ON SUPERCLASSES OF δ -OPEN SETS IN TOPOLOGICAL SPACES

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Abstract: The paper considers properties of δ -preopen, δ -semiopen, a -open and e^* -open sets in topological spaces. Particularly, we extend various results concerning these types of sets and give some more new results. Furthermore, we introduce two more subclasses of δ -semiopen sets, and study them.

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1. Introduction and Preliminaries

In 1968 [7], the class of δ -open subsets of a topological space was first introduced by Velicko. This class of sets plays an important role in the study of various properties in topological spaces. Since then many authors used this class to define new classes of sets in topological spaces. In 1993, Raychaudhuri and Mukherjee [5] introduced and investigated a new class of sets called δ -preopen.

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Later, in 1997, Park et al. [4] defined a class of δ -semiopen sets. Recently, Ekici [1, 2] introduced and studied two more classes of sets named a -open and e^* -open. In this paper, we study further properties of these types of sets. Among these properties we strengthen several results of [1, 2, 4, 5]. We also define two new types of sets under the names of δ -semi-regular and δ -semi- θ -open.

Throughout the paper, spaces X and Y mean topological spaces. For a subset A of a space X , $\text{Cl}_X(A)$ and $\text{Int}_X(A)$ (or simply $\text{Cl}(A)$ and $\text{Int}(A)$ if no confusion arises) represent the closure of A and the interior of A on X , respectively. A subset A of a space X is said to be regular open [6] if $A = \text{Int}(\text{Cl}(A))$. A subset A of a space X is called δ -open if for each $x \in A$, there exists an open set U such that $x \in U \subset \text{Int}(\text{Cl}(U)) \subseteq A$. The δ -interior of a subset A of X , denoted by $\delta\text{-Int}(A)$ [7], is the union of all regular open sets of X contained in A . A subset A is called δ -open if $A = \delta\text{-Int}(A)$. The δ -closure [7] of a set A in a space (X, τ) is defined by $\delta\text{-Cl}(A) = \{x \in X : A \cap \text{Int}(\text{Cl}(B)) \neq \emptyset, B \in \tau \text{ and } x \in B\}$. It is well-known that the family of all δ -open sets in (X, τ) forms a topology which is weaker than τ .

Definition 1.1. A subset A of a space X is said to be

1. δ -preopen [5] if $A \subseteq \text{Int}(\delta\text{-Cl}(A))$.
2. δ -semieopen [4] if $A \subseteq \text{Cl}(\delta\text{-Int}(A))$.
3. a -open [1] if $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$.
4. e^* -open [2] if $A \subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(A)))$.

The family of all regular open (resp. δ -open, δ -preopen, δ -semiopen, a -open and e^* -open) subsets of a space X is denoted by $\text{RO}(X)$ (resp. $\delta\text{O}(X)$, $\delta\text{PO}(X)$, $\delta\text{SO}(X)$, $a\text{O}(X)$ and $e^*\text{O}(X)$). The complement of each regular open (resp. δ -open, δ -preopen, δ -semiopen, a -open and e^* -open) set is regular closed (resp. δ -closed, δ -preclosed, δ -semiclosed, a -closed and e^* -closed). The family of all regular closed (resp. δ -closed, δ -preclosed, δ -semiclosed, a -closed and e^* -closed) subsets of X is denoted by $\text{RC}(X)$ (resp. $\delta\text{C}(X)$, $\delta\text{PC}(X)$, $\delta\text{SC}(X)$), $a\text{C}(X)$ and $e^*\text{C}(X)$).

Remark 1.2. [1] The relation between the above mentioned sets is given

in following diagram:

Remark 1.3. For any space X , the family $\delta O(X)$ forms a topology, [7, Lemma 3].

Lemma 1.4. Let A be a subset of a space X . Then

1. $Cl(A) \cap U \subseteq Cl(A \cap U)$, for any open set U in X .
2. $Int(A \cup F) \subseteq Int(A) \cup F$, for any closed set F in X .
3. $\delta-Cl(A) \cap U \subseteq \delta-Cl(A \cap U)$, for any δ -open set U in X .
4. $\delta-Int(A \cup F) \subseteq \delta-Int(A) \cup F$, for any δ -closed set F in X .

Proof. (1) & (2) are easy to follow. (3) & (4) are in [4, Lemma 1]. □

Lemma 1.5. [7] For a subset A of a topological space (X, τ) , the following properties hold:

1. If A is open in (X, τ) , then $\delta-Cl(A) = Cl(A)$.
2. If A is closed in (X, τ) , then $\delta-Int(A) = Int(A)$.

Remark 1.6. In the sequel, we have used the above results repeatedly either in an explicit or implicit way.

