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B-BIMORPHISMS

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Abstract: Let X be an Archimedean vector lattice and it has a separating order dual $X^{\sim}$. By $(X^{\sim})_n^{\sim}$ we denote the order continuous bidual of X. In this paper, we define a b-bimorphism of X and we extend it to the order continuous bidual of X.

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1. Introduction

The lattice ordered algebras (or Riesz algebras) were introduced in [1, 7] by the authors Aliprantis, Burkinshw and Meyer-Nieberg. Their order biduals of lattice ordered algebras were studied by using Arens multiplications in [3, 6]. f-algebras, almost f-algebras, d-algebras and b-algebras in lattice ordered algebras were studied and their order biduals and order continuous biduals were investigated by the mathematicians in [3, 6, 9] using Arens products. The Arens triadjoint of bilinear mappings on products of vector lattices has been investigated by some mathematicians, for example, A. Toumi [8], R. Yilmaz

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[10, 11]. In this paper, we study the Arens triadjoints of bilinear mappings which are b-bimorphisms on vector lattices. Let us recall the following definitions of some classes of bilinear mappings. Let X be a vector lattice. X^+ denotes the positive cone of X.

Definition 1. ([4, 5]) Let X be a vector lattice. A bilinear mapping $T : X \times X \to X$ is called a biorthomorphism if $x \wedge y = 0$ implies $T(z, x) \wedge y = 0$ for all $z \in X^+$ and it is separately order bounded.

The triadjoint of biorthomorphism is also biorthomorphism by Yilmaz [10]. He proved that the extensions of biorthomorphism to order bidual and order continuous bidual are biorthomorphism.

Definition 2. Let X and Y be vector lattices. A bilinear mapping $T : X \times X \to Y$ is called an orthosymmetric if $x \wedge y = 0$ implies $T(x, y) = 0$ for all $x, y \in X$.

The triadjoint of orthosymmetric bimorphism is also an orthosymmetric bimorphism by Toumi [8]. Toumi's proof is focused that the triadjoint of orthosymmetric bilinear mapping is defined on the order continuous bidual of vector lattice. In [10], Yilmaz extends this result to whole order bidual of a vector lattice.

Definition 3. Let X and Y be vector lattices. A bilinear mapping $T : X \times X \to Y$ is called a d-bimorphism if $x \wedge y = 0$ implies $T(z, x) \wedge (T(z, y) = 0$ for all $z \in X^+$.

The definition of d-bimorphism is due to Yilmaz [11]. He proved that triadjoint of d-bimorphism is also a d-bimorphism on the order continuous bidual of a vector lattice by using Arens multiplication.

We can see from the definitions that every biorthomorphism is both orthosymmetric and d-bimorphism.

2. The Arens triadjoints of b-bimorphsim

In this section we define the b-bimorphism on a vector lattice and we prove that the triadjoint T''' of a b-bimorphism $T : X \times X \to X$ is also a b-bimorphism. Let X be a vector lattice with separating order dual $X^\sim$. By $(X^\sim)^\sim_n$ we denote the order continuous bidual of X. We can embedd X into second order dual $X^{\sim\sim}$ by means of canonical mapping. That is, $\sigma : X \to X^{\sim\sim}$ is defined by $\sigma(x) = x'' : x''(f) = f(x)$ for all $f \in X^{\sim}$ and $\sigma(x)$ defines an order continuous lattice homomorphism on $X^{\sim}$ for every $x \in X$.Consider $\sigma(X)$ is a subalgebra in the second order continuous bidual $(X^{\sim})_n^{\sim}$ of X. Next, the band generated by $\sigma(X)$ is an order dense in the order continuous bidual $(X^{\sim})_n^{\sim}$ of X. The order ideal generated by $\sigma(X)$ in $(X^{\sim})_n^{\sim}$ is given by the formula

$$I_{\sigma(X)} = \{F \in (X^{\sim})_n^{\sim} : |F| \leq x'' \ \exists \ x \in X^+\}.$$

By Meyer, for every $0 < F \in (X^{\sim})_n^{\sim}$ there exist an upward directed net $0 < (F_\alpha)$ in $I_{\sigma(X)}$ such that $F_\alpha \uparrow F$.

Definition 4. Let X be a vector lattice. A bilinear mapping $T : X \times X \to X$ is called a b-bimorphism if $x \wedge y = 0$ and $x \wedge z = 0$ in X imply $x \wedge T(y, z) = 0$ for all $x, y, z \in X$.

By using the definitions we obtain the following theorem.

Theorem 5. *Every biorthomorphism is a b-bimorphism.*

Theorem 6. *Let X be a vector lattice and $T : X \times X \to X$ be a b-bimorphism. Then, the triadjoint of T is a b-bimorphism.*

Proof. We can establish the following adjoints of the mapping T by means of the Arens multiplication, [2].

$$T : X \times X \longrightarrow X, (x, y) \to T(x, y), \tag{1}$$
$$T' : X^{\sim} \times X \longrightarrow X^{\sim}, (f, x) \to T'(f, x)y = f(T(x, y)), \tag{2}$$
$$T'' : (X^{\sim})_n^{\sim} \times X^{\sim} \to X', (F, f) \to T''(F, f)x = F(T'(f, x)), \tag{3}$$
$$T''' : (X^{\sim})_n^{\sim} \times (X^{\sim})_n^{\sim} \to (X^{\sim})_n^{\sim}, (F, G) \to T'''(F, G)f = F(T''(G, f)), \tag{4}$$

for all $x, y \in X$, $f \in X^{\sim}$ and $F, G \in (X^{\sim})_n^{\sim}$.

