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A NOTE ON p-ADIC LINDEMANN-WEIERSTRASS

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Abstract: In this paper we apply Ax-Schanuel's Theorem to the ultraproduct of the $p-$adic fields in order to prove a weak form of the p-adic Lindemann-Weierstrass conjecture for almost all primes.

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1. Introduction

Let $\mathbb{Q}_p$ be the field of p-adic numbers, for p a prime number. Given an algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ of $\mathbb{Q}_p$, it naturally comes equipped with a norm $|.|_p$, uniquely extending the usual norm on $\mathbb{Q}_p$. Recall that the standard normalization for $|.|_p$ is $|p|_p = p^{-1}$.

Denote by $\mathbb{C}_p$ the completion of $\mathbb{Q}_p^{\mathrm{alg}}$ with respect to the norm $|.|_p$. Then, $\mathbb{C}_p$ is also algebraically closed. It is called a complex p-adic field.

The p-adic exponential map is defined as:

$$\exp_p : E_p \to \mathbb{C}_p^{\times}, x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

where E_p is the set $E_p = \{x \in \mathbb{C}_p : |x|_p < p^{-\frac{1}{p-1}}\}$ (the domain of convergence of the defining power series of the exponential).

While in the case of complex exponentiation many transcendence results are already known, most of these do not have a counterpart in the p-adic setting.

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Most notable among these results is the Lindemann-Weierstrass conjecture (a theorem in the complex case!), which can be stated as follows:

Conjecture 1.1. (L-W) *Let* $x_1, ..., x_n \in E_p$ *be algebraic numbers over* $\mathbb{Q}$ *and let* $\mathbb{Q}(\exp_p(\overline{x}))$ *denotes the field* $\mathbb{Q}(\exp_p(x_1), ..., \exp_p(x_n))$. *If* $x_1, ..., x_n$ *are* $\mathbb{Q}$*- linearly independent, then*

$$\mathrm{td}_{\mathbb{Q}}\mathbb{Q}(\exp_p(\overline{x})) = n.$$

In this paper, we apply the ultraproduct construction and basic model theory in order to obtain some results in the above direction.

We embed $\mathbb{C}_p$ as a valued subfield in the $p-$adic Malcev-Neumann field $\mathbb{L}_p$.

The main theorem can be obtained by applying Ax-Schanuel's theorem, [1], to a non-principal ultraproduct of $\mathbb{C}_p$, and it reads as:

Theorem 1.1. *Let* $p \in \mathbb{P}$, $\alpha \in \mathbb{Q} \cap (1, 2)$, $n, N \in \mathbb{N}$ *and let* $x_{1,p}, ..., x_{n,p} \in \mathbb{C}_p \hookrightarrow \mathbb{L}_p$ *be* $p-$*adic algebraic numbers over* $\mathbb{Q}$ *with* $\mathrm{ord}_p(x_{i,p}) = 0$ *and* $\mathrm{Supp}(x_{i,p}) \subseteq \frac{1}{N}\mathbb{Z}[\frac{1}{p}]$. *Consider the elements* $z_{i,p} := p^{\frac{1}{p-\alpha}} x_{i,p}, i = 1, 2, .., n$. *For almost all* $p \in \mathbb{P}$, *if* $z_{1,p}, ..., z_{n,p}$ *are* $\mathbb{Q}$*-linearly independent, then*

$$\mathrm{td}_{\mathbb{Q}}\mathbb{Q}(\exp_p(\overline{z}_p)) = n.$$

Theorem 1.1 will be proved in Section 5 after several preliminary sections.

2. Background

It is well-known that the field $\mathbb{C}_p$ is the completion (with respect to the norm $|.|_p$) of an algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers. One may consider instead the additive valuation ord_p defined on $\mathbb{C}_p$. This valuation is defined through the relation:

$$|z|_p = p^{-\mathrm{ord}_p(z)}.$$

Recall that a *derivation* over a (commutative) field K is a map $D : K \to K$ satisfying additivity ($D(x + y) = Dx + Dy$) and Leibniz rule ($D(xy) = xDy + yDx$). The *field of constants for* D is the set of $x \in K$ for which $Dx = 0$. Using additivity and Leibniz rule, one can see that C is indeed a subfield of K.

