability of nonautonomous fractional order systems. On the other hand, in [1, 3, 12, 2] continuous Lyapunov functions, Dini-like fractional derivative operators, initial time difference have been introduced for the stability analysis of fractional order systems.

In this paper, we attempt to construct the continuously differentiable time-varying Lyapunov functions and see the implication of such functions for the Lyapunov stability analysis of nonautonomous fractional order systems. First, we propose an inequality which involves the Caputo fractional derivative of the product of two continuously differentiable functions. Based on this inequality, we develop few of its properties and establish several new inequalities. By presenting a few illustrative examples, we show that it is possible to ensure the Lyapunov stability of nonautonomous fractional order systems based on the time-varying Lyapunov functions. Further, by considering time varying quadratic Lyapunov function along with the fractional Lyapunov direct method, we propose two stability theorems from which one can ensure the asymptotic stability (Mittag-Leffler stability) of certain class of nonautonomous fractional order systems. Finally, we demonstrate the applications of these theorems via illustrative examples.

2. Preliminaries

Let us denote by $\mathbb{Z}^+$ the set of positive integers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\Re(z)$ the real part of complex number z, $\mathbb{R}^n$ the n-dimensional Euclidean space, X^T the transpose of a vector or matrix X, $\|x\|$ the Euclidean norm of a vector x, For given symmetric matrices X, Y: the notation $X \leq Y$ ($X \geq Y$) means the matrix $X - Y$ is negative (positive) semi-definite, and $X < Y$ ($X > Y$) means the matrix $X - Y$ is negative (positive) definite.

Definition 1. ([17, 11]) The Euler Gamma function Γ is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{1}$$

where $z \in \mathbb{C}$ and $\Re(z) > 0$.

Definition 2. ([17, 11]) The Riemann-Liouville fractional integral of order α of function $x : [t_0, T] \to \mathbb{R}$ is defined as

$$^{RL}D_{t_0,t}^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau, \tag{2}$$

where $\alpha \in \mathbb{R}^+$.

Definition 3. ([17, 11]) The Caputo fractional derivative of order α of function $x : [t_0, T] \to \mathbb{R}$ is defined as

$$ {}^{C}D^{\alpha}_{t_0,t}x(t) = \begin{cases} {}^{RL}D^{-(n-\alpha)}_{t_0,t}\left(\frac{d^n x(t)}{dt^n}\right), & \text{if } \alpha \in (n-1, n), \\ \frac{d^n x(t)}{dt^n}, & \text{if } \alpha = n, \end{cases} \tag{3} $$

where $\alpha \in \mathbb{R}^+$, and $n \in \mathbb{Z}^+$.

Consider the initial value problem of nonautonomous fractional order system

$$ {}^{C}D^{\alpha}_{0,t}x(t) = g(t, x(t)), \quad x(0) = x_0, \tag{4} $$

where $\alpha \in (0, 1]$, and $g : [0, \infty) \times \Omega \to \mathbb{R}^n$ is continuously differentiable and $\Omega \subseteq \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

Definition 4. The origin $x = 0$ is an equilibrium point of (4) if $g(t, 0) = 0$, $\forall\, t \geq 0$.

Definition 5. The zero solution of the system (4) is said to be:

(i) stable if for each $\epsilon > 0$, $\exists\, \delta = \delta(\epsilon)$ such that

$$ \|x(0)\| < \delta \implies \|x(t)\| \leq \epsilon, \quad \forall\, t \geq 0, \tag{5} $$

(ii) asymptotically stable if it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.

Definition 6. ([9]) The one parameter Mittag-Leffler function is defined as

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{R}^+, \quad z \in \mathbb{C}. \tag{6} $$

Definition 7. ([14]) The zero solution of the system (4) is said to be Mittag-Leffler stable if

$$ \|x(t)\| \leq \left[m\left(x(0)\right) E_\alpha\left(-\lambda t^\alpha\right)\right]^b \tag{7} $$

where $\alpha \in (0, 1]$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$ and $m(x)$ is locally Lipschitz on $x \in \Omega \subseteq \mathbb{R}^n$ with Lipschitz constant m_0.

Definition 8. ([10]) A continuous function $\gamma : [0, a) \to [0, \infty)$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$.

Theorem 9. [14] *Let $x = 0$ be an equilibrium point for the nonautonomous fractional order system (4) and $\Omega \subseteq \mathbb{R}^n$ be the domain that contains the origin $x = 0$. Suppose there exists a continuously differentiable function $V : [0, \infty) \times \Omega \to \mathbb{R}^n$ and class-$\mathcal{K}$ functions γ_i $(i = 1, 2, 3)$ such that*

$$\gamma_1 (\|x\|) \leq V(t, x) \leq \gamma_2 (\|x\|), \tag{8}$$

and

$$^{C}D^{\alpha}_{0,t} V(t, x(t)) \leq -\gamma_3 (\|x\|), \tag{9}$$

where $\alpha \in (0, 1]$. Then, the origin $x = 0$ of system (4) is asymptotically stable.

3. Inequalities

In this section, we develop few inequalities which not only will provide a way of searching various types of time-varying Lyapunov functions but also will allow us to utilize the Lyapunov theory for the stability analysis of nonautonomous fractional order systems.

Lemma 10. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a non-negative and continuously differentiable function. Then, the inequality*

$$^{C}D^{\alpha}_{t_0,t} \{\phi(t)x(t)\} \leq \phi(t)\,^{C}D^{\alpha}_{t_0,t} x(t), \;\; \forall t \geq t_0, \;\; \forall \alpha \in (0, 1], \tag{10}$$

holds.

