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CHARACTERIZATION OF SPAN OF BASE $\mathcal{B}$-INDUCED 1-UNIFORM DCSL GRAPHS

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Abstract: A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \rightarrow 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \rightarrow 2^X \setminus \{\phi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen. A dcsl f of G is k-uniform if all the constants of proportionality with respect to f are equal to k, and if G admits such a dcsl then G is called a k-uniform dcsl graph. Let $\mathcal{F}$ be a family of subsets of a set X. A tight path between two distinct sets P and Q in $\mathcal{F}$ is a sequence $P_0 = P, P_1, P_2 \ldots P_n = Q$ in $\mathcal{F}$ such that $d(P,Q) =| P \triangle Q |= n$ and $d(P_i, P_{i+1}) = 1$ for $0 \leq i \leq n-1$. The family $\mathcal{F}$ is well-graded family, if there is a tight path between any two of its distinct sets. In this paper we characterize problem of determining those $\mathcal{F}$-induced graph $G_{\mathcal{F}}$ in which the base $\mathcal{B}$-induced graph is 1-uniform dcsl.

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1. Introduction

Throughout this paper by a graph we mean a connected, finite, simple graph. Unless otherwise mentioned, for all terminology in graph theory the reader is referred to [4]. Acharya [1] introduced the notion of vertex set valuation as a set analogue of number valuation. For a graph $G = (V, E)$ and a non empty set X, Acharya defined a set valuation of G as an injective set valued function $f : V(G) \to 2^X$, and he defined a set-indexer as a set valuation such that the function $f^{\oplus} : E(G) \to 2^X \setminus \{\phi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina [2], having studying topological set valuation, introduced the particular kind of set valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them [2]. In otherwords, the question is whether one can determine those graphs $G = (V, E)$ that admit an injective function $f : V \to 2^X$, X being a non empty ground set such that the cardinality of the symmetric difference $f^{\oplus}(uv)$ is proportional to the usual path distance $d_G(u, v)$ between u and v in G, for each pair of distinct vertices u and v in G. They called f a *distance compatible set labeling* (dcsl) of G, and the ordered pair (G, f), a distance compatible set labeled (dcsl) graph.

Definition 1. ([2]) Let $G = (V, E)$ be any connected graph. A distance compatible set labeling (dcsl) of a graph G is an injective set assignment $f : V(G) \to 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\phi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u, v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen.

A distance compatible set labeling f of G is k-uniform if all the constants of proportionality with respect to f in Definition 1 are equal to k, and if G admits such a distance compatible set labeling then G is called a k-uniform distance compatible set labeled graph.

Listed below are the definitions and known results which are used in this paper.

Definition 2. ([5]) A family of sets $\mathcal{F}$ is well-graded if any two sets in $\mathcal{F}$ can be connected by a sequence of sets formed by single element insertion and deletion, without redundant operations, such that all intermediate sets in the sequence belong to $\mathcal{F}$.

Definition 3. ([5]) Let $\mathcal{F}$ be a family of subsets of a set X. A tight path between two distinct sets P and Q(or from P to Q) in $\mathcal{F}$ is a sequence $P_0 = P, P_1, P_2 \dots P_n = Q$ in $\mathcal{F}$ such that $d(P,Q) = |P \oplus Q| = n$ and $d(P_i, P_{i+1}) = 1$ for $0 \leq i \leq n-1$.

The family $\mathcal{F}$ is well-graded family (or wg-family), if there is a tight path between any two of its distinct sets.

Definition 4. ([5]) Any family $\mathcal{F}$ of subsets of X defines a graph $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$,where $E_{\mathcal{F}} = \{\{P,Q\} \subseteq \mathcal{F} : |P \oplus Q| = 1\}$ and we call $G_{\mathcal{F}}$, an $\mathcal{F}$-induced graph.

Definition 5. ([5]) A family of sets $\mathcal{F}$ is closed under union or $\bigcup-$closed if for any nonempty $\mathcal{H} \subseteq \mathcal{F}$, we have $U\mathcal{H} \in \mathcal{F}$. The span of a family of sets $\mathcal{F}$ is the family $\mathcal{F}'$ containing any set which is the union of some nonempty subfamily of $\mathcal{F}$.