2. δ -Semiopen Sets

Lemma 2.1. [4] *A subset A of a space X is δ -semiopen if and only if there exists a δ -open set U such that $U \subseteq A \subseteq \text{Cl}(U)$.*

We recall that a point $x \in X$ is said to be in the δ -semiclosure [4] (resp. δ -preclosure [5] and e^* -closure) of a set $A \subset X$ if and only if $A \cap U \neq \emptyset$, for every δ -semiopen (resp. δ -preopen and e^* -open) U containing x and is denoted by $\delta\text{-sCl}(A)$ (resp. $\delta\text{-pCl}(A)$ and $e^*\text{-Cl}(A)$).

Lemma 2.2. [4] *Let A be a subset of a topological space (X, τ) . Then $\delta\text{-sCl}(\delta\text{-Int}(A)) = \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$, or equivalently, $\delta\text{-sCl}(U) = \text{Int}(\text{Cl}(U))$ for each δ -open set U of X .*

Theorem 2.3. *Let A, B be subsets of a space (X, τ) . If $A \in \delta\text{SO}(X)$ and $A \subseteq B \subseteq \text{Cl}(A)$, then $B \in \delta\text{SO}(X)$.*

Proof. Since A is δ -semiopen, then there exists a δ -open set U such that $U \subseteq A \subseteq \text{Cl}(U)$ and then $U \subseteq B$. But $\text{Cl}(A) = \text{Cl}(U)$ and thus $B \subseteq \text{Cl}(U)$. Therefore, $U \subseteq B \subseteq \text{Cl}(U)$ and hence $B \in \delta\text{SO}(X)$. $\square$

Theorem 2.4. *Let A, B be subsets of (X, τ) . If $A \in \delta\text{SC}(X)$ and $\text{Int}(A) \subseteq B \subseteq A$, then $B \in \delta\text{SC}(X)$.*

Proof. Since A is δ -semiclosed, then $X \setminus A$ is δ -semiopen. By hypothesis $\text{Int}(A) \subseteq B \subseteq A$, so $X \setminus \text{Int}(A) \supseteq X \setminus B \supseteq X \setminus A$. But $X \setminus \text{Int}(A) = \text{Cl}(X \setminus A)$. Therefore $X \setminus A \subseteq X \setminus B \subseteq \text{Cl}(X \setminus A)$, and hence, by 2.3, $X \setminus B$ is δ -semiopen. Thus B is δ -semiclosed. $\square$

Theorem 2.5. *Let A be a subset of a topological space (X, τ) . If $A \in \delta\text{SO}(X)$, then $\delta\text{-sCl}(A)$ is both δ -semiopen and δ -semiclosed.*

Proof. Since $\delta\text{-sCl}(A)$ is δ -semiclosed, it suffices to show that $\delta\text{-sCl}(A) \in \delta\text{SO}(X)$. Again since $A \in \delta\text{SO}(X)$, then $U \subseteq A \subseteq \text{Cl}(U)$ for some δ -open set U of X . Therefore, we have $U \subseteq \delta\text{-sCl}(U) \subseteq \delta\text{-sCl}(A) \subseteq \delta\text{-sCl}(\text{Cl}(U)) = \text{Cl}(U)$ and hence, by 2.1, $\delta\text{-sCl}(A) \in \delta\text{SO}(X)$. $\square$

Lemma 2.6. [3] *Let A, B be subsets of a topological space (X, τ) . If*

$A \in \delta SO(X)$ and $B \in \delta O(X)$, then $A \cap B \in \delta SO(X)$.

Lemma 2.7. [3] Let A, B be subsets of a topological space (X, τ) . If $A \in \delta SO(X)$ and $B \in \delta O(X)$, then $A \cap B \in \delta SO(B)$.

Lemma 2.8. [3] Let A, Y be subsets of a topological space (X, τ) such that $A \subset Y \in \delta O(X)$. Then, $A \in \delta SO(X)$ if and only if $A \in \delta SO(Y)$.

We strengthen the above three results, as follows:

Theorem 2.9. In a topological space (X, τ) , if $A \in aO(X)$, $B \in \delta SO(X)$, then $A \cap B \in \delta SO(X)$.

Proof. By assumption, $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ and $B \subseteq \text{Cl}(\delta\text{-Int}(B))$. Therefore, by 1.4 and 1.3, we have that

$$\begin{aligned} A \cap B &\subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Cl}(\delta\text{-Int}(B)) \\ &= \text{Cl}[\text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \delta\text{-Int}(B)] \\ &\subseteq \text{Cl}[\text{Cl}(\delta\text{-Int}(A)) \cap \delta\text{-Int}(B)] \\ &= \text{Cl}[\text{Cl}[\delta\text{-Int}(A) \cap \delta\text{-Int}(B)]] \\ &= \text{Cl}(\delta\text{-Int}(A \cap B)). \end{aligned}$$

Thus, $A \cap B \in \delta SO(X)$. □

Theorem 2.10. Let A, B be subsets of a topological space (X, τ) . If $A \in \delta PO(X)$ and $B \in \delta SO(X)$, then $A \cap B \in \delta SO(A)$.