In [8], Ax-Schanuel's theorem was restated as follows:

Theorem 2.1. *Let K be a field of characteristic zero endowed by a derivation D with the constant field C and let $y_1, \ldots, y_n, z_1, \ldots, z_n \in K^\times$ be such that $Dy_k = \frac{Dz_k}{z_k}$ for $k = 1, \ldots, n$.*

If $\mathrm{td}_C C(y_1, \ldots, y_n, z_1, \ldots, z_n) \leq n$, then $\sum\limits_{i=1}^{n} m_i y_i \in C$ for some $m_1, \ldots, m_n \in \mathbb{Q}$ not all zero.

3. The fields $\mathbb{K}_{\mathcal{U}}$ and $k_{\mathcal{U}}((t^\Gamma))$

We firstly recall some basics concerning the field of generalized power series, [9].

Let k be a field and Γ be a totally ordered Abelian group. A generalized series with coefficients in k and exponents in Γ is a map $a : \Gamma \longrightarrow k$ denoted by $a = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with its support $\mathrm{Supp}(a) = \{\gamma \in \Gamma : a_\gamma \neq 0\}$ well-ordered.

We denote by $k((t^\Gamma))$ the set of generalized power series which is actually a field endowed by componentwise sum and convolution product.

The field $k((t^\Gamma))$ is endowed by a valuation

$$\begin{aligned} \mathrm{ord} : (k((t^\Gamma)))^\times &\longrightarrow \Gamma \\ a &\longmapsto \min(\mathrm{Supp}(a)). \end{aligned}$$

The valuation ring is the elements in $k((t^\Gamma))$ with positive exponents, while its maximal ideal is the elements a in $k((t^\Gamma))$ with $\min(\mathrm{Supp}(a)) > 0$ and the residue field is k. It is well-known that if Γ is divisible and k is algebraically closed, then $k((t^\Gamma))$ is also algebraically closed.

Poonen, [10], constructed a new type of generalized power series fields in order to obtain a maximal valued field of mixed characteristic. More precisely, he proved among the valued fields whose value group is $\mathbb{Q}$ (considered to be divisible), the residue class field is the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ (considered to be algebraically closed) and the restriction of the valuation to $\mathbb{Q}$ is the $p-$adic valuation, there exists a filed, denoted by $\mathbb{L}_p$, that has the following properties:

1) $\mathbb{L}_p$ is maximal, i.e, for each valued field R with these properties, there exists a value-preserving embedding from R to $\mathbb{L}_p$.

2) $\mathbb{L}_p$ is algebraically closed and spherically complete (equivalently, $\mathbb{L}_p$ is pseudo-complete).

3) $\mathbb{L}_p$ consists of the elements of the form

$$x = \sum_{g\in\mathbb{Q}} \alpha_g p^g,$$

where $\{\alpha_g\}$ is the set of representatives of $\overline{\mathbb{F}}_p$ and the set $\mathrm{Supp}(x) = \{g \in \mathbb{Q}; \alpha_g \neq 0\}$ is well-ordered subset in $\mathbb{Q}$. In particular, there exists a value-preserving embedding from $\mathbb{C}_p$ to $\mathbb{L}_p$. Furthermore, Poonen [10] (Corollary 8), characterized the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as follows: if $\alpha \in \overline{\mathbb{Q}}_p$, then there exist $N, m \in \mathbb{N}$ such that $\mathrm{Supp}(\alpha) \subseteq \frac{1}{N}\mathbb{Z}[\frac{1}{p}]$ and the residue classes of the coefficients in the expansion of α are in $\mathbb{F}_{p^m}$.

Let $\mathbb{P}$ be the set of all prime numbers, and let $\mathcal{U}$ be a non-principle ultrafilter on $\mathbb{P}$.