Proof. Let

$$F_\alpha(t) = {}^{C}D^{\alpha}_{t_0,t} \{\phi(t)x(t)\} - \phi(t)\,^{C}D^{\alpha}_{t_0,t} x(t). \tag{11}$$

In order to prove that the inequality (10) holds, it is sufficient to check $F_\alpha(t) \leq 0$, $\forall t > t_0$, $\forall \alpha \in (0, 1)$. This is because the inequality (10) holds for the case when $\alpha = 1$, and to the case, when $t = t_0$ for $\alpha \in (0, 1)$. Note that by the definition of Caputo fractional derivative, we have

$$^{C}D^{\alpha}_{t_0,t} \{\phi(t)x(t)\} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \psi(\tau, x(\tau)) d\tau, \tag{12}$$

where $\psi(t,x(t)) = \frac{d\phi(t)}{dt} \cdot x(t) + \frac{dx(t)}{dt} \cdot \phi(t)$, and

$$\phi(t)^C D^{\alpha}_{t_0,t} \{x(t)\} = \frac{\phi(t)}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{dx(\tau)}{d\tau} d\tau. \tag{13}$$

Substituting (12) and (13) into the right hand side of (11), gives

$$F_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{d}{d\tau} \{(\phi(\tau)-\phi(t))\, x(\tau)\}\, d\tau. \tag{14}$$

Let $u(\tau) = (\phi(\tau) - \phi(t))\, x(\tau)$. After substituting it into (14), and then integrating by parts, yields

$$F_\alpha(t) = \left[\frac{u(\tau)}{\Gamma(1-\alpha)(t-\tau)^\alpha}\right]_{\tau=t} - \left[\frac{u(t_0)}{\Gamma(1-\alpha)(t-t_0)^\alpha}\right] - \frac{\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{u(\tau)}{(t-\tau)^{\alpha+1}} d\tau. \tag{15}$$

Since the first term in the right hand side of (15) is in $\frac{0}{0}$ form, by L' Hôpital rule, it follows that the first term has a limiting value 0. Due to the monotonic decreasing property of function ϕ, and non-negativity of function x, the second and third terms of the right hand side of (15) become negative. Therefore, we have $F_\alpha(t) \leq 0$ for $t > t_0$. As a result, the inequality (10) holds. □

Lemma 11. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a monotonically increasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a non-negative and continuously differentiable function. Then, the inequality*

$$^C D^{\alpha}_{t_0,t} \{\phi(t)x(t)\} \geq \phi(t)^C D^{\alpha}_{t_0,t} x(t), \;\; \forall t \geq t_0, \;\; \forall \alpha \in (0,1], \tag{16}$$

holds.

Proof. Let the function $\psi(t) = -\phi(t)$. Then, it follows from Lemma 10 that

$$^C D^{\alpha}_{t_0,t} \{\psi(t)x(t)\} \leq \psi(t)^C D^{\alpha}_{t_0,t} x(t), \;\; \forall t \geq t_0, \;\; \forall \alpha \in (0,1]. \tag{17}$$

As a consequence, the inequality (16) holds. □

Lemma 12. *Let $\phi : [t_0, \infty) \to \mathbb{R}^n$ be a monotonically decreasing and continuously differentiable vector function. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a non-negative and continuously differentiable vector function. Then, the inequality*

$$^C D^{\alpha}_{t_0,t} \left\{\phi^T(t)x(t)\right\} \leq \phi^T(t)^C D^{\alpha}_{t_0,t} x(t), \;\; \forall t \geq t_0, \;\; \forall \alpha \in (0,1], \tag{18}$$

holds.

Proof. Let

$$\phi(t) = (\phi_1(t), \cdots, \phi_n(t))^T, \text{ and } x(t) = (x_1(t), \cdots, x_n(t))^T.$$

Since $\phi^T(t)x(t) = \sum_{i=1}^{n} \phi_i(t)x_i(t)$, by the linearity of Caputo fractional derivative operator, we have

$$^C D^{\alpha}_{t_0,t} \left\{\phi^T(t)x(t)\right\} = \sum_{i=1}^{n} {}^C D^{\alpha}_{t_0,t} \{\phi_i(t)x_i(t)\}. \tag{19}$$

Note that the functions ϕ_i's are monotonically decreasing and continuously differentiable for $i = 1, 2, \cdots, n$, and x_i's are non-negative and continuously differentiable for $i = 1, 2, \cdots, n$. Then, applying Lemma 10 to the equation (19), we get

$$^C D^{\alpha}_{t_0,t} \left\{\phi^T(t)x(t)\right\} \leq \sum_{i=1}^{n} \phi_i(t) {}^C D^{\alpha}_{t_0,t} x_i(t) = \phi^T(t) {}^C D^{\alpha}_{t_0,t} x(t). \tag{20}$$

This completes the proof. □

Lemma 13. ([4]) *Let $x : [t_0, \infty) \to \mathbb{R}$ be a continuous and derivable function. Then, for any $t \geq t_0$, the following inequality holds*

$$^C D^{\alpha}_{t_0,t} x^2(t) \leq 2x(t) {}^C D^{\alpha}_{t_0,t} x(t), \ \ \forall\, \alpha \in (0, 1). \tag{21}$$

Lemma 14. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a continuously differentiable function. Then, the inequality*

$$^C D^{\alpha}_{t_0,t} \left\{\phi(t)x^2(t)\right\} \leq 2\phi(t)x(t) {}^C D^{\alpha}_{t_0,t} x(t), \ \ \forall t \geq t_0, \ \ \forall \alpha \in (0, 1], \tag{22}$$

holds.