Proof. Consider

$$\begin{aligned} A \cap B &\subseteq \text{Int}(\delta\text{-Cl}(A)) \cap \text{Cl}(\delta\text{-Int}(B)) \\ &\subseteq \text{Cl}[\text{Int}(\delta\text{-Cl}(A)) \cap \delta\text{-Int}(B)] \\ &\subseteq \text{Cl}[\delta\text{-Cl}(A) \cap \delta\text{-Int}(B)] \\ &\subseteq \text{Cl}[\delta\text{-Cl}[A \cap \delta\text{-Int}(B)]] \\ &= \text{Cl}(A \cap \delta\text{-Int}(B)), \end{aligned}$$

so $A \cap B \subseteq \text{Cl}(A \cap \delta\text{-Int}(B)) \cap A = \text{Cl}_A(A \cap \delta\text{-Int}(B))$. But $A \cap \delta\text{-Int}(B)$ is a δ -open set in A , therefore $A \cap B \subseteq \text{Cl}_A(A \cap \delta\text{-Int}(B)) = \text{Cl}_A[\delta\text{-Int}_A[A \cap \delta\text{-Int}(B)]] \subseteq \text{Cl}_A(\delta\text{-Int}_A(A \cap B))$. Hence $A \cap B \in \delta SO(A)$. □

Theorem 2.11. *Let (X, τ) be a topological space, and let $A \subseteq Y \in \delta SO(X)$. Then $A \in \delta SO(X)$ if and only if $A \in \delta SO(Y)$.*

Proof. If $A \in \delta SO(X)$, then there is a δ -open set U in X such that $U \subseteq A \subseteq \text{Cl}_X(U)$. By assumption, $U \subseteq A \subseteq Y$. Therefore $U \subseteq A \subseteq \text{Cl}_X(U) \cap Y = \text{Cl}_Y(U)$. Since U is also δ -open in Y , so $A \in \delta SO(Y)$.

Conversely, let $A \in \delta SO(Y)$. There is a δ -open set U in Y such that $U \subseteq A \subseteq \text{Cl}_Y(U)$. Since U is δ -open in Y , there exists a δ -open set V in X such that $U = V \cap Y$. Then $V \cap Y \subseteq A \subseteq \text{Cl}_Y(V \cap Y) \subseteq \text{Cl}_X(V \cap Y)$. Since $V \cap Y$ is δ -semiopen, by 2.3, $A \in \delta SO(X)$. $\square$

Proposition 2.12. *Let A be a subset of a topological space (X, τ) . Then A is δ -semiopen if and only if $\text{Cl}(A) = \text{Cl}(\delta\text{-Int}(A))$.*

Proof. If A is δ -semiopen, then $A \subseteq \text{Cl}(\delta\text{-Int}(A))$ and so $\text{Cl}(A) \subseteq \text{Cl}(\delta\text{-Int}(A))$. For other side of inclusion, we always have $\delta\text{-Int}(A) \subseteq A$. Therefore $\text{Cl}(\delta\text{-Int}(A)) \subseteq \text{Cl}(A)$. Thus $\text{Cl}(A) = \text{Cl}(\delta\text{-Int}(A))$.

Conversely, assume that $\text{Cl}(A) = \text{Cl}(\delta\text{-Int}(A))$, but $A \subseteq \text{Cl}(A)$ always, so $A \subseteq \text{Cl}(\delta\text{-Int}(A))$. Hence $A \in \delta SO(X)$. $\square$

Remark 2.13. We remark from the above result that the δ -interior of any nonempty δ -semiopen is not empty. Otherwise, if $\emptyset \neq A \in \delta SO(X)$ such that $\delta\text{-Int}(A) = \emptyset$. By 2.12, we would have $\text{Cl}(A) = \emptyset$ which implies that $A = \emptyset$. Contradiction!

Theorem 2.14. *let A, Y be subsets of topological space (X, τ) such that $A \subseteq Y$. If Y is a -open, then $\delta\text{-sCl}(A) \cap Y = \delta\text{-sCl}_Y(A)$.*

Proof. Let $x \in \delta\text{-sCl}(A) \cap Y$. Then $x \in \delta\text{-sCl}(A)$ and $x \in Y$. Take any $U \in \delta SO(Y)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta SO(X)$, then by 2.11, $U \in \delta SO(X)$ containing x . By assumption $U \cap A \neq \emptyset$, therefore $x \in \delta\text{-sCl}_Y(A)$. Thus, $\delta\text{-sCl}(A) \cap Y \subseteq \delta\text{-sCl}_Y(A)$.