In [3], it was shown that the field $\mathbb{K}_{\mathcal{U}} := \prod_{p\in\mathbb{P}} \mathbb{C}_p/\mathcal{U}$ forms a valued field. The valuation map on this field is defined as follows:

$$\begin{aligned} \mathrm{val} : \mathbb{K}_{\mathcal{U}}^{\times} &\longrightarrow \Gamma \\ [(x_p)_{p\in\mathbb{P}}] &\longmapsto [(\mathrm{ord}_p(x_p))_{p\in\mathbb{P}}], \end{aligned}$$

where $\Gamma := \prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$ and the residue field is $\prod_{p\in\mathbb{P}} \overline{\mathbb{F}}_p/\mathcal{U}$. Also, it was shown that $\mathbb{K}_{\mathcal{U}}$ admits an exponential map E defined as follows:

$$\begin{aligned} E : \mathbb{K}_{\mathcal{U}} &\longrightarrow \mathbb{K}_{\mathcal{U}}^{\times} \\ [(x_p)_{p\in\mathbb{P}}] &\longmapsto [(\mathrm{EXP}_p(x_p))_{p\in\mathbb{P}}], \end{aligned}$$

where EXP_p is some extension of the usual exponential map $\exp_p$ to $\mathbb{C}_p$, [11].

The embedding $\mathbb{C}_p \hookrightarrow \mathbb{L}_p$ induces an embedding $\prod_{p\in\mathbb{P}} \mathbb{C}_p/\mathcal{U} \hookrightarrow \prod_{p\in\mathbb{P}} \mathbb{L}_p/\mathcal{U}$. Using the same argument, we find that the field
$\prod_{p\in\mathbb{P}} \mathbb{L}_p/\mathcal{U}$ admits a valuation with the value group $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$ and the residue field $\prod_{p\in\mathbb{P}} \overline{\mathbb{F}}_p/\mathcal{U}$, [2] (p.173). Hence, both $\prod_{p\in\mathbb{P}} \mathbb{L}_p/\mathcal{U}$ and its residue field have characteristic zero. Applying Kaplansky Theorem, [7], to the field $\prod_{p\in\mathbb{P}} \mathbb{L}_p/\mathcal{U}$, we find that there exists a value-preserving embedding

$$\sigma : \prod_{p\in\mathbb{P}} \mathbb{L}_p/\mathcal{U} \hookrightarrow k_{\mathcal{U}}((t^{\Gamma})), \tag{3.1}$$

where $k_{\mathcal{U}}$ is $\prod_{p\in\mathbb{P}} \overline{\mathbb{F}}_p/\mathcal{U}$ and Γ is $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$.

4. Statement of Lindemann-Weierstrass Property

Let $(K, v, \exp, D)$ be a differential valued exponential field, [8], with the domain E of the exponential function and the constant field C. Let $x_1, .., x_n \in C$ and $t \in K - C$ be such that $tx_1, ..., tx_n \in E$. Using the same argument in [6] (p. 278), and making obvious changes in appropriate places, one can obtain the following equivalent statements:

Proposition 4.1. *Keep the notation as above. Then, the following are equivalent:*
(a) If $tx_1, ..., tx_n$ are $\mathbb{Q}$-linearly independent, then the elements $\exp(tx_1), ..., \exp(tx_n)$ are algebraically independent over C.
(b) If $tx_1, ..., tx_n$ are mutually distinct, then the set $\{1, \exp(tx_1), ..., \exp(tx_n)\}$ is linearly independent over C.

We will show that the statement (a) is true if the domain of exp is contained in $\mathcal{M}$ (the maximal ideal of the valuation ring of K).

In fact, Statement (a) can be rephrased as follows:

If $\exp(tx_1), ..., \exp(tx_n)$ are algebraically dependent over C, then the elements $tx_1, ..., tx_n$ are $\mathbb{Q}$-linearly dependent.