Proof. It follows from Lemma 10 that

$$^C D^{\alpha}_{t_0,t} \left\{\phi(t)x^2(t)\right\} \leq \phi(t) {}^C D^{\alpha}_{t_0,t} x^2(t), \ \ \forall t \geq t_0, \ \ \forall \alpha \in (0, 1]. \tag{23}$$

Then, by applying Lemma 13 to the inequality (23), we obtain the inequality (22). □

Lemma 15. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a continuously differentiable vector function. Then, for any $t \geq t_0$ and $\alpha \in (0, 1]$, the following inequality holds*

$$\begin{aligned} &{}^C D^{\alpha}_{t_0,t} \left\{ a\phi(t)x^2(t) + bx^2(t) \right\} \\ &\leq 2a\phi(t)x(t)\, {}^C D^{\alpha}_{t_0,t} x(t) + 2bx(t)\, {}^C D^{\alpha}_{t_0,t} x(t), \end{aligned} \tag{24}$$

where $a > 0$ and $b > 0$.

Proof. Since the Caputo fractional derivative operator is linear, we have

$${}^C D^{\alpha}_{t_0,t} \left\{ a\phi(t)x^2(t) + bx^2(t) \right\} = a\, {}^C D^{\alpha}_{t_0,t} \left\{ \phi(t)x^2(t) \right\} + b\, {}^C D^{\alpha}_{t_0,t} \left\{ x^2(t) \right\}. \tag{25}$$

Applying Lemma 14 and Lemma 13, to the inequality (25), we get the inequality (24). □

Lemma 16. [7] *Let $x : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function. Then, for any $t \geq 0$, the following inequality holds*

$${}^C D^{\alpha}_{0,t} x^p(t) \leq p x^{p-1}(t)\, {}^C D^{\alpha}_{0,t} x(t), \quad \forall\, \alpha \in (0, 1), \tag{26}$$

where $p = 2^n$ for $n \in \mathbb{Z}^+$.

Lemma 17. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a continuously differentiable vector function. Then, for any $t \geq t_0$ and $\alpha \in (0, 1]$, the following inequality holds*

$${}^C D^{\alpha}_{t_0,t} \left\{ a\phi(t)x^p(t) + bx^p(t) \right\} \leq p(a\phi(t) + b)x^{p-1}(t)\, {}^C D^{\alpha}_{t_0,t} x(t), \tag{27}$$

where $a > 0$ and $b > 0$.

Proof. Application of Lemma 10 and Lemma 16, leads to the inequality (27). □

Lemma 18. *Let $\phi_i : [t_0, \infty) \to \mathbb{R}$ are non-negative, monotonically decreasing and continuously differentiable functions for $i = 1, 2, \cdots, n$. Suppose $x_i : [t_0, \infty) \to \mathbb{R}$ are continuously differentiable functions for $i = 1, 2, \cdots, n$. Let $h_i(t) = c_i\phi_i(t) + d_i$, where $c_i,\ d_i > 0$ for $i = 1, 2, \cdots, n$. Then, for any $t \geq t_0$, and $\alpha \in (0, 1]$, the following inequality holds*

$${}^C D^{\alpha}_{t_0,t} \left\{ \sum_{i=1}^{n} h_i(t) x_i^{2^i}(t) \right\} \leq \sum_{i=1}^{n} 2^i h_i(t) x_i^{2^i - 1}(t)\, {}^C D^{\alpha}_{t_0,t} x_i(t), \tag{28}$$

for $i = 1, 2, \cdots, n$.

Proof. By using Lemma 17, one can easily obtain the inequality (28). □

Lemma 19. *Let* $\phi : [t_0, \infty) \to \mathbb{R}$ *be a non-negative, monotonically decreasing and continuously differentiable function. Suppose* $x : [t_0, \infty) \to \mathbb{R}^n$ *is a continuously differentiable vector function. Then,* $\forall t \geq t_0$, $\forall \alpha \in (0, 1]$, *the following inequality holds*

$$ {}^C D^{\alpha}_{t_0,t} \left\{ \phi(t) x^T(t) P x(t) \right\} \leq 2\phi(t) x^T(t) P \, {}^C D^{\alpha}_{t_0,t} x(t), \tag{29} $$

where $P \in \mathbb{R}^{n \times n}$ *is a constant, symmetric, and positive definite matrix.*

Proof. Since $P \in \mathbb{R}^{n\times n}$ is a constant symmetric and positive definite matrix, we can write $x^T(t)Px(t) = x^T(t)U\Lambda U^T x(t)$, where U is an orthogonal matrix and Λ is an diagonal matrix. Let $y(t) = U^T x(t)$. Then, we have

$$ x^T(t)Px(t) = y^T(t)\Lambda y(t) = \sum_{i=1}^{n} \lambda_{ii} y_i^2(t), \tag{30} $$

where $y(t) = (y_1(t), \cdots, y_n(t))^T$ and λ_{ii}'s are the diagonal elements of Λ. Note that

$$ {}^C D^{\alpha}_{t_0,t} \left\{ \phi(t) x^T(t) P x(t) \right\} = \sum_{i=1}^{n} \lambda_{ii} \, {}^C D^{\alpha}_{t_0,t} \left\{ \phi(t) y_i^2(t) \right\}. \tag{31} $$

