International Journal of Applied Mathematics

Volume 32 No. 1 2019, 35-43
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v32i1.3

ON THE MULTIPLE SOLUTIONS OF A NONHOMOGENEOUS STURM-LIOUVILLE EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract: In this work, we are concerned with a nonlocal boundary value problem of nonhomogeneous Sturm-Liouville equation. Then existence of at least one solution will be proved. The spectral properties of the problem will be studied. The multiple solutions of the nonhomogeneous equation with the nonlocal boundary condition will be given.

AMS Subject Classification: 34A55, 34B10, 34B15, 34B18, 34L10, 34L40, 34K10
Key Words: Sturm-Liouville problem; nonlocal boundary conditions; existence of solutions; nonhomogeneous problem

1. Introduction

A.V. Bitsadze and A.A. Samarskii considered a certain class of spatial nonlocal problems [1]. Later, many authors used Bitsadze and Samarskii type condition (see [3]-[7]).

Received: July 12, 2018

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Here we are concerning with the nonhomogeneous differential equation problem

$$x''(t) + m(t) = \ -\lambda^2 x(t), \qquad t \in (0, \pi) \tag{1}$$

with Bitsadze and Samarskii nonlocal type boundary conditions

$$x(0) = 0, \qquad x'(\xi) + \lambda x(\xi) = 0. \tag{2}$$

The existence of at least one solution of problem (1)-(2) will be proved. As spacial case of the problem (1)-(2) when $m(t) = 0$, we consider the Sturm-Liouville problem of the second order-differential equation

$$-x''(t) \ = \ \lambda^2 \ x(t), \qquad t \in (0, \pi) \tag{3}$$

with Bitsadze and Samarskii nonlocal type boundary conditions (2).

We study the existence of eigenvalues and eigenfunctions.Then, we provide the multiple solutions of the problem (1)-(2).

2. Existence of solution

Let $m : [0, \pi] \to [0, \pi]$ be continuous, and consider the nonlocal problem (1)-(2).

2.1. Integral representation

For the integral representation of the solution of (1)-(2) we have the following lemma.

Lemma 2.1. *The solution of the problem (1)-(2), if it exists, can be represent by the integral equation*

$$x(t) = \frac{t}{1 + \lambda\xi} \int_0^\xi (1 + \lambda(\xi - s))(m(s) + \lambda^2 x(s))ds$$

$$- \int_0^t (t - s)(m(s) - \lambda^2 x(s))ds. \tag{4}$$

Proof. Integrating both sides of equation (1) twice, we obtain

$$x'(t) - x'(0) = - \int_0^t m(s)ds - \lambda^2 \int_0^t x(s)ds \tag{5}$$

and

$$x(t) - x(0) - tx'(0) = -\int_0^t (t-s)m(s)ds - \lambda^2 \int_0^t (t-s)x(s)ds. \tag{6}$$

For $x(0) = 0$ we obtain

$$x(t) = tx'(0) - \int_0^t (t-s)m(s)ds - \lambda^2 \int_0^t (t-s)x(s)ds. \tag{7}$$

Applying the nonlocal condition (5), we obtain

$$x'(\xi) = x'(0) - \int_0^\xi (m(s)ds + \lambda^2 x(s))ds \tag{8}$$

and

$$\lambda x(\xi) = \lambda \xi x'(0) - \lambda \int_0^\xi (\xi - s)(m(s) + \lambda^2 x(s))ds. \tag{9}$$

Now from (8)-(9), we can get

$$x'(0) = \frac{1}{1+\lambda\xi} \int_0^\xi (1 + \lambda(\xi - s))(m(s) + \lambda^2 x(s))ds. \tag{10}$$

Hence, from (10) and (7), we obtain (4).

3. Existence of at least one solution

Now for the existence of at least one continuous solution of the integral equation (4) we have the following theorem.

Theorem 3.1. *Let $m : [0, \pi] \to R$ be continuous, then a sufficient condition for the existence of at least one solution $x \in C[0, \pi]$ of the nonlocal boundary value problem (1)-(2) is*

$$\lambda^2 < \frac{1}{4\pi^2}.$$

Proof. Define the subset $Q_r \subset C[0, \pi]$ by $Q_r = \{x \in C : \|x\| \leq r\}$,

$$r = \frac{(3\pi^2 + \lambda\pi^3)\|m\|}{2 - \lambda^2(3\pi^2 + \lambda\pi^3)}.$$