On the other hand, let $x \in \delta\text{-sCl}_Y(A)$, so that $x \in Y$. Let $U \in \delta SO(X)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta PO(X)$, by 2.10, $x \in U \cap Y \in \delta SO(Y)$. By assumption $\emptyset \neq A \cap (U \cap Y) \subseteq A \cap U$, so $x \in \delta\text{-sCl}(A)$ which implies that $x \in \delta\text{-sCl}(A) \cap Y$. Therefore $\delta\text{-sCl}_Y(A) \subseteq \delta\text{-sCl}(A) \cap Y$. Consequently $\delta\text{-sCl}_Y(A) = \delta\text{-sCl}(A) \cap Y$. $\square$

In the above theorem, the term a -open cannot be replaced with δ -semiopen,

as shown in the following example:

Example 2.15. Let $X = \{a, b, c, d\}$. For a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\delta O(X) = \tau$. If $Y = \{b, c, d\}$, which is δ -semiopen but not a -open, and $A = \{b, c\}$, then $\delta\text{-sCl}(A) \cap Y = Y$ but $\delta\text{-sCl}_Y(A) = A$.

3. δ -Preopen Sets

Lemma 3.1. [5] *In a topological space (X, τ) , if $A \in \delta PO(X)$, $B \in \delta O(X)$, then $A \cap B \in \delta PO(X)$.*

Lemma 3.2. [5] *In a topological space (X, τ) , if $A \in \delta PO(X)$, $B \in \delta O(X)$, then $A \cap B \in \delta PO(B)$.*

Lemma 3.3. [5] *Let A, Y be subsets of a topological space (X, τ) such that $A \subseteq Y \in \delta O(X)$. If $A \in \delta PO(Y)$, then $A \in \delta PO(X)$.*

Below the extension of the above results.

Theorem 3.4. *In a topological space (X, τ) , if $A \in \delta PO(X)$, $B \in aO(X)$, then $A \cap B \in \delta PO(X)$.*

Proof. By assumption we have that $A \subseteq \text{Int}(\delta\text{-Cl}(A))$ and $B \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(B)))$. Therefore

$$\begin{aligned}
 A \cap B &\subseteq \text{Int}(\delta\text{-Cl}(A)) \cap \text{Int}(\text{Cl}(\delta\text{-Int}(B))) \\
 &= \text{Int}[\text{Int}(\delta\text{-Cl}(A)) \cap \text{Int}(\text{Cl}(\delta\text{-Int}(B)))] \\
 &\subseteq \text{Int}[\text{Int}(\delta\text{-Cl}(A)) \cap \text{Cl}(\delta\text{-Int}(B))] \\
 &\subseteq \text{Int}[\text{Cl}[\text{Int}(\delta\text{-Cl}(A)) \cap \delta\text{-Int}(B)]] \\
 &= \text{Int}[\delta\text{-Cl}[\text{Int}(\delta\text{-Cl}(A)) \cap \delta\text{-Int}(B)]] \\
 &\subseteq \text{Int}[\delta\text{-Cl}[\delta\text{-Cl}(A) \cap \delta\text{-Int}(B)]] \\
 &= \text{Int}[\delta\text{-Cl}[\delta\text{-Cl}[A \cap \delta\text{-Int}(B)]]] \\
 &\subseteq \text{Int}[\delta\text{-Cl}[\delta\text{-Cl}[A \cap B]]] \\
 &= \text{Int}(\delta\text{-Cl}(A \cap B)).
 \end{aligned}$$

Hence, $A \cap B \in \delta PO(X)$. □

Theorem 3.5. *Let A, B be subsets of a topological space (X, τ) . If $A \in \delta SO(X)$ and $B \in \delta PO(X)$, then $A \cap B \in \delta PO(A)$.*

Proof. Given the sets A, B , then

$$\begin{aligned}
 A \cap B &\subseteq A \cap \text{Int}(\delta - \text{Cl}(B)) \\
 &\subseteq \text{Int}_A[A \cap \text{Int}(\delta - \text{Cl}(B))] \\
 &\subseteq \text{Int}_A[\text{Cl}(\delta - \text{Int}(A)) \cap \text{Int}(\delta - \text{Cl}(B))] \\
 &\subseteq \text{Int}_A[\text{Cl}[\delta - \text{Int}(A) \cap \text{Int}(\delta - \text{Cl}(B))]] \\
 &= \text{Int}_A[\delta - \text{Cl}[\delta - \text{Int}(A) \cap \text{Int}(\delta - \text{Cl}(B))]] \\
 &\subseteq \text{Int}_A[\delta - \text{Cl}[\delta - \text{Int}(A) \cap \delta - \text{Cl}(B)]] \\
 &\subseteq \text{Int}_A[\delta - \text{Cl}[\delta - \text{Cl}[\delta - \text{Int}(A) \cap B]]] \\
 &\subseteq \text{Int}_A[\delta - \text{Cl}[\delta - \text{Cl}[A \cap B]]] \\
 &\subseteq \text{Int}_A(\delta - \text{Cl}(A \cap B)).
 \end{aligned}$$