Since $\lambda_{ii} > 0$ for $i = 1, \cdots, n$, the application of Lemma 14 to the right hand side of expression (31) gives

$$ \begin{aligned} {}^C D^{\alpha}_{t_0,t} \left\{ \phi(t) x^T(t) P x(t) \right\} &\leq \sum_{i=1}^{n} 2\lambda_{ii} \phi(t) y_i(t) \, {}^C D^{\alpha}_{t_0,t} \left\{ y_i(t) \right\} \\ &= 2\phi(t) y^T(t) \Lambda \, {}^C D^{\alpha}_{t_0,t} y(t) = 2\phi(t) x^T(t) U \Lambda U^T \, {}^C D^{\alpha}_{t_0,t} x(t) \\ &= 2\phi(t) x^T(t) P \, {}^C D^{\alpha}_{t_0,t} x(t). \end{aligned} \tag{32} $$

□

Lemma 20. [8] *Let* $x : [t_0, \infty) \to \mathbb{R}^n$ *be a vector of differentiable function. Then, for any* $t \geq t_0$, *the following relationship holds*

$$ {}^C D^{\alpha}_{t_0,t} \left(x^T(t) P x(t) \right) \leq 2x^T(t) P \; {}^C D^{\alpha}_{t_0,t} x(t), \;\; \forall \, \alpha \in (0, 1), \tag{33} $$

where $P \in \mathbb{R}^{n\times n}$ *is a constant, square, symmetric and positive definite matrix*

Lemma 21. *Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a continuously differentiable vector function. Then, for any $t \geq t_0$ and $\alpha \in (0, 1]$, the following inequality holds*

$$
\begin{aligned}
&{}^{C}D^{\alpha}_{t_0,t}\left\{\phi(t)x^T(t)Px(t) + x^T(t)Qx(t)\right\} &(34)\\
&\leq 2\phi(t)x^T(t)P\,{}^{C}D^{\alpha}_{t_0,t}x(t) + 2x^T(t)Q\,{}^{C}D^{\alpha}_{t_0,t}x(t), &(35)
\end{aligned}
$$

where $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ are constant, symmetric, and positive definite matrices.

Proof. The proof follows from the application of Lemma 19, and Lemma 20. □

Definition 22. A symmetric matrix $P(t) \in \mathbb{R}^{n\times n}$ is said to be positive definite matrix if for each $t \geq t_0$, the inequality $x^T P(t)x > 0$, holds $\forall x \neq 0$.

Assumption 23. *Let $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ be a continuously differentiable, symmetric and positive definite matrix function such that*

(i) the matrix $P(t) = U\Lambda(t)U^T$, where U is constant orthogonal matrix, and $\Lambda(t) = diag\,(\lambda_{11}(t), \lambda_{22}(t), \cdots, \lambda_{nn}(t))$.

(ii) the scalar functions $\lambda_{ii}(t)$ are monotonically decreasing and continuously differentiable for all $i = 1, 2 \cdots, n$.

Lemma 24. *Let Assumption 23 holds. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a continuously differentiable function. Then, the inequality*

$$
{}^{C}D^{\alpha}_{t_0,t}\{x^T(t)P(t)x(t)\} \leq 2x^T(t)P(t)\,{}^{C}D^{\alpha}_{t_0,t}x(t),\ \forall\alpha \in (0, 1],\ \forall t \geq t_0, \qquad (36)
$$

holds.

Proof. Since $P(t) = U\Lambda(t)U^T$ (by condition (i) of Assumption 23), and $x(t)$ is a vector function, we can write

$$
x^T(t)P(t)x(t) = x^T(t)U\Lambda(t)U^T x(t). \qquad (37)
$$

Let $y(t) = U^T x(t)$. Then, the equation (37) becomes

$$
x^T(t)P(t)x(t) = y^T(t)\Lambda(t)y(t). \qquad (38)
$$

Since $y^T(t)\Lambda(t)y(t) = \sum_{i=1}^{n} \lambda_{ii}(t)y_i^2(t)$, where $y(t) = (y_1(t), \cdots, y_n(t))^T$, we have

$$ {}^C D_{t_0,t}^{\alpha}\{y^T(t)\Lambda(t)y(t)\} = \sum_{i=1}^{n} {}^C D_{t_0,t}^{\alpha}\left(\lambda_{ii}(t)y_i^2(t)\right), \ \forall \alpha \in (0,1], \ \forall t \geq t_0. \tag{39} $$

Since $\lambda_{ii}(t)$ are monotonically decreasing and continuously differentiable for all $i = 1, 2 \cdots, n$ (by the condition (ii) of Assumption 23), and $y_i^2(t)$ are non-negative continuously differentiable function for $i = 1, 2 \cdots, n$, by applying Lemma 10 (or Lemma 12) to the equation (39), we get

$$ {}^C D_{t_0,t}^{\alpha}\{y^T(t)\Lambda(t)y(t)\} \leq \sum_{i=1}^{n} \lambda_{ii}(t) {}^C D_{t_0,t}^{\alpha}\left(y_i^2(t)\right), \ \forall \alpha \in (0,1], \ \forall t \geq t_0. \tag{40} $$

