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NEW IMPROVEMENTS IN OLD APPROXIMATIONS TO THE NORMAL CDF

Pedro Mota
Center of Mathematics and Applications (CMA), FCT, UNL
and Department of Mathematics, FCT, UNL
2829-516 Caparica, PORTUGAL

Abstract: The list of approximations to the Normal cumulative distribution function is long and, eventually, not fully known due to the large number of published articles in the last decades. In this paper we will present new improvements in some well known approximations, without increasing the complexity of the formulas.

AMS Subject Classification: 60E05, 62Q99
Key Words: approximation, improvement, maximum absolute error, normal cumulative distribution function

1. Introduction

We will introduce improvements in some of the simpler approximations to the cumulative distribution function (CDF) of the standard normal distribution, such as, the approximations in [1, 2, 3, 4, 5] and, as far as we know, the proposed improvements are new.

We will show that with small order polynomials and with a small number of decimals in the coefficients there is still room to improve the old approximations. We can use this new approximations in a pocket calculator and some of them

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have maximum error of the same degree of many of the tables that are provided, for instance, to the students in many statistics courses. The computation of the approximations presented in this paper, is accomplished by implementing a minimax procedure, that is, by computing the maximum error between the standard normal CDF (obtained from the wolfram Mathematica software) and the proposed approximations and selecting the ones with the smaller maximum error.

2. Known approximations and new improvements

In the following and from the symmetry of the standard normal distribution that implies that $\phi(z) = 1 - \phi(z), \forall z < 0$, we will compute the maximum absolute error between $\phi(z)$ and the considered approximations, only for $z \geq 0$.

We will represent the error function ϵ_i as the difference between the CDF of the standard normal ϕ and the approximations $\phi_i, i = 1, ..., 11$ by:

$$\epsilon_i(z) = \phi(z) - \phi_i(z), \quad z \geq 0. \tag{1}$$

Plots for the error functions are presented in Figures 1 to 6 and in Table 1, details regarding the maximum absolute errors are summarized.

We first start by considering the well known approximation from Hart, that can be found in [3],

$$\phi_1(z) = 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{z^2}{2}}}{z + 0.8e^{-0.4z}}, \quad z \geq 0 \tag{2}$$

and the proposed improvement,

$$\phi_2(z) = 1 - \frac{e^{-\frac{z^2}{2}}}{2.53z + 2e^{-0.45z}}, \quad z \geq 0. \tag{3}$$

In Figure 1 we plot the error functions ϵ_1 and ϵ_2 with maximum absolute errors, $\max(\epsilon_1) = 4.30 \times 10^{-3}$ and $\max(\epsilon_2) = 7.85 \times 10^{-4}$.

Next, we improve the Hamaker's [2] approximation,

$$\phi_3(z) = 0.5\left(1 + \sqrt{1 - e^{-(0.806z(1-0.018z))^2}}\right), \quad z \geq 0 \tag{4}$$

by considering the function

$$\phi_4(z) = 0.5\left(1 + \sqrt{1 - e^{-(0.803z(1-0.015z))^2}}\right), \quad z \geq 0. \tag{5}$$