In this case, we have

$$\mathrm{td}_C(tx_1, ..., tx_n, \exp(tx_1), ..., \exp(tx_n)) \leq 1 + (n - 1) = n.$$

Using Ax Theorem (namely Theorem 2.1), we find that the elements $tx_1, ..., tx_n$ are linearly dependent modulo the constant field C. Therefore, there exist rationals $m_1, ..., m_n \in \mathbb{Q}$ (not all zero) such that $m_1tx_1 + ... + m_ntx_n \in C$. By multiplying by a suitable integer, we can assume that the coefficients m_i are rational integers (where we have used the fact that C contains $\mathbb{Q}$). Since $tx_i \in \mathcal{M}$, it follows that $m_1tx_1 + ... + m_ntx_n \in \mathcal{M}$. Therefore, $m_1tx_1 + ... + m_ntx_n = 0$ (because $\mathcal{M} \cap C = \{0\}$). Hence, the elements $tx_1, ..., tx_n$ are $\mathbb{Q}-$linearly dependent. Thus, we obtain the following:

Theorem 4.1. *Let $(K, v, D, \exp)$ be a differential valued exponential field with the domain $E \subseteq \mathcal{M}$ of exp and the constant field C. Let $x_1, ..., x_n \in C$ and let $t \in K - C$ be such that $tx_1, ..., tx_n \in E$. If $1, \exp(tx_1), ..., \exp(tx_n)$ are linearly dependent over C, then not all of the elements $x_1, ..., x_n$ are distinct.*

5. The Main Results

Recall that Lindemann-Weierstrass Conjecture can be restated as follows: Let $x_{1,p}, ..., x_{n,p}$ be $p-$ adic algebraic numbers over $\mathbb{Q}$ in the domain of $\exp_p$. If $1, \exp_p(x_{1,p}), ..., \exp_p(x_{n,p})$ are linearly dependent over $\mathbb{Q}$, then not all of the elements $x_{1,p}, ..., x_{n,p}$ are distinct, see [6] (p. 278) (the proof in the p-adic setting still works with some slight modifications).

In [3], the field $k_{\mathcal{U}}((t^{\Gamma}))$ (for any non-principle ultrafilter $\mathcal{U}$ on $\mathbb{P}$) has been made into a differential valued exponential field where the constant field is $k_{\mathcal{U}}$ and the exponential map exp is defined as follows:

$$\begin{aligned} \exp : & k_{\mathcal{U}}((t^{\Gamma>0})) \longrightarrow k_{\mathcal{U}}((t^{\Gamma})) \\ & \epsilon \longmapsto \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}. \end{aligned}$$

Furthermore, it was proved in [3] (Theorem 3.1), that if $x = [(x_p)_{p\in\mathbb{P}}] \in \mathbb{K}_{\mathcal{U}}$ with $\mathrm{ord}_p(x_p) > \frac{1}{p-1}$, then

$$\sigma(E(x)) = \exp(\sigma(x)). \tag{5.1}$$

We need the following lemmas:

Lemma 5.1. *Consider $\Gamma = \prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$ as a $\mathbb{Q}$-vector space (which is possible since $\mathbb{Q}$ is a subfield in $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$). Let $\Gamma' = \{r.b; r \in \mathbb{Q}, b \in \frac{1}{N}\prod_{p\in\mathbb{P}} \mathbb{Z}[\frac{1}{p}]/\mathcal{U}\}$. Then, Γ' forms $\mathbb{Q}$-vector subspace which is strictly contained in Γ.*

Proof. It is clear that Γ' forms a $\mathbb{Q}$-vector subspace. For each $p \in \mathbb{P}$, let $p' := \min\{q \in \mathbb{P} : q > p\}$. Since N is fixed, it follows that the element $[(\frac{1}{p'})_{p\in\mathbb{P}}]$ is not in Γ'. □

Using the same argument in [3], we prove the following:

Lemma 5.2. *Let Γ^* be the complement of Γ' as a $\mathbb{Q}$-vector subspace in Γ and let $K := k_{\mathcal{U}}((t^{\Gamma'}))$. Then, the generalized power series field $K((t^{\Gamma^*}))$ forms a differential valued exponential field with the constant field K.*

Proof. Since Γ^* is $\mathbb{Q}$-vector subspace in Γ, it follows that Γ^* is an Abelian group. Also, we have Γ totally ordered. This implies that Γ^* (which is a subset in Γ) is also totally ordered. So, Γ^* forms a totally ordered Abelian group.