Now by using Lemma 13 in (40), and since $\lambda_{ii}(t) > 0$, we obtain

$$ {}^C D_{t_0,t}^{\alpha}\{y^T(t)\Lambda(t)y(t)\} \leq 2\sum_{i=1}^{n} \lambda_{ii}(t)y_i(t) {}^C D_{t_0,t}^{\alpha}\left(y_i(t)\right), \tag{41} $$

$\forall \alpha \in (0,1]$, and $\forall t \geq t_0$. Note that $\lambda_{ii}(t)$ are the diagonal entries of matrix $\Lambda(t)$. We can write

$$ \sum_{i=1}^{n} \lambda_{ii}(t)y_i(t) {}^C D_{t_0,t}^{\alpha}\left(y_i(t)\right) = y^T(t)\Lambda(t) {}^C D_{t_0,t}^{\alpha} y(t). \tag{42} $$

Substituting the equation (42) into the inequality (41), we get

$$ {}^C D_{t_0,t}^{\alpha}\{y^T(t)\Lambda(t)y(t)\} \leq 2y^T(t)\Lambda(t) {}^C D_{t_0,t}^{\alpha} y(t), \ \ \forall \alpha \in (0,1], \ \ \forall t \geq t_0. \tag{43} $$

From the inequality (43), it follows that the inequality (36) is true. □

Assumption 25. *Let $P : [t_0, \infty) \to \mathbb{R}^{n \times n}$ be a continuously differentiable, symmetric and positive definite matrix function such that*

(i) the matrix $P(t) = U(t)\Lambda(t)U^T(t)$, where $U(t)$ is continuously differentiable orthogonal matrix, and the matrix $\Lambda(t) = diag\,(\lambda_{11}(t), \cdots, \lambda_{nn}(t))$.

(ii) the scalar functions $\lambda_{ii}(t)$ are monotonically decreasing and continuously differentiable for all $i = 1, 2 \cdots, n$.

(iii) the real valued functions $u_{ij}(t)$ of matrix $U(t)$ are non-negative and monotonically decreasing for all $i, j = 1, 2, \cdots, n$.

Lemma 26. *Let Assumption 25 holds. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a non-negative continuously differentiable function. Then, the inequality*

$$^{C}D^{\alpha}_{t_0,t}\{x^T(t)P(t)x(t)\} \leq 2x^T(t)P(t)\,^{C}D^{\alpha}_{t_0,t}x(t), \;\; \forall\alpha \in (0,1], \;\; \forall t \geq t_0, \tag{44}$$

holds.

Proof. Based on the condition (i) of Assumption 25, we can write

$$x^T(t)P(t)x(t) = x^T(t)U(t)\Lambda(t)U^T(t)x(t). \tag{45}$$

Let us define the transformation by $y(t) = U^T(t)x(t)$. Then, it follows from the proof of Lemma 24 (where the condition (ii) of Assumption 25 is utilized) that

$$^{C}D^{\alpha}_{t_0,t}\{y^T(t)\Lambda(t)y(t)\}\leq 2y^T(t)\Lambda(t)\,^{C}D^{\alpha}_{t_0,t}y(t), \;\; \forall\alpha \in (0,1], \;\; \forall t \geq t_0. \tag{46}$$

Note that under the condition (iii) of Assumption 25, the application of Lemma 12 gives

$$^{C}D^{\alpha}_{t_0,t}y(t) = \,^{C}D^{\alpha}_{t_0,t}\left\{U^T(t)x(t)\right\} \leq U^T(t)\,^{C}D^{\alpha}_{t_0,t}x(t). \tag{47}$$

Then, by substituting (47) into (46), one obtains the inequality (44). □

4. Main discussion

In the previous section, we have established various inequalities which involve fractional derivatives. In this section, we analyse the stability of zero solution to nonautonomous fractional order systems and establish few sufficient conditions for such systems.

Here, first, we will present a few motivating examples. We shall try to verify the conditions of fractional extension of Lyapunov theorem by considering time-varying Lyapunov functions and using appropriate inequalities.

Example 27. Consider the following scalar nonautonomous nonlinear fractional differential equation

$$^{C}D^{\alpha}_{0,t}x(t) = -x^3(t) - e^t x^3(t), \;\; x(0) = x_0, \;\; 0 < \alpha \leq 1. \tag{48}$$

Let $V(t,x) = x^2 + e^{-t}x^2$ be the function, which depends on time t and variable x. Then, by using Lemma 15, we obtain the Caputo fractional derivative of $V(t,x)$ along the solution $x(t)$ to (48) as follows

$$\begin{aligned} {}^{C}D^{\alpha}_{0,t}V(t,x(t)) &\le \left[-2x^4(t) - 2e^t x^4(t)\right] + \left[-2e^{-t}x^4(t) - 2x^4(t)\right] \\ &= -2\left(1+e^{-t}\right)x^4(t) - 2\left(1+e^{t}\right)x^4(t) \\ &\le -4x^4(t), \quad \forall\, x \in \mathbb{R}, \quad \forall\, t \ge 0. \end{aligned} \tag{49}$$

Note that $x^2 \le V(t,x) \le 2x^2$, $\forall x \in \mathbb{R}$, $\forall t \ge 0$. Let $\gamma_1(\rho) = \rho^2$, $\gamma_2(\rho) = 2\rho^2$ and $\gamma_3(\rho) = 4\rho^4$, where $\rho = |x|$. Then, we see that all the assumptions of Theorem 9 are satisfied. This observation confirms that $V(t,x)$ is indeed a time-varying Lyapunov function. Hence, it follows from Theorem 9 that the zero solution is asymptotically stable.