The following universal theorem has been established in [2].

Theorem 6. ([2]) *Every graph admits a distance compatible set labeling.*

Germina and Jinto [7] established some interesting results relating 1-uniform distance compatible set labeled graphs and well-graded family of sets.

Theorem 7. ([7]) *A graph G is a 1-uniform distance compatible set labeled graph if and only if there exists a family of subsets $\mathcal{F}$ of X, which is well-graded.*

Theorem 8. ([8]) *The set of vertex labelings of any 1-uniform dcsl graph form a well-graded family of sets $\mathcal{F}$, there exist the $\mathcal{F}$- induced graph $G_{\mathcal{F}}$, which is 1-uniform dcsl graph.*

Definition 9. ([7]) The span of a 1- uniform dcsl graph G is the graph induced by the span of the vertex labelings of G, denoted by $SpanG$.

Theorem 10. ([8]) *If $\mathcal{F}'$ is the span of a well-graded family of sets $\mathcal{F}$, then the $\mathcal{F}'$- induced graph $G_{\mathcal{F}'}$ is 1-uniform dcsl graph.*

Theorem 11. ([8]) *A* $1-$ *uniform distance compatible set labeled graph* G *can be isometrically embedded in its span.*

Germina and Jinto [7] while discussing the $\mathcal{F}$-induced graph $G_{\mathcal{F}}$, paused an open problem [8] of determining those $\mathcal{F}$-induced graph $G_{\mathcal{F}}$, in which the base $\mathcal{B}$-induced graph is 1-uniform dcsl. In this paper we discuss this open problem and completely characterize the classes of $\mathcal{F}$-induced graph $G_{\mathcal{F}}$, in which the base $\mathcal{B}$-induced graph is a 1-uniform dcsl.

2. Main Results

Towards establishing this open problem, we first study the structure of $\mathcal{B}$-induced 1-uniform distance compatible set labeled graph with $\mathcal{B}$ as the basis and, the structure of the respective $\mathcal{F}$-induced graph.

One may note that a family $\mathcal{B}$, spanning a family $\mathcal{F}$, is a base of $\mathcal{F}$, if and only if, none of the sets in $\mathcal{B}$ is the union of some other sets in $\mathcal{B}$. Let $\mathcal{F}$ be a $\bigcup$-closed wg-family, and $\mathcal{B}$ be a base of $\mathcal{F}$. Then $\mathcal{B}$ need not necessarily be a wg-family and hence $G_{\mathcal{B}}$, the $\mathcal{B}$-induced graph in general, need not necessarily be a 1-uniform dcsl graph. In particular, the $\mathcal{B}$-induced graph may even be disconnected so that it is not 1-uniform dcsl graph.

Theorem 12. *The base* $\mathcal{B}$*-induced graph* $G_{\mathcal{B}}$ *is* 1- *uniform distance compatible set labeled graph if and only if for every pair of vertices* $u, v \in V(G_{\mathcal{B}})$ *with* $f(u) = \{\emptyset\}$, *there exist a tight path between* u *and* v.

Proof. Suppose that the base $\mathcal{B}-$induced graph $G_{\mathcal{B}}$ is 1- uniform dcsl. Let $u \in V(G_{\mathcal{B}})$ with $f(u) = \{\emptyset\}$ and v be any vertex in $G_{\mathcal{B}}$. Since $G_{\mathcal{B}}$ is 1-uniform distance compatible set labeled graph there exist a tight path between every pair of vertices u_i, v_j and in particular between u, v.

