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CHARACTERISTIC DECOMPOSITIONS FOR THE UNSTEADY TRANSONIC SMALL DISTURBANCE EQUATION

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Abstract: We consider a Riemann problem for the unsteady transonic small disturbance equation resulting in interacting rarefaction waves. We rewrite the problem in self-similar coordinates and we derive characteristic decomposition equations for the inclination angle variables.

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1. Introduction

In this paper we continue the efforts from our earlier work in [7, 8] on the study of Riemann problems for the unsteady transonic small disturbance equation using characteristic decomposition equations. We are interested in Riemann initial data that results in interacting rarefaction waves and gives rise to Goursat-type boundary value problems in self-similar coordinates.

The Goursat-type boundary value problems appear in the study of the expansion of a wedge of gas into vacuum and have been considered using two main approaches: the hodograph transform and the direct characteristic decomposition. An overview of ideas and results for various systems of gas dynamics equations could be found in [15] by Zheng.

Li [10] modeled the problem using the unsteady isentropic Euler equations and proved existence of a global smooth solution in the hodograph plane. In

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[12], Li and Zheng showed that the hodograph transform was non-degenerate for non-simple waves and, therefore, the solutions in the hodograph plane in [10] could be transformed back to the self-similar plane. The existence of a global continuous solution to the problem of gas expansion into a vacuum was proved for the pressure-gradient equations using hodograph transform in [14], by Yang and Zhang. The interaction of steady rarefaction waves for the isentropic Euler equations resulting in Goursat and generalized Goursat problems was studied, also using hodograph transform, by Chen and Qu in [1].

The expansion of a wedge of gas into vacuum problem was considered using characteristic forms for pressure-gradient equations by Dai and Zhang in [3], where they proved the global existence of a smooth solution to the degenerate Goursat problem up to the vacuum boundary. The characteristic decompositions for the isentropic Euler equations and potential flow were derived in [2], by Chen and Zheng, in [11], by Li, Zhang, and Zheng, and in [13], by Li, Yang, and Zheng. These characteristic decompositions were used to prove that any wave adjacent to a constant state for the adiabatic Euler equations was a simple wave and to prove existence of of a smooth solution to the interaction of two planar rarefaction waves for the isentropic gas dynamics equations. The characteristic decomposition approach was also utilized in the study of the Goursat-type problems for the isothermal Euler equations in [5], by Hu, Li, and Sheng, for the Euler equations for the generalized Chaplygin gas in [4], by Ge and Sheng, and for the nonlinear wave system in [6], by Hu and Wang.

The goal of this paper is to establish the characteristic decomposition equations for the unsteady transonic small disturbance equation following the studies in [2, 11]. The paper is organized as follows. In §2 we state the unsteady transonic small disturbance equation, we derive the characteristic form in the self-similar plane, and in §3 we derive the second order differential equations for the velocity u along the characteristic curves. The inclination angle of characteristics variables are introduced in §4, where we derive angle forms for the directional derivatives of u in the characteristic directions. In Theorem 6 we obtain the angle form second order differential equations for u in the characteristic directions. We conclude with §5, where we derive the second order differential equations for the inclination angle variables.

2. Motivation and the characteristic form

An overview of two-dimensional Riemann problems for various gas dynamics equations was presented by Keyfitz in [9]. In this paper we consider the unsteady transonic small disturbance equation

$$\begin{aligned} &u_t + uu_x + v_y = 0, \\ &-v_x + u_y = 0, \end{aligned} \tag{1}$$

where $t \in [0, \infty)$, $(x, y) \in \mathbb{R}^2$, and $(u, v) : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity vector.

We are interested in Riemann initial data which results in interacting rarefaction waves and one such problem was posed in [8]. Two possible structures of the solution were conjectured and both involve a Goursat-type problem in part of the domain (see the region $ABMB_1$ in Figures 3 and 4 in [8]). To study existence of solutions to such Goursat-type boundary value problems, in this paper we derive characteristic decomposition equations, following the ideas in [2, 11].