Example 28. Consider the following nonautonomous linear fractional order system

$$\begin{aligned} {}^{C}D^{\alpha}_{0,t}x_1(t) &= -x_1(t) - h(t)x_2(t), \quad x_1(0) = x_{10} \\ {}^{C}D^{\alpha}_{0,t}x_2(t) &= x_1(t) - x_2(t), \quad x_2(0) = x_{20} \end{aligned} \tag{50}$$

where $0 < \alpha \le 1$, $h(t)$ is a monotonically decreasing, continuously differentiable and satisfies

$$0 \le h(t) \le M, \quad \forall t \ge 0. \tag{51}$$

Let us choose the time-varying Lyapunov function $V(t,x) = x_1^2 + x_2^2 + h(t)x_2^2$. Note that $x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+M)x_2^2$, $\forall x = (x_1,x_2)^T \in \mathbb{R}^2$, where the condition (51) is used. Then, the application of Lemma 21, enables us to calculate the Caputo fractional derivative of $V(t,x)$ along the solution $x(t)$ to (50)

$$\begin{aligned} {}^{C}D^{\alpha}_{0,t}V(t,x(t)) &\le \left[-2x_1^2(t) - 2h(t)x_1(t)x_2(t)\right] \\ &\qquad + \left[(2+2h(t))x_1(t)x_2(t) - (2+2h(t))x_2^2(t)\right] \\ &= -2x_1^2(t) + 2x_1(t)x_2(t) - (2+2h(t))x_2^2(t) \\ &\le -2x_1^2(t) + 2x_1(t)x_2(t) - 2x_2^2(t) \\ &= -x^T(t)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t), \quad \forall\, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \end{aligned} \tag{52}$$

Since $x^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x$ is a positive definite quadratic function, it follows from (52) that

$$ {}^{C}D_{0,t}^{\alpha}V(t,x(t)) \leq -\|x(t)\|^2. \tag{53} $$

Let $\gamma_1(\rho) = \rho^2$, $\gamma_2(\rho) = (1+M)\rho^2$ and $\gamma_3(\rho) = \rho^2$, where $\rho = \|x\|$. Then, all the assumptions of Theorem 9 are satisfied. Hence, the zero solution is asymptotically stable. Further, from (52), we deduce

$$ V(t,x(t)) \leq E_{\alpha}\left(-\frac{1}{M+1}t^{\alpha}\right)V(0,x(0)), \quad \forall t \geq 0. \tag{54} $$

Thus, it follows that

$$ \|x(t)\| \leq \left[(1+M)E_{\alpha}\left(-\frac{1}{M+1}t^{\alpha}\right)\|x(0)\|^2\right]^{1/2}, \quad \forall t \geq 0. \tag{55} $$

As a result, the zero solution is Mittag-Leffler stable.

Example 29. Consider the nonautonomous nonlinear fractional order system

$$ \begin{aligned} {}^{C}D_{0,t}^{\alpha}x_1(t) &= -x_1(t) + \frac{3}{1+t}x_2^3(t),\ x_1(0) = x_{10} \\ {}^{C}D_{0,t}^{\alpha}x_2(t) &= -x_2(t) - \frac{1}{1+t}x_1^3(t),\ \ x_2(0) = x_{20} \end{aligned} \tag{56} $$

We observe that $V(t,x) = 2x_1^4 + 6x_2^4 + \frac{1}{1+t}x_1^4 + \frac{3}{1+t}x_2^4$ is the time-varying Lyapunov function. In fact, by Lemma 18, it follows that the Caputo derivative of $V(t,x)$ along the solution $x(t)$ to (56) is

$$ \begin{aligned} {}^{C}D_{0,t}^{\alpha}V(t,x(t)) &\leq 4\left(2+\frac{1}{1+t}\right)\left(-x_1^4(t) + \frac{3}{1+t}x_1^3(t)x_2^3(t)\right) \\ &\quad + 4\left(6+\frac{3}{1+t}\right)\left(-x_2^4(t) - \frac{1}{1+t}x_1^3(t)x_2^3(t)\right) \\ &\leq -8\left(x_1^4(t) + 3x_2^4(t)\right),\ \ \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,\ \ \forall t \geq 0 \end{aligned} \tag{57} $$

Let $\gamma_1(\rho) = \rho^4$, $\gamma_2(\rho) = 9\rho^4$ and $\gamma_3(\rho) = 4\rho^4$, where $\rho = \sqrt{x_1^2 + x_2^2}$. Interestingly, all the assumptions of Theorem 9 are satisfied. Therefore, we conclude from Theorem 9 that the zero solution is asymptotically stable.

In the above mentioned examples, we notice that the time-varying Lyapunov functions are indeed good choice of candidate functions for the Lyapunov stability analysis of nonautonomous fractional order systems. The discussion indicates that the choice of time-varying Lyapunov functions and inequalities are very essential for the Lyapunov stability analysis of nonautonomous fractional order systems. By a similar analysis to these examples, one may wish to construct various types of time-varying Lyapunov functions and try to investigate the stability of nonautonomous fractional order systems.

Here, we are interested in the stability of nonautonomous fractional order system (4). Mainly, we consider the system (4) described in the following form

$$ {}^{C}D^{\alpha}_{0,t}x(t) = A(t)x(t) + f(t, x(t)), \quad x(0) = x_0, \tag{58} $$

where $\alpha \in (0, 1]$, $A(t) \in \mathbb{R}^{n\times n}$ is continuous with its elements are bounded, and $f : [0, \infty) \times \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function with $f(t, 0) = 0$ for all $t \geq 0$.