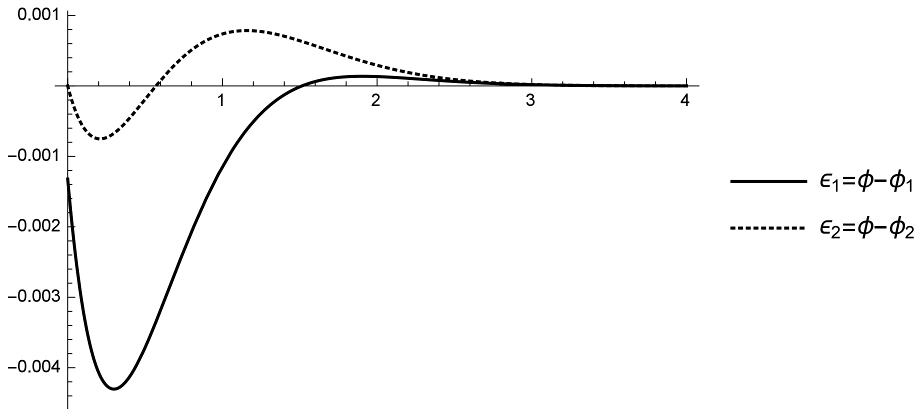

Figure 1: ϵ_1 and ϵ_2 plots

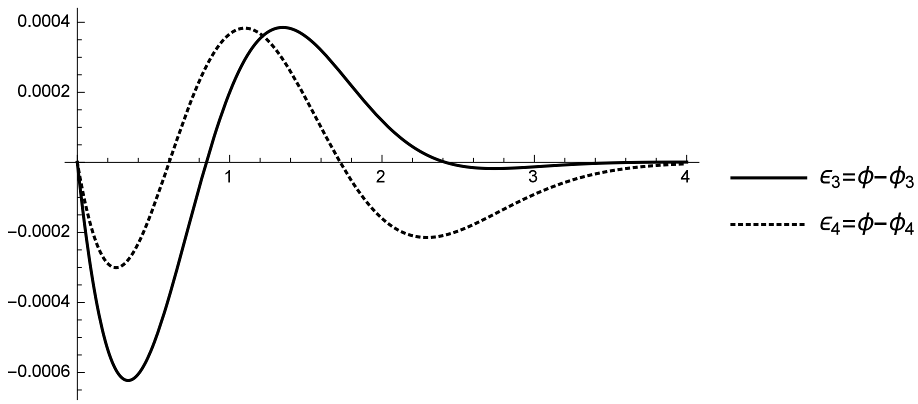

Figure 2: ϵ_3 and ϵ_4 plots

The error functions ϵ_3 and ϵ_4 are plotted in Figure 2 with $\max(\epsilon_3) = 6.23 \times 10^{-4}$ and $\max(\epsilon_4) = 3.83 \times 10^{-4}$.

We also define a new approximation corresponding to an increase in the order of the polynomial in the Hamaker's approximation,

$$\phi_5(z) = 0.5\left(1 + \sqrt{1 - e^{-(0.798z - 0.002z^2 - 0.004z^3)^2}}\right) \tag{6}$$

and in Figure 3, we plot the three error functions (with $\max(\epsilon_5) = 1.18 \times 10^{-4}$).

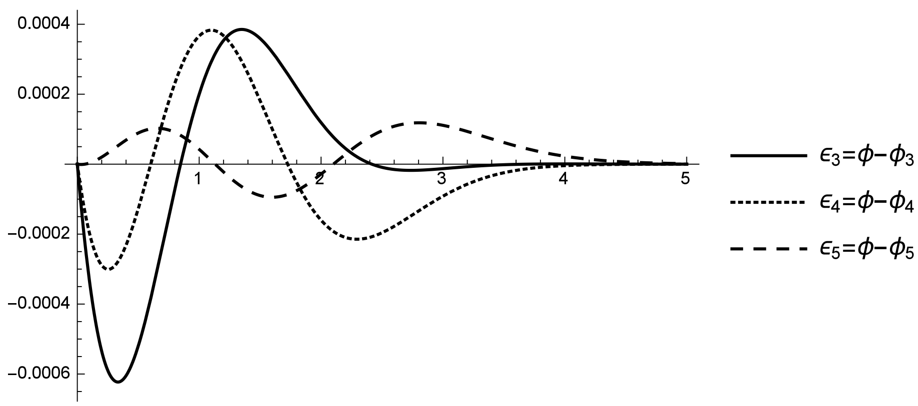

Figure 3: ϵ_3, ϵ_4 and ϵ_5 plots

In Lin [4], it is introduced the approximation

$$\phi_6(z) = 1 - 0.5e^{-0.717z - 0.416z^2}, \tag{7}$$

and our improvement to this approximation is

$$\phi_7(z) = 1 - 0.5e^{-0.778z - 0.375z^2}. \tag{8}$$

In Figure 4 are plotted the error functions ϵ_6 and ϵ_7 with maximum absolute errors, $\max(\epsilon_6) = 6.59 \times 10^{-3}$ and $\max(\epsilon_7) = 8.46 \times 10^{-4}$.