We write (1) in self-similar coordinates $\xi = x/t$ and $\eta = y/t$, and obtain

$$\begin{aligned} &(u - \xi)u_\xi - \eta u_\eta + v_\eta = 0, \\ &-v_\xi + u_\eta = 0, \end{aligned} \tag{2}$$

which can be written as

$$\begin{bmatrix} u \\ v \end{bmatrix}_\xi + \begin{bmatrix} -\frac{\eta}{u-\xi} & \frac{1}{u-\xi} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_\eta = \vec{0}. \tag{3}$$

The characteristic equation of the matrix in (3) is

$$(\xi - u)\Lambda^2 - \eta\Lambda - 1 = 0. \tag{4}$$

We find the eigenvalues

$$\Lambda_\pm = \frac{\eta \pm \sqrt{\eta^2 + 4(\xi - u)}}{2(\xi - u)}, \tag{5}$$

and note that

$$\Lambda_+ + \Lambda_- = \frac{\eta}{\xi - u} \qquad \text{and} \qquad \Lambda_+\Lambda_- = -\frac{1}{\xi - u}. \tag{6}$$

The left eigenvectors are given by

$$\vec{l}_\pm = [\Lambda_\pm(u - \xi), \quad 1] = [\Lambda_\mp^{-1}, \quad 1].$$

We multiply (3) on the left by $\vec{l}_\pm$ and we obtain the characteristic form

$$\partial^\pm u + \Lambda_\mp\, \partial^\pm v = 0, \tag{7}$$

where $\partial^\pm := \partial_\xi + \Lambda_\pm\, \partial_\eta$ denote the directional derivatives along the characteristic directions in the self-similar plane.

3. Characteristic decomposition equations for u

In this section we derive the directional derivatives of eigenvalues $\Lambda_\pm$ along the characteristic directions in the self-similar plane and we use the characteristic decomposition (Theorem 2.1) from [15] to obtain the second order directional derivatives for u.

Proposition 1. *There hold*

$$\partial^\pm \Lambda_\pm = \frac{(\Lambda_\pm)^3 \Lambda_\mp}{\Lambda_\mp - \Lambda_\pm} \partial^\pm u \ \ \textit{and} \ \ \partial^\pm \Lambda_\mp = \frac{(\Lambda_\mp)^3 \Lambda_\pm}{\Lambda_\pm - \Lambda_\mp} \partial^\pm u + \Lambda_\pm (\Lambda_\mp)^2. \tag{8}$$

Proof. We recall the characteristic equation (4) and we regard Λ as a function of ξ, η and u. Differentiating (4) with respect to u, ξ, and η we obtain

$$\partial_u \Lambda = -\frac{\Lambda^2}{2\Lambda(u-\xi)+\eta}, \qquad \partial_\xi \Lambda = \frac{\Lambda^2}{2\Lambda(u-\xi)+\eta},$$

and

$$\partial_\eta \Lambda = -\frac{\Lambda}{2\Lambda(u-\xi)+\eta},$$

respectively. From (5) we have

$$\Lambda_+ - \Lambda_- = \frac{\sqrt{\eta^2 + 4(\xi - u)}}{\xi - u} = -\Lambda_+ \Lambda_- \sqrt{\eta^2 + 4(\xi - u)},$$

where the last equality follows from (6). Hence,

$$2\Lambda_\pm (u - \xi) + \eta = \mp \sqrt{\eta^2 + 4(\xi - u)} = \frac{\Lambda_\pm - \Lambda_\mp}{\Lambda_\pm \Lambda_\mp}.$$