We specifically focus on the application of time-varying quadratic Lyapunov function $x^T P(t)x$ and aim to establish some sufficient conditions for the system (58). Here, we make the following assumptions on the function f:

Assumption 30. $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$, $L > 0$, $\forall x, y \in \mathbb{R}^n$, $\forall t \geq 0$.

Assumption 31. $\|f(t, x)\| \leq \ell\|x\|^2$, $\ell > 0$, $\forall \|x\| \leq r$, $r > 0$, $\forall t \geq 0$.

Theorem 32. *Under Assumption 30, if there exists a matrix $P(t)$ such that the following conditions are satisfied*

(i) the matrix $P(t)$ satisfies Assumption 23,

(ii) the matrix $P(t)$ is bounded, i.e. $0 < k_1 I \leq P(t) \leq k_2 I$, $\forall t \geq 0$,

(iii) there exists a constant $\mu > 0$ such that

$$ P(t)A(t) + A^T(t)P(t) + L^2 I + P^2(t) \leq -\mu I, \tag{59} $$

then, the zero solution of system (58) *is globally Lyapunov asymptotically stable (globally Mittag-Leffler stable).*

Proof. Let us consider the time-varying quadratic Lyapunov function $V(t, x) = x^T P(t)x$, where the matrix $P(t)$ satisfies the condition (i). Then, by Lemma 24, it follows that the Caputo fractional derivative of $V(t, x)$ along the solution to system (58) is

$$ \begin{aligned} {}^{C}D^{\alpha}_{0,t}V(t, x(t)) &\leq 2x^T(t)P(t)\,{}^{C}D^{\alpha}_{t_0,t}x(t) \\ &= x^T(t)\left[P(t)A(t) + A^T(t)P(t)\right]x(t) + 2f^T(t, x(t))P(t)x(t) \end{aligned} $$

$$\leq x^T(t)\left[P(t)A(t) + A^T(t)P(t)\right]x(t)$$
$$+ f^T(t, x(t))f(t, x(t)) + x^T(t)P^2(t)x(t)$$
$$\leq x^T(t)\left[P(t)A(t) + A^T(t)P(t) + L^2 I + P^2(t)\right]x(t) \tag{60}$$
$$\leq -\mu x^T(t)x(t), \quad \forall\, x \in \mathbb{R}^n, \tag{61}$$

where in (60) the condition (iii) is used. Note that $k_1\|x\|^2 \leq V(t,x) \leq k_2\|x\|^2$, where k_1, k_2 are positive constants (by condition (ii)). Hence, by fractional order Lyapunov Theorem 9, we conclude that the zero solution is asymptotically stable. Further, the inequality (61) becomes

$$^{C}D^{\alpha}_{0,t}V(t, x(t)) \leq -\frac{\mu}{k_2}V(t, x(t)). \tag{62}$$

From (62), we deduce

$$V(t, x(t)) \leq E_\alpha\left(-\frac{\mu}{k_2}t^\alpha\right)V(0, x(0)) \leq E_\alpha\left(-\frac{\mu}{k_2}t^\alpha\right)k_2\|x(0)\|^2. \tag{63}$$

Then, from (63), we get

$$\|x(t)\| \leq \sqrt{\frac{k_2}{k_1}E_\alpha\left(-\frac{\mu}{k_2}t^\alpha\right)\|x(0)\|^2}. \tag{64}$$

Hence, it follows from (64) that the zero solution is Mittag-Leffler stable. □

Theorem 33. *Under Assumption 31, if there exists a matrix $P(t)$ such that the following conditions are satisfied*

(i) the matrix $P(t)$ satisfies Assumption 23,

(ii) the matrix $P(t)$ is bounded, i.e. $0 < k_1 I \leq P(t) \leq k_2 I, \quad \forall t \geq 0$,

(iii) there exists a constant $\mu > 0$ such that

$$P(t)A(t) + A^T(t)P(t) \leq -\mu I, \tag{65}$$

then, we have the following estimation for solution

$$\|x(t)\| \leq \sqrt{\frac{k_2}{k_1}E_\alpha\left(-\frac{(\mu - 2k_2 r\ell)}{k_2}t^\alpha\right)\|x(0)\|^2}, \tag{66}$$

where $r < \frac{\mu}{2\ell k_2}$. Thus, the zero solution to system (58) is locally asymptotically stable (locally Mittag-Leffler stable).

Proof. Let $V(t,x) = x^T P(t)x$ be the time varying quadratic Lyapunov function, where the matrix $P(t)$ satisfies the condition (i). Then, by using Lemma 24, the Caputo fractional derivative of $V(t,x)$ along the solution $x(t)$ to system (58) is estimated as given below:

$$
\begin{aligned}
{}^C D^\alpha_{0,t} V(t,x(t)) &\le x^T(t)\left[P(t)A(t) + A^T(t)P(t)\right]x(t) \\
&\quad + 2x^T(t)P(t)f(t,x(t)) \\
&\le x^T(t)\left[P(t)A(t) + A^T(t)P(t)\right]x(t) + 2k_2\ell\|x(t)\|^3 \qquad (67)\\
&\le -\mu\|x(t)\|^2 + 2k_2 r\ell\|x(t)\|^2 \qquad (68)\\
&\le -\left(\mu - 2k_2 r\ell\right)\|x(t)\|^2, \quad \forall\ \|x\| < r, \qquad (69)
\end{aligned}
$$