Again Lin, in [5] introduces a different approximation,

$$\phi_8(z) = \left(1 + e^{-\frac{4.2\pi z}{9-z}}\right)^{-1}, \quad 0 \leq z \leq 9, \tag{9}$$

and we improve it by

$$\phi_9(z) = \left(1 + e^{-\frac{18.48z}{12-z}}\right)^{-1}, \quad 0 \leq z \leq 12. \tag{10}$$

In Figure 5 are plotted ϵ_8 and ϵ_9 and the maximum absolute errors are: $\max(\epsilon_8) = 6.69 \times 10^{-3}$ and $\max(\epsilon_9) = 1.66 \times 10^{-3}$.

Finally, the approximation from Bryc [1]

$$\phi_{10}(z) = 1 - \frac{(4-\pi)z + \sqrt{2\pi}(\pi - 2)}{(4-\pi)\sqrt{2\pi}z^2 + 2\pi z + 2\sqrt{2\pi}(\pi - 2)} e^{-\frac{z^2}{2}} \tag{11}$$

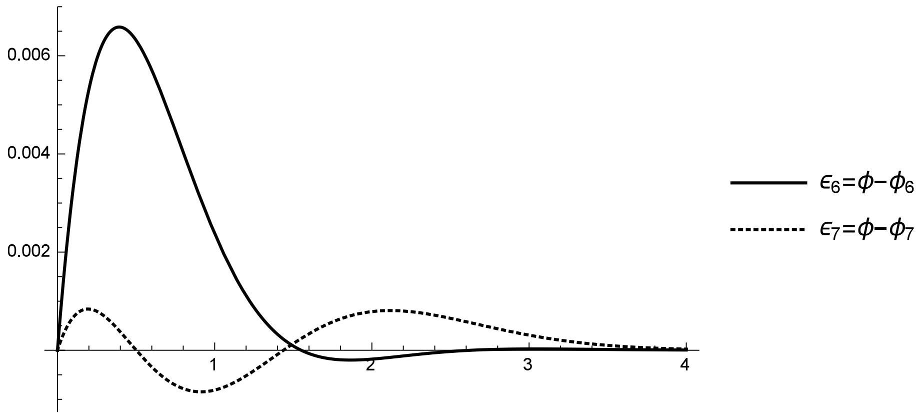

Figure 4: ϵ_6 and ϵ_7 plots

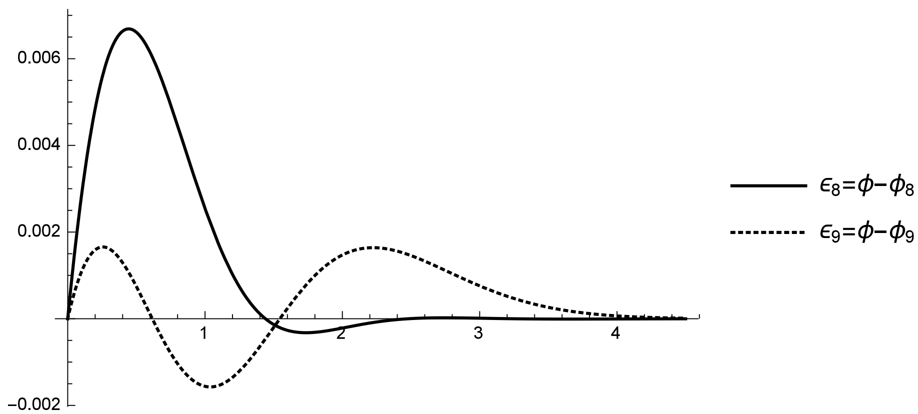

Figure 5: ϵ_8 and ϵ_9 plots

can be improved by

$$\phi_{11}(z) = 1 - \frac{0.878z + 2.91271}{2.27929z^2 + 6.387z + 5.82542} e^{-\frac{z^2}{2}}. \tag{12}$$