Let Δ be the set of archimedean classes of Γ^*. Consider the element $\alpha^* = [(\frac{1}{p'})_{p\in\mathbb{P}}]$ (defined in the previous lemma). Then, α^* is an infinitesimal positive element in Γ^*. Let Φ be the archimedean class of α^*. Therefore, the map

$$\begin{aligned} \sigma^* : \Delta &\longrightarrow \Delta \\ \delta &\longmapsto \Phi.\delta \end{aligned}$$

is a right-shift map and preserves the order on Δ. Using the same argument in [8] (Example 6, Case 1), one can define a series derivation D' on $K((t^{\Gamma^*}))$ in which the constant field is K.

We can endow $K((t^{\Gamma^*}))$ by the exponentiation $\exp'$, defined on $K((t^{\Gamma^{*>0}}))$ by the series $\exp'(\epsilon) := \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}, \forall \epsilon \in K((t^{\Gamma^{*>0}}))$. Since D' is a series derivation, it follows that $D'(\epsilon) = \frac{D'(\exp'(\epsilon))}{\exp'(\epsilon)}, \forall \epsilon \in K((t^{\Gamma^{*>0}}))$. It is clear that $\exp'$ is the restriction of exp to $\Gamma^{>0} \cap \Gamma^*$. Thus, $(K((t^{\Gamma^*})), \exp', D')$ forms a differential valued exponential field where the constant field is K. Hence, each element of the form $\sum_\gamma \alpha_\gamma t^\gamma, \gamma \in \Gamma', \alpha_\gamma \in k_{\mathcal{U}}$ is considered a constant element in the field $K((t^{\Gamma^*}))$. □

As an application of Kaplansky Theorem, we prove the following

Claim. *Let a be an arbitrary non-zero element in $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U} \subseteq \mathbb{K}_{\mathcal{U}}$. Then, $\sigma(a) \in K$. That is, $\sigma(a)$ is a constant element in the differential valued exponential field $K((t^{\Gamma^*}))$.*

Proof. We have $(\mathbb{Q}, \mathrm{ord_p})$ forming a valued field with the value group $\mathbb{Z}$ and the residue field $\mathbb{F}_p$. Hence, $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$ is a valued field with the value group $\prod_{p\in\mathbb{P}} \mathbb{Z}/\mathcal{U}$ and the residue field $\prod_{p\in\mathbb{P}} \mathbb{F}_p/\mathcal{U}$. Using Kaplansky Theorem, we find that $\prod_{p\in\mathbb{P}} \mathbb{Q}/\mathcal{U}$ is embedded in $\prod_{p\in\mathbb{P}} \mathbb{F}_p/\mathcal{U}((t^{\prod_{p\in\mathbb{P}} \mathbb{Z}/\mathcal{U}}))$. So, $\mathrm{Supp}(\sigma(a)) \subseteq \prod_{p\in\mathbb{P}} \mathbb{Z}/\mathcal{U}$. It is clear that $\prod_{p\in\mathbb{P}} \mathbb{Z}/\mathcal{U} \subseteq \Gamma'$. Therefore, $\sigma(a)$ takes the form $\sum_\gamma \alpha_\gamma t^\gamma, \gamma \in \Gamma', \alpha_\gamma \in k_{\mathcal{U}}$. Hence, $\sigma(a) \in k_{\mathcal{U}}((t^{\Gamma'})) = K$. □

Keeping the notation of the previous lemmas, we prove the main theorem as follows:

5.1. Proof of Theorem 1.1

Proof. Assume the theorem is not true. This implies the existence of an infinite subset $S \subseteq \mathbb{P}$ such that for each $p \in S$ the elements $1, \exp_p(p^{\frac{1}{p-\alpha}} y_{1,p}), ..., \exp_p(p^{\frac{1}{p-\alpha}} y_{n,p})$ satisfy a linear dependence relation over $\mathbb{Q}$ of the form:

$$a_{0,p} + a_{1,p} \exp_p(p^{\frac{1}{p-\alpha}} y_{1,p}) + ... + a_{n,p} \exp_p(p^{\frac{1}{p-\alpha}} y_{n,p}) = 0$$

and the elements $y_{1,p}, ..., y_{n,p}$ are mutually distinct. Let $\mathcal{U}$ be a non-principle ultrafilter on $\mathbb{P}$ such that $S \in \mathcal{U}$. Using the construction above, we consider the fields $\mathbb{K}_{\mathcal{U}} \hookrightarrow k_{\mathcal{U}}((t^\Gamma))$ and define the elements:

$$x_1 = [(x_{1,p})_{p\in\mathbb{P}}], ..., x_n = [(x_{n,p})_{p\in\mathbb{P}}], r = [(p^{\frac{1}{p-\alpha}})_{p\in\mathbb{P}}] \in \mathbb{K}_{\mathcal{U}} \hookrightarrow k_{\mathcal{U}}((t^\Gamma)),$$

where $x_{i,p} = y_{i,p}$ if $p \in S$. Otherwise, let $x_{i,p}$ be arbitrary algebraic numbers over $\mathbb{Q}$ of order zero and $\mathrm{Supp}(x_{i,p}) \subseteq \frac{1}{N}\mathbb{Z}[\frac{1}{p}]$. Using the same argument in the previous claim, we find that

$$\mathrm{Supp}(\sigma(x_i)) \subseteq \frac{1}{N} \prod_{p \in \mathbb{P}} \mathbb{Z}[1/p]/\mathcal{U} \text{ (since } N \text{ is fixed).} \tag{5.2}$$

Consider the $\mathbb{Q}-$ vector subspaces Γ' and Γ^* in Γ as defined in the previous Lemmas. So, $\Gamma = \Gamma' \oplus \Gamma^*$. Consider the field $K = k_{\mathcal{U}}((t^{\Gamma'}))$. Using Lemma 5.2, we find that $K((t^{\Gamma^*}))$ forms a differential valued exponential field with the constant field K. From (5.2), we find that $\mathrm{Supp}(\sigma(x_i)) \subseteq \Gamma'$. So, each element $\sigma(x_i)$ can be written in the form $\sigma(x_i) = \sum_\gamma a_\gamma t^\gamma$, where $\gamma \in \Gamma'$ and $a_\gamma \in k_{\mathcal{U}}$. Hence, $\sigma(x_i) \in K, \forall i = 1, 2, .., n$. For each $p \in \mathbb{P}$, we have $\mathrm{ord}_p(p^{\frac{1}{p-\alpha}}) = \frac{1}{p-\alpha}$. Let p runs over $\mathbb{P}$. Then, $\mathrm{val}(r) = [(\frac{1}{p-\alpha})_{p \in \mathbb{P}}]$. Using the embedding σ and noting that it preserves the valuation, we find that

$$\mathrm{ord}(\sigma(r)) = \mathrm{val}(r) = \left[\left(\frac{1}{p-\alpha}\right)_{p \in \mathbb{P}}\right] \notin \Gamma'.$$

So, $\sigma(r) \notin K$. Therefore, $\sigma(rx_i) \in K((t^{\Gamma^* > 0})), \forall i = 1, .., n$ since $\mathrm{ord}(\sigma(rx_i)) = \mathrm{val}(rx_i) = [(\frac{1}{p-\alpha})_{p \in \mathbb{P}}] > 0$. Hence, $\exp'(\sigma(rx_i))$ is well-defined and we have

$$\exp(\sigma(rx_i)) = \exp'(\sigma(rx_i)). \tag{5.3}$$

Using o Theorem, we get a linear relation over $\prod_{p \in \mathbb{P}} \mathbb{Q}/\mathcal{U}$ of the form:

$$a_0 + a_1 E(rx_1) + ... + a_n E(rx_n) = 0.$$

Using (3.1), (5.1) and (5.3), we obtain the following

$$\sigma(a_0) + \sigma(a_1)\exp'(\sigma(rx_1)) + ... + \sigma(a_n)\exp'(\sigma(rx_n)) = 0.$$