Next, we note $\partial^\pm \xi = 1$ and $\partial^\pm \eta = \Lambda^\pm$, and by the chain rule, we have

$$\begin{aligned} \partial^\pm \Lambda_\pm &= \partial_u \Lambda_\pm \partial^\pm u + \partial_\xi \Lambda^\pm \partial^\pm \xi + \partial_\eta \Lambda^\pm \partial^\pm \eta \\ &= \partial_u \Lambda_\pm \partial^\pm u = \frac{(\Lambda_\pm)^3 \Lambda_\mp}{\Lambda_\mp - \Lambda_\pm} \partial^\pm u, \end{aligned}$$

and

$$\begin{aligned} \partial^\pm \Lambda^\mp &= \partial_u \Lambda_\mp \partial^\pm u + \partial_\xi \Lambda^\mp \partial^\pm \xi + \partial_\eta \Lambda^\mp \partial^\pm \eta \\ &= \partial_u \Lambda_\mp \partial^\pm u + \frac{\Lambda_\mp (\Lambda_\mp - \Lambda_\pm)}{2\Lambda_\mp (u - \xi) + \eta} = \frac{(\Lambda_\mp)^3 \Lambda_\pm}{\Lambda_\pm - \Lambda_\mp} \partial^\pm u + \Lambda_\pm (\Lambda_\mp)^2. \end{aligned}$$

□

Theorem 2. *There holds*

$$\begin{aligned}&\partial^{\pm}\partial^{\mp}u + \frac{\Lambda_{\pm}\Lambda_{\mp}^{2}}{(\Lambda_{\pm}-\Lambda_{\mp})^{2}}(\Lambda_{\mp}-2\Lambda_{\pm})\partial^{\pm}u\partial^{\mp}u\\ &+\frac{\Lambda_{\pm}^{3}\Lambda_{\mp}}{(\Lambda_{\pm}-\Lambda_{\mp})^{2}}(\partial^{\mp}u)^{2}-\Lambda_{\pm}\Lambda_{\mp}\partial^{\mp}u=0.\end{aligned}$$

Proof. The result follows from Theorem 2.1 in [15], which gives the characteristic decomposition

$$\partial^{\pm}\partial^{\mp}u+\frac{\partial^{+}\Lambda_{-}-\partial^{-}\Lambda_{+}}{\Lambda_{+}-\Lambda_{-}}\partial^{\mp}u=\frac{\Lambda_{+}\Lambda_{-}}{\Lambda_{+}-\Lambda_{-}}\left(\frac{\partial^{-}\Lambda_{-}}{\Lambda_{-}^{2}}\partial^{+}u-\frac{\partial^{+}\Lambda_{+}}{\Lambda_{+}^{2}}\partial^{-}u\right),$$

and by substituting the expressions in (8) for the derivatives of $\Lambda^{\pm}$. □

4. Characteristic decomposition equations for u in terms of the inclination angle variables

We introduce the inclination angle variables α and β by

$$\tan\alpha=\Lambda_{+}\quad\text{and}\quad\tan\beta=\Lambda_{-}. \tag{9}$$

From the second property in (6) we obtain

$$u=\xi+\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}, \tag{10}$$

which expresses u in terms of these new variables α and β. In this section we derive the characteristic decomposition equations for u in terms of the inclination angle variables α and β.

Proposition 3. *There hold*

$$\begin{aligned}\partial^{\pm}u&=1-\frac{\cos\beta}{\sin^{2}\alpha\sin\beta}\partial^{\pm}\alpha-\frac{\cos\alpha}{\sin\alpha\sin^{2}\beta}\partial^{\pm}\beta, \qquad (11)\\ \partial^{+}v&=-\frac{\cos\beta}{\sin\beta}\left(1-\frac{\cos\beta}{\sin^{2}\alpha\sin\beta}\partial^{+}\alpha-\frac{\cos\alpha}{\sin\alpha\sin^{2}\beta}\partial^{+}\beta\right),\\ \partial^{-}v&=-\frac{\cos\alpha}{\sin\alpha}\left(1-\frac{\cos\beta}{\sin^{2}\alpha\sin\beta}\partial^{-}\alpha-\frac{\cos\alpha}{\sin\alpha\sin^{2}\beta}\partial^{-}\beta\right).\end{aligned}$$