First, we will prove that if $x \in X^+$ and $0 \leq F, G, H \in (X^{\sim})_n^{\sim}$ hold $F, G, H \leq \sigma(x)$ and $F \wedge G = F \wedge H = 0$, then $F \wedge T'''(G, H) = 0$. Let $0 \leq f \in X^{\sim}$ and $N_F = \{f \in X^{\sim} : |F||f| = 0\}$ is a band. This implies $X^{\sim} = N_F \oplus C_F$, where $C_F = N_F^d$. So, $C_F \subseteq N_G$ and $C_F \subseteq N_H$ satisfy. For this, let $f \in (X^{\sim})^+$ and $x \in X^+$. Then, there exist $g, h \in X^{\sim}$ with $g \wedge h = 0$ and $F(g) = G(h) = H(h) = 0$ such that $T'(f, x) = g + h$ by [3].

By the Riesz-Kantarovich formula,

$$(g \wedge h)(x) = 0 = inf\{g(y) + h(z) : y + z = x, y, z \in X^+\}.$$

This implies that for a given any $\epsilon > 0$, there exist $y, z \in X^+$ such that $x = y + z, g(y) < \epsilon/2$ and $h(z) < \epsilon/2$. Define the order continuous linear functionals K , L and M on $X^\sim$ by $K = F \wedge \sigma(y - y \wedge z)$ and $L = G \wedge \sigma(z - y \wedge z)$ and $M = H \wedge \sigma(z - y \wedge z)$. Then, $0 \leq F - K = F - (F \wedge \sigma(y - y \wedge z) \leq 2\sigma(z)$, and by the same way $G - L \leq 2\sigma(y), H - M \leq 2\sigma(y)$. Since $(y'' - z'')^+ \wedge (z'' - y'')^+ = 0$ and T is a b-bimorphism and T,T''' coincide on $X^{\sim\sim}$, we obtain

$$(y'' - z'')^+ \wedge T'''((z'' - y'')^+, (z'' - y'')^+) = 0.$$

Therefore, $0 \leq K \wedge T'''(L, M) \leq (y'' - z'')^+ \wedge T'''((z'' - y'')^+, (z'' - y'')^+) = 0$. That is, $K \wedge T''(L, M) = 0$. We obtain $K \wedge T'''(F, H) = 0$ and also $K \wedge T'''(G, H) = 0$. From here by using the similar technique in [11], we find $F \wedge T'''(G, H) = 0$.

Secondly, we prove that if $0 \leq F, G, H \in (X^\sim)^\sim_n, F \wedge G = 0, F \wedge H = 0$, then $F \wedge T'''(G, H) = 0$. If $0 \leq F, G, H \in I_{\sigma(X)}$, then by the definition of order ideal generated by $\sigma(X)$,we have $0 \leq F \leq x'', 0 \leq G \leq y'', 0 \leq H \leq z''$ for some $x, y, z \in X$. Hence $0 \leq F, G, H \leq x'' + y'' + z''$. By the first part of the proof, $F \wedge T'''(G, H) = 0$. If we take $0 \leq F, G, H \in (X^\sim)^\sim_n$ as arbitrary, by the order densness there are upwards directed nets $F_\alpha, G_\alpha, H_\alpha$ in the order ideal $I_{\sigma(X)}$ generated by $\sigma(X)$ such that $0 \leq F_\alpha \uparrow F, 0 \leq G_\alpha \uparrow G, 0 \leq H_\alpha \uparrow H$.By the same technique in the paper Bernau and Huijsmans [3], we can obtain $F \wedge T'''(G, H) = 0$. □

Definition 7. An Archimedean lattice ordered algebra A is called a b-algebra if $a \wedge (bc) = 0$ for all $a, b, c \in A$ with $a \wedge b = a \wedge c = 0$

Definition 8. A lattice ordered algebra A is said to be a d-algebra if $a \wedge b = 0$ in A implies $ac \wedge bc = 0 = ca \wedge cb$ for every $c \in A^+$.

Definition 9. A lattice ordered algebra A is said to be an almost f-algebra if $a \wedge b = 0$ in A implies $ab = 0$.

Any f-algebra is a d-algebra, an almost f-algebra and a b-algebra. If A has a unit element then d-algebra and almost f-algebra are f-algebra.

Any Archimedean b-algebra is not necessary a d-algebra and almost f-algebra, and f-algebra, [9].

As a result of this work, we obtain that if A is a b-algebra, then the order continuous bidual of A is a b-algebra.Also, if we add to b-algebra a property called positive square closedness, then the order bidual becomes a b-algebra with respect to Arens product. The following result is presented, [9].

Theorem 10. *If a b-algebra A has positive square, then the order bidual of A is also a b-algebra.*

Proof. Let A be a b-algebra with positive squares. Define a map $T : A \times A \to A$ by $T(a, b) = a.b$ for every $a, b \in A$. Here, T is a b-bimorphism. triadjoint of T is also a b-bimorphism. □

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