Since $\text{Int}_A(\delta - \text{Cl}(A \cap B))$ is an open set in A , so $\text{Int}_A(\delta - \text{Cl}(A \cap B)) \cap A = \text{Int}_A(\delta - \text{Cl}(A \cap B) \cap A)$, and hence

$$A \cap B \subseteq \text{Int}_A(\delta - \text{Cl}(A \cap B) \cap A) = \text{Int}_A(\delta - \text{Cl}_A(A \cap B)).$$

This shows that $A \cap B$ is δ -preopen in A . □

Theorem 3.6. *Let A, Y be subsets of a topological space (X, τ) and let $A \subseteq Y \in \delta PO(X)$. Then $A \in \delta PO(X)$ if and only if $A \in \delta PO(Y)$.*

Proof. Assume that $A \in \delta PO(X)$. Then $A = A \cap Y \subseteq \text{Int}(\delta - \text{Cl}(A)) \cap Y$. Since $\text{Int}(\delta - \text{Cl}(A)) \cap Y$ is open in Y , we have $A \subseteq \text{Int}(\delta - \text{Cl}(A)) \cap Y \subseteq \text{Int}_Y[\text{Int}(\delta - \text{Cl}(A)) \cap Y] \subseteq \text{Int}_Y[\delta - \text{Cl}(A) \cap Y] = \text{Int}_Y(\delta - \text{Cl}_Y(A))$. Hence A is δ -preopen in Y .

Conversely, suppose that $A \in \delta PO(Y)$. So $A \subseteq \text{Int}_Y(\delta - \text{Cl}_Y(A))$. By 1.5 (2), $\text{Int}_Y(\delta - \text{Cl}_Y(A)) = \delta - \text{Int}_Y(\delta - \text{Cl}_Y(A))$, which implies that $\text{Int}_Y(\delta - \text{Cl}_Y(A))$ is δ -open in Y . By 1.3, there a δ -open set U in X such that $\text{Int}_Y(\delta - \text{Cl}_Y(A)) = U \cap Y$. By 3.1, $\text{Int}_Y(\delta - \text{Cl}_Y(A))$ is δ -preopen in X . Therefore

$$\begin{aligned}
 A &\subseteq \text{Int}_Y(\delta - \text{Cl}_Y(A)) \\
 &\subseteq \text{Int}_X(\delta - \text{Cl}_X(\text{Int}_Y(\delta - \text{Cl}_Y(A)))) \\
 &= \text{Int}_X(\delta - \text{Cl}_X(\text{Int}_Y[\delta - \text{Cl}_X(A) \cap Y])) \\
 &\subseteq \text{Int}_X(\delta - \text{Cl}_X[\delta - \text{Cl}_X(A) \cap Y])
 \end{aligned}$$

$$\begin{aligned} &\subseteq \text{Int}_X(\delta\text{-Cl}_X[\delta\text{-Cl}_X(A)]) \\ &= \text{Int}_X(\delta\text{-Cl}_X(A)). \end{aligned}$$

This shows that $A \in \delta PO(X)$. $\square$

Theorem 3.7. *Let A, Y be subsets of a topological space (X, τ) with $A \subseteq Y$. If Y is a -open, then $\delta\text{-pCl}(A) \cap Y = \delta\text{-pCl}_Y(A)$.*

Proof. Let $x \in \delta\text{-pCl}(A) \cap Y$. Then $x \in \delta\text{-pCl}(A)$ and $x \in Y$. Let $U \in \delta PO(Y)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta PO(X)$, by 3.6, $U \in \delta PO(X)$ containing x . By assumption $U \cap A \neq \emptyset$, therefore $x \in \delta\text{-pCl}_Y(A)$. Thus, $\delta\text{-pCl}(A) \cap Y \subseteq \delta\text{-pCl}_Y(A)$.

On the other hand, let $x \in \delta\text{-pCl}_Y(A)$, clearly $x \in Y$. Let $U \in \delta PO(X)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta SO(X)$, by 3.5, $x \in U \cap Y \in \delta PO(Y)$. By assumption $\emptyset \neq A \cap (U \cap Y) \subseteq A \cap U$, so $x \in \delta\text{-pCl}(A)$ which implies that $x \in \delta\text{-pCl}(A) \cap Y$. Therefore $\delta\text{-pCl}_Y(A) \subseteq \delta\text{-pCl}(A) \cap Y$. Consequently $\delta\text{-pCl}_Y(A) = \delta\text{-pCl}(A) \cap Y$. $\square$

In the above theorem, the condition a -open cannot be replaced with δ -preopen, as shown in the following example:

Example 3.8. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ be a topology on X . Then $\delta O(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. If $Y = \{a, b, d\}$, which is δ -preopen but not a -open, and $A = \{b\}$, then $\delta\text{-pCl}(A) \cap Y = \{b, d\}$ but $\delta\text{-pCl}_Y(A) = A$.