Conversely, let $G_{\mathcal{B}}$ be the $\mathcal{B}$- induced graph with $\mathcal{B}$ as the basis with a vertex u identified as $f(u) = \{\emptyset\}$ and, which is minimal with respect to the set inclusion. Suppose that there exist a tight path $P = P_0, P_1, P_2, \ldots, P_n = Q$ between every pair of vertices u, v of $G_{\mathcal{B}}$ and let $u = u_0, u_1, u_2, \ldots, u_n = v$ be the shortest path between u and v. Without loss of generality, let $P_0 = \{\emptyset\} = f(u)$. and $P_1 = f(u_1), P_2 = f(u_2), \ldots, f(u_n) = f(v) = Q$. Hence, for $u \in G_{\mathcal{B}}$, let $u - v_j\ (j = 1, 2, 3, \ldots, k)$ be the tight path between u and $v_j\ (j = 1, 2, 3, \ldots, k)$. That is, there exist $P_0 = \{\emptyset\} = f(u), Q_1 = f(v_1), Q_2 = f(v_2), \ldots, f(v_j) = Q$ is the $1-$ uniform dcsl labeling of the path $u - v_j$. Let $u_i, v_j \in V(G_{\mathcal{B}})$ be any two arbitrary vertices. If $u_iv_j \in E(G_{\mathcal{B}})$ then, $|f(u_i) \oplus f(v_j)| = 1$. Otherwise, if u_iv_j

is not an edge in $G_{\mathcal{B}}$, then $|f(u_i)\oplus f(v_j)| = d(u_i, v_j) = |Q_i\oplus Q_j| = j - i$. □

Corollary 13. *The base $\mathcal{B}-$induced graph $G_{\mathcal{B}}$ is a 1-uniform distance compatible labeled path P_n if and only if there exists exactly one path $u, v \in V(G_{\mathcal{B}})$ with $f(u) = \{\emptyset\}$ and of length n. Also the $\mathcal{B}-$induced graph $G_{\mathcal{B}}$ is isomorphic to the path P_n.*

Theorem 14. *Let $G_{\mathcal{B}}$ be the base $\mathcal{B}$- induced 1-uniform dcsl graph. Then the $\mathcal{F}$-induced graph $G_{\mathcal{F}}$ is 1-uniform dcsl if and only if, there exists an isometric embedding of $G_{\mathcal{B}}$ to $G_{\mathcal{F}}$.*

Proof. Invoking Theorem 12 and Corollary 13, if there exists a vertex u with $f(u) = \{\emptyset\}$ and if the $u - v$ path with $f(u) = \{\emptyset\}$ is a unique 1-uniform dcsl path, then the $\mathcal{B}$-induced graph $G_{\mathcal{B}}$ is isomorphic to the path P_n, which is clearly an isometric embedding of $G_{\mathcal{B}}$ to $G_{\mathcal{F}}$. On the other hand, if the path is not unique and there exists a unique vertex u with $f(u) = \{\emptyset\}$ with all the $u-v$ paths with initial vertex as u, is 1 uniform dcsl then, since $G_{\mathcal{F}}$ is U-closed, we get $G_{\mathcal{F}}$ as an isometric embedding of $G_{\mathcal{B}}$, whose span is clearly 1-uniform dcsl.

Conversely, let $u - w$ path be the path of maximum length say n. Then $f(u) = \{\emptyset\}$, and $|f(u_i)|$ should necessarily equal to i ; $1 \leq i \leq n$, where n is the largest length of the path. This necessarily implies that in $G_{\mathcal{B}}$ all the vertices adjacent to u should necessarily be assigned with singleton set and all the vertices adjacent with vertices of assignments, as the sets of cardinality 1 will have the set assignment with cardinality 2 and so on. Now, $\mathcal{B}$ is a basis and minimal with respect to set inclusion. Hence, to move from the initial vertex u to the immediate next vertex in any $u - v$ path, the vertex assignment should defer by one in cardinality. Also any 1- uniform dcsl graph G can be isometrically embedded in its span. Now the span of $\mathcal{B}$ is $\mathcal{F}$ and the $\mathcal{B}-$ induced graph $G_{\mathcal{B}}$ is 1-uniform and is an induced subgraph of $G_{\mathcal{F}}$. By defining the isometry from $\mathcal{B}$ to $\mathcal{F}$ by the identity map we get $G_{\mathcal{F}}$, an isometric embedding of $G_{\mathcal{B}}$. Hence the $\mathcal{F}-$ induced graph $G_{\mathcal{F}}$ with $G_{\mathcal{B}}$ the base $\mathcal{B}$-induced 1$-$uniform dcsl are the graphs nothing but the isometric embedding of $G_{\mathcal{B}}$ to $G_{\mathcal{F}}$. □