Proof. We differentiate (10) to find

$$\partial_\xi u = 1 - \frac{\cos\beta}{\sin^2\alpha\sin\beta}\alpha_\xi - \frac{\cos\alpha}{\sin\alpha\sin^2\beta}\beta_\xi \tag{12}$$

and

$$\partial_\eta u = -\frac{\cos\beta}{\sin^2\alpha\sin\beta}\alpha_\eta - \frac{\cos\alpha}{\sin\alpha\sin^2\beta}\beta_\eta. \tag{13}$$

Therefore

$$\partial^\pm u = u_\xi + \Lambda_\pm u_\eta = 1 - \frac{\cos\beta}{\sin^2\alpha\sin\beta}\partial^\pm\alpha - \frac{\cos\alpha}{\sin\alpha\sin^2\beta}\partial^\pm\beta.$$

From the characteristic form (7) we have

$$\partial^\pm v = -(\Lambda_\mp)^{-1}\partial^\pm u,$$

implying

$$\partial^+ v = -\frac{\cos\beta}{\sin\beta}\partial^+ u \quad \text{and} \quad \partial^- v = -\frac{\cos\alpha}{\sin\alpha}\partial^- u,$$

which completes the proof. □

Proposition 4. *There hold*

$$\bar\partial^+\alpha = \sin^3\alpha - \frac{\sin^2\alpha}{\sin^2\beta}\bar\partial^+\beta \quad \textit{and} \quad \bar\partial^-\beta = \sin^3\beta - \frac{\sin^2\beta}{\sin^2\alpha}\bar\partial^-\alpha.$$

where $\bar\partial^+ := \cos\alpha\,\partial_\xi + \sin\alpha\,\partial_\eta$ *and* $\bar\partial^- := \cos\beta\,\partial_\xi + \sin\beta\,\partial_\eta$.

Proof. Using the definition (9) of α and (8) we have

$$\begin{aligned}\partial^+\alpha &= \partial^+(\arctan\Lambda_+) = \frac{1}{1+\Lambda_+^2}\partial^+\Lambda_+ \\ &= \frac{1}{1+\Lambda_+^2}\frac{\Lambda_+^3\Lambda_-}{\Lambda_- - \Lambda_+}\partial^+ u = -\frac{\sin^3\alpha\sin\beta}{\sin(\alpha-\beta)}\partial^+ u.\end{aligned}$$

Substituting (11) we obtain

$$\partial^+\alpha = -\frac{\sin^3\alpha\sin\beta}{\sin(\alpha-\beta)}\left(1 - \frac{\cos\beta}{\sin^2\alpha\sin\beta}\partial^+\alpha - \frac{\cos\alpha}{\sin\alpha\sin^2\beta}\partial^+\beta\right),$$

implying

$$\partial^+\alpha = \frac{\sin^3\alpha}{\cos\alpha} - \frac{\sin^2\alpha}{\sin^2\beta}\partial^+\beta.$$

Multiplying this equation by $\cos\alpha$, we obtain

$$\bar\partial^+\alpha = \sin^3\alpha - \frac{\sin^2\alpha}{\sin^2\beta}\bar\partial^+\beta.$$

Similarly, we obtain the expression for $\bar\partial^-\beta$ in terms of $\bar\partial^-\alpha$. □