4. a -Open Sets

Lemma 4.1. [1] *For a topological space (X, τ) , the family of all a -open sets of X forms a topology.*

Lemma 4.2. [1] *Let X be a topological space. Then $aO(X) = \delta SO(X) \cap \delta PO(X)$.*

Theorem 4.3. *For any subset A of a space (X, τ) , the following statements are equivalent:*

1. A is a -open.

2. There exists a δ -open set U such that $U \subseteq A \subseteq \text{Int}(\text{Cl}(U))$.
3. There exists a δ -open set U such that $U \subseteq A \subseteq \delta\text{-sCl}(U)$.

Proof. (1) $\Rightarrow$ (2). Let $A \in aO(X)$, then $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$. Now, we have $\delta\text{-Int}(A) \subseteq A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$. So if $U = \delta\text{-Int}(A)$, then we have $U \subseteq A \subseteq \text{Int}(\text{Cl}(U))$ and U is δ -open. This completes the proof.

(2) $\Leftrightarrow$ (3). Follows directly from 2.2.

(2) $\Rightarrow$ (1). Let $U \subseteq A \subseteq \text{Int}(\text{Cl}(U))$ for some δ -open set U . But $U \subseteq \delta\text{-Int}(A)$ and thus $\text{Int}(\text{Cl}(U)) \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ implies that $A \subseteq \text{Int}(\text{Cl}(U)) \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$. Therefore $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$. This proves that A is a -open. $\square$

Theorem 4.4. Let A, B be subsets of (X, τ) . If $A \in aO(X)$ and $A \subseteq B \subseteq \text{Int}(\text{Cl}(A))$, then $B \in aO(X)$.

Proof. Since $A \in aO(X)$, then $B \subseteq \text{Int}(\text{Cl}(A)) \subseteq \text{Int}(\text{Cl}(\text{Int}(\text{Cl}(\delta\text{-Int}(A)))) = \text{Int}(\text{Cl}(\delta\text{-Int}(A))) \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(B)))$. This implies that $B \in aO(X)$. $\square$

5. e^* -Open Sets

Lemma 5.1. [1] Let X be a topological space. If $A \in \delta SO(X)$ and $B \in \delta PO(X)$, then $\delta SO(X) \cap \delta PO(X) \in e^*O(X)$.

Lemma 5.2. [2] Let A be a subset of a topological space X . The following are equivalent:

1. $A \in e^*O(X)$.
2. There exists $U \in \delta PO(X)$ such that $U \subseteq A \subseteq \delta\text{-Cl}(U)$.
3. $\delta\text{-Cl}(A)$ is regular closed.

Theorem 5.3. Every e^* -open δ -semiclosed set is δ -semiopen.

Proof. Let $A \in e^*O(X)$ and $A \in \delta SC(X)$. Then $A \subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(A)))$ and $\text{Int}(\delta\text{-Cl}(A)) \subseteq A$. Thus $\text{Int}(\delta\text{-Cl}(A)) \subseteq A \subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(A)))$. Set $U = \text{Int}(\delta\text{-Cl}(A))$. By 1.5, $U = \delta\text{-Int}(\delta\text{-Cl}(A))$, so U is δ -open and $U \subseteq A \subseteq \text{Cl}(U)$. Therefore A is δ -semiopen. $\square$

Theorem 5.4. *In a topological space (X, τ) , if $A \in aO(X)$ and $B \in e^*O(X)$, then $A \cap B \in e^*O(X)$.*

Proof. Since $A \in aO(X)$ and $B \in e^*O(X)$, then $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ and $B \subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(B)))$ respectively. Now, we have

$$\begin{aligned}
 A \cap B &\subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Cl}(\text{Int}(\delta\text{-Cl}(B))) \\
 &\subseteq \text{Cl}[\text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &\subseteq \text{Cl}[\text{Cl}(\delta\text{-Int}(A)) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &\subseteq \text{Cl}[\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &= \text{Cl}[\text{Int}(\delta\text{-Int}(A)) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &= \text{Cl}[\text{Int}[\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))]] \\
 &\subseteq \text{Cl}[\text{Int}[\delta\text{-Int}(A) \cap \delta\text{-Cl}(B)]] \\
 &= \text{Cl}[\text{Int}[\delta\text{-Cl}[\delta\text{-Int}(A) \cap B]]] \\
 &\subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(A \cap B))).
 \end{aligned}$$