Theorem 15. *Let $G_{\mathcal{B}}$ be the $\mathcal{B}$-base induced graph. If $G_{\mathcal{B}} \cong P_n$ then $G_{\mathcal{F}}$ is isomorphic to an induced subgraph of a partial cube or is isomorphic to a partial cube.*

Proof. Let $\mathcal{B}$ be the base of a $u-v$ path P_n and $\mathcal{B}$ is a minimal with respect to the inclusion. If $G_{\mathcal{B}}$ is 1-uniform dcsl with $f(u) = \{\emptyset\}$, then the result is obvious as $G_{\mathcal{F}} \cong P_n$. If $G_{\mathcal{B}}$ is 1-uniform dcsl with any other vertex other than the initial vertex u receives the labeling $\{\emptyset\}$,in which case the collection of vertex assignments need not form a basis. But, since $\mathcal{F}$ is $U-$closed, union of all pairs of vertices that are not in $\mathcal{B}$ should necessarily be assigned to some isolated vertices in $G_{\mathcal{F}}$. Define the edges so that the induced graph is 1-uniform dcsl so that the span of $\mathcal{B}$ is a 1-uniform dcsl graph which can be isometrically embedded in its span. Thus, for every pair of vertices u, w in $G_{\mathcal{F}}$, there exist a tight path with $|f(u_i) \oplus f(u_{i+1})| = 1$ and $|f(u) \oplus f(w)| = d(u, w)$, which in turn implies that either $G_{\mathcal{F}}$ is an induced subgraph of partial cube or is isomorphic to a partial cube. □

Theorem 16. *If the base $\mathcal{B}$ induced graph $G_{\mathcal{B}} \cong K_{1,n}$ then $G_{\mathcal{F}}$ isomorphic to partial cube of dimension n or $n+1$.*

Proof. Let $\mathcal{B}$ be the base of a star graph $K_{1,n}$. If $\{\emptyset\} \in \mathcal{B}$, and is assigned to some vertex of $K_{1,n}$, then the other vertices of $K_{1,n}$ should necessarily be labeled with distinct singleton sets or with the sets such that they are not union of any two vertex assignments. Now consider the span $\mathcal{F}$ of $\mathcal{B}$, and add enough number of isolated vertices so as to assign the sets in $\mathcal{F}$ that are not assigned to any vertex of $G_{\mathcal{B}}$, to these isolated vertices. If $\{\emptyset\} \notin \mathcal{B}$, then add an isolated vertex to assign $\{\emptyset\}$. Also, since, $\mathcal{B}$ is a minimal with respect to the inclusion, $G_{\mathcal{B}}$ is a $\mathcal{B}$-induced 1-uniform dcsl graph. Hence, by defining edges with the isolated vertices one may find that the span $\mathcal{F}$ is nothing but the power set 2^n or 2^{n+1} according as $\{\emptyset\} \in \mathcal{B}$ or $\{\emptyset\} \notin \mathcal{B}$ respectively. Hence $G_{\mathcal{F}}$ isomorphic to partial cube of dimension n or $n+1$. □

Theorem 17. *If $G_{\mathcal{B}} \cong C_n$ where n is even then, $G_{\mathcal{F}}$ isomorphic to partial cube.*

Proof. Let $\mathcal{B}$ be the base of a cycle C_n where n is even and $\mathcal{B}$ is a minimal with respect to the inclusion. Let $G_{\mathcal{B}}$ be the induced 1-uniform dcsl graph. Then, there exists exactly two distinct tight paths namely, $u, u_1, u_2, \ldots, u_{\frac{n}{2}}$ and $u, u_n, u_{n-1}, \ldots, u_{\frac{n}{2}}$. Also if $\{\emptyset\}$ is not assigned to a vertex in $V(G_{\mathcal{B}})$, then there exists exactly two distinct tight paths namely the $u, u_1, u_2, \ldots, u_{\frac{n}{2}}$ and, $u, u_n, u_{n-1}, \ldots, u_{\frac{n}{2}}$ with $|f(u)| = 1$. In both the cases $G_{\mathcal{F}}$ isomorphic to partial cube of dimension n or $n+1$ according as whether or not the $\{\emptyset\}$ is assigned to a vertex of C_n. □