In Figure 6 we plot the error functions ϵ_{10} and ϵ_{11} with maximum absolute errors, $\max(\epsilon_{10}) = 7.18 \times 10^{-4}$ and $\max(\epsilon_{11}) = 7.14 \times 10^{-5}$.

In Table 1 we summarize the results, writing down the maximum absolute error for all approximations, the already known and the proposed ones (new maximum absolute errors in bold). An approximation of the value of z where

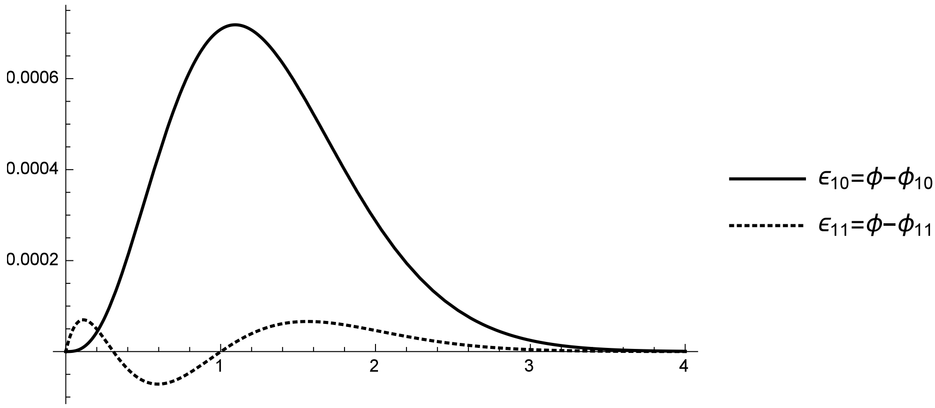

Figure 6: ϵ_{10} and ϵ_{11} plots

that maximum is attained and the percentage in the reduction of the maximum absolute error are also presented.

ϕ_i	$\max \varepsilon_i$	z	% of improvement
ϕ_1	4.30×10^{-3}	≈ 0.29892	–
ϕ_2	$\mathbf{7.85 \times 10^{-4}}$	≈ 1.15756	81.7%(to ϕ_1)
ϕ_3	6.23×10^{-4}	≈ 0.33368	–
ϕ_4	$\mathbf{3.83 \times 10^{-4}}$	≈ 1.09926	38.4%(to ϕ_3)
ϕ_5	$\mathbf{1.18 \times 10^{-4}}$	≈ 2.80390	81.0%(to ϕ_3)
ϕ_6	6.59×10^{-3}	≈ 0.39286	–
ϕ_7	$\mathbf{8.46 \times 10^{-4}}$	≈ 0.91278	87.2%(to ϕ_6)
ϕ_8	6.69×10^{-3}	≈ 0.44402	–
ϕ_9	$\mathbf{1.66 \times 10^{-3}}$	≈ 0.25576	75.2%(to ϕ_8)
ϕ_{10}	7.18×10^{-4}	≈ 1.09427	–
ϕ_{11}	$\mathbf{7.14 \times 10^{-5}}$	≈ 0.59549	90.2%(to ϕ_{10})

Table 1: Maximum Absolute Error and Percentual Improvement

3. Conclusion

In conclusion, from simple changes we can reduce the maximum absolute error from 38.4% in the worst scenario to 90.2% in the best scenario, without increasing the complexity of the approximations.

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