Remark 5. Note that using the previous proposition in (11) we can express derivatives $\partial^\pm u$ in terms of directional derivatives of only one inclination angle variable. More precisely, we have

$$\begin{aligned}\partial^+ u &= 1 - \frac{\cos\beta}{\cos\alpha\sin^2\alpha\sin\beta}\bar\partial^+\alpha - \frac{1}{\sin\alpha\sin^2\beta}\bar\partial^+\beta\\ &= 1 - \frac{\cos\beta}{\cos\alpha\sin^2\alpha\sin\beta}\left(\sin^3\alpha - \frac{\sin^2\alpha}{\sin^2\beta}\bar\partial^+\beta\right) - \frac{1}{\sin\alpha\sin^2\beta}\bar\partial^+\beta\\ &= \frac{\sin(\beta-\alpha)}{\sin\beta\cos\alpha}\left(1 - \frac{1}{\sin\alpha\sin^2\beta}\bar\partial^+\beta\right)\end{aligned}$$

and, similarly,

$$\partial^- u = \frac{\sin(\alpha-\beta)}{\sin\alpha\cos\beta}\left(1 - \frac{1}{\sin^2\alpha\sin\beta}\bar\partial^-\alpha\right). \tag{14}$$

Theorem 6. *There hold*

$$\begin{aligned}\partial^+\partial^- u = \sin\alpha\sin\beta\,\partial^- u\Bigg\{&\frac{1}{\cos\alpha\cos\beta} - \frac{1}{\sin^2(\alpha-\beta)}\times\\ &\left(\frac{\sin\beta(\sin(\beta-\alpha)-\sin\alpha\cos\beta)}{\cos\beta}\partial^+ u + \frac{\sin^2\alpha\cos\beta}{\cos\alpha}\partial^- u\right)\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\partial^-\partial^+ u = \sin\alpha\sin\beta\,\partial^+ u\Bigg\{&\frac{1}{\cos\alpha\cos\beta} - \frac{1}{\sin^2(\alpha-\beta)}\times\\ &\left(\frac{\sin\alpha(\sin(\alpha-\beta)-\sin\beta\cos\alpha)}{\cos\alpha}\partial^- u + \frac{\sin^2\beta\cos\alpha}{\cos\beta}\partial^+ u\right)\Bigg\}.\end{aligned}$$

Proof. The results follow from Theorem 2 and definitions (9) of α and β. □

5. Characteristic decomposition equations for the inclination angle variables

We derive the second order differential equations for the inclination angle variables α and β along the characteristic curves.

Proposition 7. *For any $I(\xi,\eta)$ there holds the identity*

$$\begin{aligned}
&\bar\partial^-\bar\partial^+ I-\bar\partial^+\bar\partial^- I\\
&=\bar\partial^+ I\left\{\frac{1}{\cos\alpha}\left(\frac{\cos\beta}{\sin(\alpha-\beta)}-\sin\alpha\right)\bar\partial^-\alpha-\frac{1}{\sin(\alpha-\beta)}\bar\partial^+\beta\right\}\\
&+\bar\partial^- I\left\{-\frac{1}{\sin(\alpha-\beta)}\bar\partial^-\alpha+\frac{1}{\cos\beta}\left(\sin\beta+\frac{\cos\alpha}{\sin(\alpha-\beta)}\right)\bar\partial^+\beta\right\}.
\end{aligned}$$

Proof. Using the definitions of the derivatives $\bar\partial^\pm$ in Proposition 4 we have