Theorem 18. *Let $G_{\mathcal{B}}$ be the base $\mathcal{B}-$ induced 1-uniform dcsl graph. Then the $\mathcal{F}$-induced graph $G_{\mathcal{F}}$-a partial cube if there exists a vertex u in $V(G_{\mathcal{B}})$ such that degree of u is equal to the eccentricity of the induced graph $G_{\mathcal{B}}$.*

Proof. Let there exist a vertex $u \in V(G_{\mathcal{B}})$ such that the degree of u is equal to eccentricity of $G_{\mathcal{B}}$. Let eccentricity of $G_{\mathcal{B}} = n$, so that is there exist a $u - v$ path of length n in $G_{\mathcal{B}}$. Since $G_{\mathcal{B}}$ being 1-uniform dcsl, this $u - v$ path should necessarily be a tight path. Hence $f(u) = \{\emptyset\}$,$f(u_1) = A_1$ with $|A_1| = 1, f(u_2) = A_2$ with $|A_2| = 2, \ldots, f(v) = f(u_n) = A_n$, with $|A_n| = n$. Also $\mathcal{B}$ being the basis, it is minimal with respect to set inclusion and hence $A_i \subset A_{i+1}, 1 \leq i \leq n-1$ and $|A_i \oplus A_{i+1}| = 1$ and $|A_0 \oplus A_n| = n$. If $d(u) = n$ and let $u_1 = v_1, v_2, \ldots, v_n$ be the n neighbours of u. Hence, these n vertices should necessarily be assigned with sets of cardinality 1. That is, each of the vertices in the neighbourhood of u receive the labeling of $\{1\}, \{2\}, \ldots, \{n\}$. Hence, $u, v_1, \ldots, v_j$; $u, v_2, \ldots, v_k$; $\ldots$; $u, v_n, \ldots, v_l$ are all the paths originating from u and should necessarily be a tight path of length $j, k, \ldots, l$ respectively. That is, there exists, distinct $1-$ uniform dcsl paths with a common vertex u with $f(u) = \{\emptyset\}$, and the neighbouring vertices say $u_1, u_2, \ldots, u_n$ of u with labeling singleton sets. The paths being tight path, all the vertices in the neighbourhoods of each of the vertices $v_i, 1 \leq i \leq n$ will necessarily receive the assignments with sets of cardinality 2 and so on. Since $\mathcal{F}$ is U-closed, if needed add enough number of isolated vertices and assign the isolated vertices with subsets from 2^X where $|X| = n$, which are not being assigned to the vertices of the above considered paths. Apply the isometric embedding of $G_{\mathcal{B}}$ in such a way that for every pair of vertices u, v there exist a tight path with $|f(u_i) \oplus f(u_{i+1})| = 1$ and $|f(u) \oplus f(v)| = d(u, v)$. Hence the embedded graph $G_{\mathcal{F}}$ is nothing but a partial cube. □

Now we characterize the open problem mentioned in the paper.

Theorem 19. *Let $\mathcal{B}$ be a basis and $\mathcal{G}_{\mathcal{B}}$ be induced graph induced by the basis $\mathcal{B}$. Then, classes of $\mathcal{F}$-induced graph $G_{\mathcal{F}}$ in which the base $\mathcal{B}$-induced graph is 1-uniform are isomorphic to a bipartite graph that are embedable in a partial cube.*

Proof. Let $\mathcal{B}$ be a basis and $\mathcal{G}_{\mathcal{B}}$ be the $\mathcal{B}$ induced graph. Hence, $\mathcal{B}$ contains all unions and is minimal with respect to inclusion. Assume $\mathcal{G}_{\mathcal{B}}$ is 1-uniform dcsl. Being a 1-uniform dcsl, $\mathcal{G}_{\mathcal{B}}$ is bipartite and so is the span $\mathcal{F}$. The span $\mathcal{F}$ is always 1-uniform and invoking Theorem 12, there exists a unique vertex u such that $f(u) = \{\emptyset\}$ such that all $u - v$ paths are tight. Again invoking Theorems, [15,16,17,18], these bipartite graphs are embedable in a partial cube. Converse is obvious as any induced subgraph of a 1-uniform dcsl is again 1-uniform. □

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