From the previous claim, we deduce that the coefficients $\sigma(a_i), i = 0, 1, .., n$ are in K. Thus, the elements $1, \exp'(\sigma(rx_1)), ..., \exp'(\sigma(rx_n))$ are linearly dependent over the constant field K. Applying Theorem 4.1 to the differential valued exponential field $K((t^{\Gamma^*}))$, we find that the elements $\sigma(x_1), ..., \sigma(x_n)$ are not all distinct. Since σ is injective, it follows that not all of the elements $x_1, ..., x_n$ are distinct. Using o Theorem, it implies that there exists a member $S_1 \in \mathcal{U}$ such that the elements $x_{1,p}, ..., x_{n,p}$ are not all distinct for all $p \in S_1$. Since $\mathcal{U}$ is a filter, it follows that $S \cap S_1 \neq \emptyset$ (in fact, this intersection is infinite since $\mathcal{U}$ is free). Thus, for each $p \in S \cap S_1$ we have $x_{1,p}, ..., x_{n,p}$ which are mutually distinct (since $p \in S$). On the other hand, $x_{1,p}, ..., x_{n,p}$ are not all distinct (since $p \in S_1$). This contradiction proves the theorem. $\square$

Corollary 5.1. *Let $p \in \mathbb{P}$ and $x_{1,p}, ..., x_{n,p}$ be rational integers which are relatively prime to p. Then, for almost all p, any linear dependence relation over $\mathbb{Q}$ of the elements*

$$1, \exp_p(p^{\frac{1}{p-\alpha}} x_{1,p}), ..., \exp_p(p^{\frac{1}{p-\alpha}} x_{n,p})$$

implies that not all of the elements $x_{1,p}, ..., x_{n,p}$ are distinct.

Proof. Since $x_{1,p}, ..., x_{n,p} \in \mathbb{Z}$, then the number N, defined in Theorem 1.1, is 1. Also, since $(x_{i,p}, p) = 1$, it follows that $\mathrm{ord}_p(x_{i,p}) = 0$. Applying Theorem 1.1, we prove the corollary. $\square$

Remark. Theorem 1.1 still holds true if we take the elements $x_{1,p}, ..., x_{n,p}$ from $\mathbb{Q}_p$ (or any unramified finite extension of $\mathbb{Q}_p$).

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References

[1] J. Ax, On Schanuel's conjectures, *Annals of Mathematics, 2nd Ser.*, **3**, No 2 (Mar. 1971), 252-268.

[2] J. Antonio and A. Prestel, *Valued Fields*, Springer Monographs in Mathematics, 2005.

[3] A. Bleybel, A note on complex p-adic exponential fields, *p-Adic Numbers, Ultrametric Analysis and Applications*, **10**, No 4 (2018), 267275.

[4] C.C. Chang and H.J. Keisler, *Model Theory*, North-Holland Publishing Co, Amsterdam, 1973.

[5] F.Q. Gouvea, *p-Adic Numbers: An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 2nd Ed., 2000.

[6] N. Jacobson, *Basic Algebra 1*, Yale University, 2nd Ed., 1985.

[7] I. Kaplansky, Maximal fields with valuations, *Duke Math. J.* **9** (1942), 303-321.

[8] S. Kuhlmann, M. Matusinski, and A. Shkop, A note on Schanuels conjectures for exponential logarithmic power series fields, *Arch. Math.* (Basel) **100**, No 5 (2013), 431436.

[9] M. Matusinski, On generalized series fields and exponential-logarithmic series fields with derivations, *ArXiv Preprint*, arXiv:1209.4559v3 [math.AC], 25 Oct. 2013.

[10] B. Poonen, Maximally complete fields, *L'Enseignement Mathmatique*, **39** (1993), 87-106.

[11] W.H. Schikhof, *Ultrametric Calculus*, Cambridge University Press, Cambridge, 1984.