$$\begin{aligned}
\bar\partial^+\bar\partial^- I&=\bar\partial^+(\cos\beta I_\xi+\sin\beta I_\eta)\\
&=(-\sin\beta I_\xi+\cos\beta I_\eta)\bar\partial^+\beta+\cos\alpha\cos\beta I_{\xi\xi}\\
&+\sin(\alpha+\beta)I_{\xi\eta}+\sin\alpha\sin\beta I_{\eta\eta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar\partial^-\bar\partial^+ I&=(-\sin\alpha I_\xi+\cos\alpha I_\eta)\bar\partial^-\alpha+\cos\alpha\cos\beta I_{\xi\xi}\\
&+\sin(\alpha+\beta)I_{\xi\eta}+\sin\alpha\sin\beta I_{\eta\eta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\bar\partial^-\bar\partial^+ I-\bar\partial^+\bar\partial^- I\\
&=\frac{\bar\partial^-\alpha}{\sin\alpha\cos\alpha}\left(-\sin^2\alpha\cos\alpha I_\xi+\cos^2\alpha\sin\alpha I_\eta\right)\\
&+\frac{\bar\partial^+\beta}{\sin\beta\cos\beta}\left(\sin^2\beta\cos\beta I_\xi-\cos^2\beta\sin\beta I_\eta\right)\\
&=-\frac{\sin\alpha}{\cos\alpha}\bar\partial^-\alpha\,\bar\partial^+ I+\frac{\sin\beta}{\cos\beta}\bar\partial^+\beta\,\bar\partial^- I+I_\eta\left(\frac{\bar\partial^-\alpha}{\cos\alpha}-\frac{\bar\partial^+\beta}{\cos\beta}\right),
\end{aligned}\tag{15}$$

where we substituted $\cos^2\alpha=1-\sin^2\alpha$ and $\cos^2\beta=1-\sin^2\beta$ in the first and second lines, respectively. Next, using the definitions of $\bar\partial^\pm$, we express I_η as

$$I_\eta=\frac{1}{\sin(\alpha-\beta)}\left(\cos\beta\,\bar\partial^+ I-\cos\alpha\,\bar\partial^- I\right).$$

The result follows from substituting this last expression in (15). □

Theorem 8. *There hold the second order equations*

$$\bar\partial^+\bar\partial^-\alpha + (M_1 + N_1\,\bar\partial^-\alpha + P_1\,\bar\partial^+\beta)\bar\partial^-\alpha = Q_1 + R_1\,\bar\partial^+\beta$$

and

$$\bar\partial^-\bar\partial^+\beta + (M_2 + N_2\,\bar\partial^+\beta + P_2\,\bar\partial^-\alpha)\bar\partial^+\beta = Q_2 + R_2\,\bar\partial^-\alpha$$

for some functions $M_1, N_1, P_1, Q_1, R_1, M_2, N_2, P_2, Q_2$ and R_2, depending on α and β .

Proof. By the first result of Theorem 6 we have an expression for $\partial^+\partial^-u$ in terms of α, β, and derivatives $\partial^\pm u$. We use Remark 5 and we express ∂^+u and ∂^-u using $\bar\partial^+\beta$ and $\bar\partial^-\alpha$, respectively, to get

$$\begin{aligned} &\partial^+\partial^-u = \qquad (16)\\ &\frac{1}{\cos\alpha\cos\beta}\left\{-2\sin\alpha\sin\beta + \frac{3}{\sin\alpha}\bar\partial^-\alpha - \frac{1}{\sin^3\alpha\sin\beta}\left(\bar\partial^-\alpha\right)^2\right.\\ &\left.-\frac{\sin(\beta-\alpha)-\sin\alpha\cos\beta}{\sin\alpha\sin\beta\cos\beta}\bar\partial^+\beta\left(1-\frac{1}{\sin^2\alpha\sin\beta}\bar\partial^-\alpha\right)\right\}. \end{aligned}$$

On the other hand, we differentiate (14) along Λ_+ characteristics and we obtain an expression for $\partial^+\partial^-u$ involving $\alpha, \beta, \bar\partial^+\alpha, \bar\partial^+\beta$ and $\bar\partial^+\bar\partial^-\alpha$. We use Proposition 4 to express $\bar\partial^+\alpha$ in terms of $\bar\partial^+\beta$ and, finally, we obtain