Let $x \in Q_r$, then

$$Fx(t) = \frac{t}{1+\lambda\xi}\Big[\int_0^\xi (m(s) + \lambda^2 x(s))ds$$

$$+\lambda \int_0^\xi (\xi - s)(m(s) + \lambda^2 x(s))ds\Big] - \int_0^t (t-s)(m(s) + \lambda^2 x(s))ds,$$

$$|Fx(t)| \ \leq \ |\frac{t}{1+\lambda\xi}| \ \int_0^\xi (1 + \lambda(\xi - s))|m(s) + \lambda^2 x(s)|ds$$

$$+ \int_0^t (t-s)|m(s) + \lambda^2 x(s)|ds \ \leq \ \frac{\pi}{1+\lambda\xi}(\|m\| + \lambda^2\|x\|) \int_0^\pi ds$$

$$+\frac{\pi}{1+\lambda\xi}\lambda(\|m\| + \lambda^2\|x\|) \int_0^\pi (\pi - s)ds \ + \ (\|m\| + \lambda^2\|x\|) \int_0^\pi (\pi - s)ds$$

$$\leq \left[\frac{3\pi^2 + \lambda\pi^3}{2}\right](\|m\| + \lambda^2\|x\|) \leq r.$$

Then $F : Q_r \to Q_r$ and the class $\{Fx\} \in Q_r$ and is uniformly bounded in Q_r.

In what follows we show that the class $\{Fx\}, x \subset Q_r$ is equicontinuous on Q_r.

Let $t_1, t_2 \in (0, \pi),\ t_1 < t_2$ such that $|t_2 - t_1| < \delta$, then

$$Fx(t_2) - Fx(t_1) = \frac{t_2}{1+\lambda\xi} \int_0^\xi (m(s) + \lambda^2 x(s))ds$$

$$+\lambda \int_0^\xi (\xi - s)(m(s) + \lambda^2 x(s))ds - \int_0^{t_2} (t_2 - s)(m(s) + \lambda^2 x(s))ds$$

$$-\frac{t_1}{1+\lambda\xi} \int_0^\xi (m(s) + \lambda^2 x(s))ds - \lambda \int_0^\xi (\xi - s)(m(s)$$

$$+\lambda^2 x(s))ds + \int_0^{t_1} (t_1 - s)(m(s) + \lambda^2 x(s))ds$$

$$= \frac{t_2 - t_1}{1+\lambda\xi} \int_0^\xi (m(s) + \lambda^2 x(s))ds - \int_0^{t_1} (t_2 - t_1)(m(s)$$

$$+\lambda^2 x(s))ds \ + \ \int_{t_1}^{t_2} (t_2 - s)(m(s) + \lambda^2 x(s))ds,$$

then

$$|Fx(t_2) - Fx(t_1)| \leq \frac{|t_2 - t_1|}{1+\lambda\xi} \int_0^\xi |m(s) + \lambda^2 x(s)|ds$$

$$+\int_{t_1}^{t_2}(t_2-s)|m(s)+\lambda^2 x(s)|ds+\int_0^{t_1}(t_2-t_1)|m(s)+\lambda^2 x(s)|ds.$$

Hence the class of functions $\{Fx\}, x \subset Q_r$ is equicontinuous. Hence by Arzela Theorem [2], we fined that F is relatively compact.

Now we prove that $F : Q_r \to Q_r$ is continuous.

Let $x_n \subset Q_r$ such that $x_n \to x_o \in Q_r$, then

$$Fx_n(t) = \frac{t}{1+\lambda\xi}\Big[\int_0^{\xi}(m(s)+\lambda^2 x_n(s))ds$$

$$-\lambda\int_0^{\xi}(\xi-s)(m(s)+\lambda^2 x_n(s))ds\Big]-\int_0^{t}(t-s)(m(s)+\lambda^2 x_n(s))ds,$$

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty}\Big[\frac{t}{1+\lambda\xi}\int_0^{\xi}(m(s)+\lambda^2 x_n(s))ds$$

$$-\lambda\int_0^{\xi}(\xi-s)(m(s)+\lambda^2 x_n(s))ds-\int_0^{t}(t-s)(m(s)+\lambda^2 x_n(s))ds\Big].$$

Applying Lebesgue Dominated Convergence Theorem [2], we deduce that

$$\lim_{n\to\infty} Fx_n(t) = \frac{t}{1+\lambda\xi}\int_0^{\xi} m(s)ds-\lambda^2\int_0^{\xi}\lim_{n\to\infty} x_n(s)ds$$

$$-\lambda(\int_0^{\xi}(\xi-s)m(s)ds+\lambda^2\int_0^{\xi}(\xi-s)\lim_{n\to\infty} x_n(s)ds)$$

$$-\int_0^{t}(t-s)m(s)ds+\lambda^2\int_0^{t}(t-s)\lim_{n\to\infty} x_n(s)ds$$

$$=\frac{t}{1+\lambda\xi}\int_0^{\xi} m(s)ds-\lambda^2\int_0^{\xi} x_0(s)-\lambda(\int_0^{\xi}(\xi-s)m(s)ds$$

$$+\lambda^2\int_0^{\xi}(\xi-s)x_0(s)ds)-\int_0^{t}(t-s)m(s)ds-\lambda^2\int_0^{t}(t-s)x_0(s)ds.$$

Then $Fx_n(t) \to Fx_o(t)$ which means that the operator F is continuous.