where from (67)-(68), the condition (ii), condition (iii) and Assumption 31 are used. Note that ${}^C D^\alpha_{0,t} V(t,x(t))$ is negative definite in the region $\|x\| < r$, if $r < \frac{\mu}{2\ell k_2}$ holds. Therefore, the conditions of Theorem 9 are satisfied in the region $\|x\| < r$. Thus, we conclude that the zero solution is locally asymptotically stable. Observe that $V(t,x)$ is a positive definite function. From the inequality (69), we deduce

$$
\|x(t)\| \le \sqrt{\frac{k_2}{k_1} E_\alpha\left(-\frac{(\mu - 2k_2 r\ell)}{k_2} t^\alpha\right)\|x(0)\|^2}. \qquad (70)
$$

Hence, the zero solution is Mittag-Leffler stable (asymptotically stable). □

In the next, we shall discuss two illustrative examples for the above mentioned results.

Example 34. Consider the fractional order system (58), where the coefficient matrix

$$
A(t) = \begin{pmatrix} -2 - e^{-(t+1)} & -3 \\ 3 & -\frac{3}{2} - e^{-(t+1)} \end{pmatrix}, \qquad (71)
$$

and the nonlinear function

$$
f(t,x(t)) = \left(\sin(t)\sin(x_2(t)), \sin(t)\sin(x_1(t))\right)^T. \qquad (72)
$$

In order to examine the asymptotic stability of this system, let us consider the continuously differentiable, symmetric, positive definite and bounded matrix

$$
P(t) = \begin{pmatrix} 1 + e^{-t} & 0 \\ 0 & 1 + e^{-t} \end{pmatrix}. \qquad (73)
$$

Note that the function f satisfies Assumption 30, and the matrix $P(t)$ satisfies the conditions (i) and (ii) of Theorem 32. Here, we estimate

$$P(t)A(t) + A^T(t)P(t) + L^2I + P^2(t) = \begin{pmatrix} h_1(t) + L^2 & 0 \\ 0 & h_2(t) + L^2 \end{pmatrix}, \tag{74}$$

where $L = 1$, $h_1(t) = -3 - 2e^{-t} - 2e^{-(t+1)} + e^{-2t} - 2e^{-(2t+1)}$, and $h_2(t) = -2 - e^{-t} - 2e^{-(t+1)} + e^{-2t} - 2e^{-(2t+1)}$.

Let $m_1(t) = h_1(t) + L^2$, and $m_2(t) = h_2(t) + L^2$. Since

$$\sup_t\{m_1(t)\} = -2,\ \sup_t\{m_2(t)\} = -1,\ \inf_t(-m_1(t)) = -\sup_t\{m_1(t)\},$$

and $\inf_t(-m_2(t)) = -\sup_t\{m_2(t)\}$, for

$$0 < \mu < \min\left\{\inf_t(-m_1(t)), \inf_t(-m_2(t))\right\},$$

the following relationship holds

$$P(t)A(t) + A^T(t)P(t) + L^2I + P^2(t) \le -\mu I. \tag{75}$$

Therefore, the condition (iii) of Theorem 32 is satisfied. Hence, we conclude from Theorem 32 that the zero solution is asymptotically stable.

Example 35. Consider the fractional order system (58), where the coefficient matrix

$$A(t) = \begin{pmatrix} -2 - e^{-(t+1)} & -3 \\ 3 & -\frac{3}{2} - e^{-(t+1)} \end{pmatrix}, \tag{76}$$

and the nonlinear function

$$f(t, x(t)) = \left(\sin(t)\left(x_1^2(t) + x_2^2(t)\right), \cos(t)\left(x_1^2(t) + x_2^2(t)\right)\right)^T. \tag{77}$$

Here, we assume the matrix $P(t)$ to be as given in (73). Then, we estimate

$$P(t)A(t) + A^T(t)P(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix}, \tag{78}$$

where $h_1(t) = -4 - 2e^{-(t+1)} - 4e^{-t} - 2e^{-(2t+1)}$, and $h_2(t) = -3 - 2e^{-(t+1)} - 3e^{-t} - 2e^{-(2t+1)}$. Taking the value of $\mu = 2$, and the bounds of matrix $P(t)$ as $k_1 = 1$ and $k_2 = 2$, we see that all the conditions from (i)-(iii) of Theorem 33 are satisfied. Since the function f satisfies Assumption 31 with $\ell = 1$. Then, we conclude from Theorem 33 that the zero solution is asymptotically stable in the region $\|x\| < r = \frac{1}{2}$, and we have the estimation

$$\|x(t)\| \le \sqrt{2}\sqrt{E_\alpha\left(-(1 - 2r)t^\alpha\right)}\|x(0)\|. \tag{79}$$

5. Conclusions

We have developed an elementary inequality which involves the Caputo fractional derivative of the product of two functions. It opens up an opportunity of constructing the continuously differentiable time-varying Lyapunov functions in order to analyse the stability of fractional order systems. Indeed, we have shown the use of such type of Lyapunov functions, inequalities along with fractional Lyapunov direct method for the stability analysis of fractional order systems. Finally, by taking time-varying quadratic Lyapunov function, we have proposed two stability theorems that provide sufficient conditions for the stability of certain class of nonautonomous fractional order systems.

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