$$\begin{aligned} \partial^+\partial^-u &= \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} + \left(\frac{2\sin(\alpha-\beta)}{\sin\alpha\sin\beta\cos\beta} - \frac{1}{\sin\alpha\cos\alpha\cos\beta}\right)\bar\partial^-\alpha\\ &- \frac{\sin(\alpha+\beta)}{\sin\alpha\cos\alpha\sin\beta\cos^2\beta}\bar\partial^+\beta - \frac{\sin(\alpha-\beta)}{\sin^3\alpha\cos\alpha\sin\beta\cos\beta}\bar\partial^+\bar\partial^-\alpha +\\ &\left(\frac{\sin(\alpha+\beta)}{\sin^3\alpha\cos\alpha\sin^2\beta\cos^2\beta} - \frac{\sin(\alpha-\beta)(2\sin\alpha\cos\alpha-\sin\beta\cos\beta)}{\sin^3\alpha\cos\alpha\sin^3\beta\cos\beta}\right)\\ &\times\bar\partial^-\alpha\bar\partial^+\beta. \end{aligned}$$

Setting the right-hand side of the above expression to the right-hand side of (16), we obtain

$$\begin{aligned} &\bar\partial^+\bar\partial^-\alpha = \frac{\sin^2\alpha}{\sin(\alpha-\beta)}\times\\ &\left\{3\sin^2\alpha\sin^2\beta + (2\sin(\alpha-\beta)\cos\alpha - 4\sin\beta)\bar\partial^-\alpha + \frac{1}{\sin^2\alpha}(\bar\partial^-\alpha)^2\right.\\ &-3\sin\alpha\bar\partial^+\beta+ \end{aligned}$$

$$\frac{1}{\sin\alpha\sin\beta}\left(3-\frac{\sin(\alpha-\beta)(2\sin\alpha\cos\alpha-\sin\beta\cos\beta)}{\sin\alpha\sin\beta}\right)\bar{\partial}^-\alpha\bar{\partial}^+\beta\Bigg\},$$

which could be rewritten as

$$\begin{aligned}&\bar{\partial}^+\bar{\partial}^-\alpha-\frac{\sin^2\alpha}{\sin(\alpha-\beta)}\Bigg\{2\sin(\alpha-\beta)\cos\alpha-4\sin\beta+\frac{\bar{\partial}^-\alpha}{\sin^2\alpha}\\&+\frac{1}{\sin\alpha\sin\beta}\left(3-\frac{\sin(\alpha-\beta)(2\sin\alpha\cos\alpha-\sin\beta\cos\beta)}{\sin\alpha\sin\beta}\right)\bar{\partial}^+\beta\Bigg\}\\&\times\bar{\partial}^-\alpha\\&=\frac{\sin^2\alpha}{\sin(\alpha-\beta)}\left(3\sin^2\alpha\sin^2\beta-3\sin\alpha\,\bar{\partial}^+\beta\right).\end{aligned}$$

The second results follows similarly and we obtain

$$\begin{aligned}&\bar{\partial}^-\bar{\partial}^+\beta-\frac{\sin^2\beta}{\sin(\beta-\alpha)}\Bigg\{2\sin(\beta-\alpha)\cos\beta-4\sin\alpha+\frac{\bar{\partial}^+\beta}{\sin^2\beta}\\&+\frac{1}{\sin\alpha\sin\beta}\left(3-\frac{\sin(\beta-\alpha)(2\sin\beta\cos\beta-\sin\alpha\cos\alpha)}{\sin\alpha\sin\beta}\right)\bar{\partial}^-\alpha\Bigg\}\\&\times\bar{\partial}^+\beta\\&=\frac{\sin^2\beta}{\sin(\beta-\alpha)}\left(3\sin^2\alpha\sin^2\beta-3\sin\beta\,\bar{\partial}^-\alpha\right).\end{aligned}$$

□

6. Conclusion

The main purpose of this paper is to derive characteristic decomposition equations for the inclination angle variables (Theorem 8). Our future work includes analysis of these equations in order to utilize them to show existence of a solution to the Goursat-type boundary value problems arising in interacting rarefaction waves formulated in [8].

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