Since all conditions of Schauder Fixed Point Theorem hold [2], then F has a fixed point in Q_r, and the integral equation (4) has at least one solution $x \in [0,\pi]$.

Differentiating (4), we can obtain

$$x''(t)+m(t) = -\lambda^2 x(t), \quad a.e \quad t\in(0,\pi)$$

and

$$x(0) = 0, \qquad x'(\xi) + \lambda x(\xi) = 0.$$

This proves that the nonlocal boundary value problem (1)-(2) has at least one solution $x \in C[0, \pi]$ and $x \in C^1[0, \pi]$.

4. The homogeneous problem

Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the problem (3)-(2).

Lemma 4.1. *The eigenfunction of the nonlocal boundary value problem (3)-(2) is given by*

$$x_n(t) = c_n \sin(n - \frac{1}{4})\frac{\pi}{\xi}t. \tag{11}$$

Proof. First, we prove that

$$\lambda_n = (n - \frac{1}{4})\frac{\pi}{\xi}, \qquad n = 1, 2, ...$$

The general solution of (3)-(2) is given by

$$x(t) = c_1 \sin \lambda t + c_2 \cos \lambda t. \tag{12}$$

Using the conditions of (2), then

$$x(0) = c_1 \sin \lambda 0 + c_2 \cos \lambda 0 \Rightarrow c_2 = 0.$$

Differentiating (12) and keeping in mind that $c_2 = 0$, we get

$$x'(t) = c_1 \lambda \cos \lambda t \tag{13}$$

Putting $t = \xi$ in (13), we get

$$x'(\xi) = c_1 \lambda \cos \lambda \xi \tag{14}$$

and

$$\lambda x(\xi) = c_1 \lambda \sin \lambda \xi. \tag{15}$$

Now, from (14)-(15), we can get

$$x'(\xi) + \lambda x(\xi) = 0 = c_1\lambda\Big(\cos\lambda\xi + \sin\lambda\xi\Big), \qquad c_1 \neq 0$$

and

$$\lambda\Big(\cos\lambda\xi + \sin\lambda\xi\Big) = 0.$$

Then

$$\lambda = (n - \frac{1}{4})\frac{\pi}{\xi}$$

and

$$x_n(t) = c_n \sin(n - \frac{1}{4})\frac{\pi}{\xi}t.$$

5. The nonhomogeneous problem

Here we study the existence of multiple solutions of the nonhomogeneous problem (1) and (2).

Let x_1, x_2 be two solutions of the problem (1)-(2). Let $u(t) = x_1(t) - x_2(t)$, then the function u satisfies the Sturm-Liouville equation

$$u''(t) = -\lambda^2 u(t)$$

with the nonlocal conditions

$$u(0) = 0, \qquad u'(\xi) + \lambda u(\xi) = 0.$$

So, the values, eigenvalues λ_n (spectral) for the existence of non zero solutions of (3) and (2) have the same values, eigenvalues of λ_n (spectral) for the existence of multiple solutions (eigenfunctions) of (1)-(2) are given by

$$\lambda_n = (n - \frac{1}{4})\frac{\pi}{\xi}, \qquad n = 1, 2, ...$$

Theorem 5.1. *The multiple solutions (eigenfunctions) $x_n(t)$ of problem (1) and (2) are given by*

$$x_n(t) = c_n \sin\lambda t + \int_0^t \frac{\sin(t-s)}{\lambda} m(s)ds, \qquad n = 1, 2, ... \tag{16}$$

Proof. Here we use the variation of parameter method to get the solutions of the problem (1)-(2) in the formula

$$x_n(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + x_p(t). \tag{17}$$

So, we have

$$x(t) = \cos \lambda t, \qquad y(t) = \sin \lambda t$$

are solutions of the homogeneous problem (3)-(2). The Wronskian $W(x, y) = \lambda$, then

$$x_p \quad = \quad -\cos \lambda t \int_0^t \frac{\sin \lambda s}{\lambda} m(s)ds + \sin \lambda t \int_0^t \frac{\cos \lambda s}{\lambda} m(s)ds.$$

Thus

$$x_p \quad = \quad \int_0^t \frac{\sin \lambda(t-s)}{\lambda} m(s)ds. \tag{18}$$

Hence

$$x_n(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + \int_0^t \frac{\sin \lambda(t-s)}{\lambda} m(s)ds. \tag{19}$$

Now, when $x(0) = 0$ in the equation (19), we deduce that $c_1 = 0$.
Therefore the multiple solutions of the nonlocal problem (1)-(2) is given by

$$x_n(t) = c_n \sin(n - \frac{1}{4})\frac{\pi}{\xi} t + \int_0^t \frac{\sin \lambda(t-s)}{(n-\frac{1}{4})\frac{\pi}{\xi}} m(s)ds, \qquad n = 1, 2, ...$$

To complete the proof, differentiating (2)-(3), we can obtain (1). Also equation (18) we have $x(0) = 0$, and we can show that $x'(\xi) = -\lambda x(\xi)$.

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