Hence $A \cap B \in e^*O(X)$. □

Theorem 5.5. *In a topological space (X, τ) , if $A \in aO(X)$ and $B \in e^*O(X)$, then $A \cap B \in e^*O(A)$.*

Proof. Since $A \in aO(X)$ and $B \in e^*O(X)$, then $A \subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ and $B \subseteq \text{Cl}(\text{Int}(\delta\text{-Cl}(B)))$. Now

$$\begin{aligned}
 A \cap B &\subseteq \text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Cl}(\text{Int}(\delta\text{-Cl}(B))) \\
 &\subseteq \text{Cl}[\text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &\subseteq \text{Cl}[\text{Cl}(\delta\text{-Int}(A)) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &\subseteq \text{Cl}[\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))] \\
 &\subseteq \text{Cl}(\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))) \cap A \\
 &= \text{Cl}_A(\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))).
 \end{aligned}$$

But $\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))$ is open in A , then $\text{Int}_A[\delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))] = \delta\text{-Int}(A) \cap \text{Int}(\delta\text{-Cl}(B))$. Thus

$$\begin{aligned}
 A \cap B &\subseteq \text{Cl}_A[\text{Int}_A[(\delta\text{-Int}(A) \cap \delta\text{-Cl}(B)) \cap A]] \\
 &\subseteq \text{Cl}_A[\text{Int}_A[\delta\text{-Cl}((\delta\text{-Int}(A) \cap B) \cap A)]] \\
 &\subseteq \text{Cl}_A[\text{Int}_A[\delta\text{-Cl}(A \cap B) \cap A]]
 \end{aligned}$$

$$= \text{Cl}_A(\text{Int}_A(\delta - \text{Cl}_A(A \cap B))).$$

Thus $A \cap B \in e^*O(A)$. □

Theorem 5.6. *Let A, Y be subsets of a topological space (X, τ) and let $Y \in \delta PO(X)$. Then $A \in e^*O(X)$ if and only if $A \in e^*O(Y)$.*

Proof. If $A \in e^*O(X)$, by 5.2, there is a δ -preopen set U in X such that $U \subseteq A \subseteq \delta - \text{Cl}_X(U)$. By assumption, $U \subseteq A \subseteq Y$. Therefore $U \subseteq A \subseteq \delta - \text{Cl}_X(U) \cap Y = \delta - \text{Cl}_Y(U)$. Since Y is δ -preopen in X , by part one of 3.6, U is δ -preopen in Y , so $A \in e^*O(Y)$.

Conversely, let $A \in e^*O(Y)$. Then $A \subseteq \text{Cl}_Y(\text{Int}_Y(\delta - \text{Cl}_Y(A))) = \text{Cl}_Y(\text{Int}_Y(\delta - \text{Cl}(A) \cap Y)) \subseteq \text{Cl}_Y(\text{Int}_Y(\delta - \text{Cl}(A))) = \text{Cl}_Y(\delta - \text{Int}_Y(\delta - \text{Cl}(A)))$. Obviously $\delta - \text{Int}_Y(\delta - \text{Cl}(A))$ is δ -open in Y , there is a δ -open subset G of X such that $\delta - \text{Int}_Y(\delta - \text{Cl}(A)) = G \cap Y$. Since $Y \in \delta PO(X)$, by 5.4, $G \cap Y \in e^*O(X)$, then $A \subseteq \text{Cl}_Y(G \cap Y) \subseteq \text{Cl}_Y[\text{Cl}(\text{Int}(\delta - \text{Cl}(G \cap Y)))]$. Therefore

$$A \subseteq \text{Cl}_Y[\text{Cl}(\text{Int}(\delta - \text{Cl}(\delta - \text{Int}_Y(\delta - \text{Cl}(A)))))].$$

Hence $A \subseteq \text{Cl}(\text{Int}(\delta - \text{Cl}(A)))$. This proves that $A \in e^*O(X)$. □

Theorem 5.7. *Let A, Y be subsets of topological space (X, τ) with $A \subset Y$. If Y is a -open, then $e^* - \text{Cl}(A) \cap Y = e^* - \text{Cl}_Y(A)$.*

Proof. Let $x \in e^* - \text{Cl}(A) \cap Y$. Then $x \in e^* - \text{Cl}(A)$ and $x \in Y$. Take any $U \in e^*O(Y)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta PO(X)$, by 5.6, $U \in e^*O(X)$ containing x . By assumption $U \cap A \neq \emptyset$, therefore $x \in e^* - \text{Cl}_Y(A)$. Thus $e^* - \text{Cl}(A) \cap Y \subseteq e^* - \text{Cl}_Y(A)$.

On the other hand, let $x \in e^* - \text{Cl}_Y(A)$, so that $x \in Y$. Let $U \in e^*O(X)$ containing x . Since $Y \in aO(X)$, by 5.5, $x \in U \cap Y \in e^*O(Y)$. By assumption $\emptyset \neq A \cap (U \cap Y) \subseteq A \cap U$, so $x \in e^* - \text{Cl}(A)$ which implies that $x \in e^* - \text{Cl}(A) \cap Y$. Therefore $e^* - \text{Cl}_Y(A) \subset e^* - \text{Cl}(A) \cap Y$. Consequently $e^* - \text{Cl}(A) \cap Y = e^* - \text{Cl}_Y(A)$. □

In 5.7, the word a -open cannot be replaced with e^* -open or even δ -preopen, as shown in the following example.

Example 5.8. Consider the topological space (X, τ) given in 2.15. There, the families $\delta SO(X)$ and $e^*O(X)$ are identical. The same Y and A prove our claim.

6. δ -semi-regular sets

Definition 6.1. A subset A of a space X is said to be δ -semi-regular if $A = \delta\text{-sInt}(\delta\text{-sCl}(A))$. The family of all δ -semi-regular sets of X is denoted by $\delta\text{SR}(X)$.

Lemma 6.2. [4] Let A a subset of a topological space (X, τ) . Then

1. $\delta\text{-sInt}(\delta\text{-sCl}(A))$ is both δ -semiopen and δ -semiclosed.
2. $\delta\text{-sCl}(\delta\text{-sInt}(\delta\text{-sCl}(A))) = \delta\text{-sInt}(\delta\text{-sCl}(A))$.

Lemma 6.3. [4] Let A a subset of a topological space (X, τ) . Then

1. $\delta\text{-sCl}(A) = A \cup \text{Int}(\delta\text{-Cl}(A))$.
2. $\delta\text{-sInt}(A) = A \cap \text{Cl}(\delta\text{-Int}(A))$.

Theorem 6.4. For any subset A of a topological space (X, τ) , the following statements are equivalent:

1. $A \in \delta\text{SR}(X)$.
2. $X \setminus A \in \delta\text{SR}(X)$.
3. $A = \delta\text{-sCl}(\delta\text{-sInt}(A))$.
4. $A \in \delta\text{SO}(X) \cap \delta\text{SC}(X)$.
5. There exists $U \in \text{RO}(X)$ such that $U \subseteq A \subseteq \delta\text{-Cl}(U)$.

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) follow from 2.5 and 6.2.

(4) $\Rightarrow$ (5) Suppose $A \in \delta\text{SO}(X) \cap \delta\text{SC}(X)$. Then $\text{Int}(\delta\text{-Cl}(A)) \subseteq A \subseteq \text{Cl}(\delta\text{-Int}(A)) = \delta\text{-Cl}(\delta\text{-Int}(A)) \subseteq \delta\text{-Cl}(\text{Int}(A)) \subseteq \delta\text{-Cl}(\text{Int}(\delta\text{-Cl}(A)))$. Set $U = \text{Int}(\delta\text{-Cl}(A)) = \text{Int}(\text{Cl}(A))$ which is regular open. Now, we have $U \subseteq A \subseteq \delta\text{-Cl}(U)$ and the lemma is proved.

(5) $\Rightarrow$ (1) Assume there is $U \in \text{RO}(X)$ such that $U \subseteq A \subseteq \delta\text{-Cl}(U)$. Then $\delta\text{-Cl}(U) = \delta\text{-Cl}(A)$, $U = \text{Int}(\delta\text{-Cl}(U)) = \text{Int}(\delta\text{-Cl}(A))$ and so $\text{Int}(\delta\text{-Cl}(A)) \subseteq A$. Since $U \in \text{RO}(X)$ and $U \subseteq A$, then $U \subseteq \delta\text{-Int}(A)$ and $\text{Int}(A) \subseteq \text{Int}(\delta\text{-Cl}(A)) = U$. Therefore $U = \delta\text{-Int}(A)$ and $\text{Cl}(U) = \text{Cl}(\delta\text{-Int}(A))$. This implies that $A \subseteq \text{Cl}(\delta\text{-Int}(A))$. Now, applying 6.3, $\delta\text{-sInt}(\delta\text{-sCl}(A)) = \delta\text{-sCl}(A) \cap \text{Cl}(\delta\text{-Int}(\delta\text{-sCl}(A))) = (A \cup \text{Int}(\delta\text{-Cl}(A))) \cap (\text{Cl}(\delta\text{-Int}(A \cup \text{Int}(\delta\text{-Cl}(A))))) = A \cap \text{Cl}(\delta\text{-Int}(A)) = A$. Thus $A \in \delta\text{SR}(X)$. $\square$

Theorem 6.5. *For any subset A of a topological space (X, τ) , $RO(X) \cup RC(X) \subseteq \delta SR(X) \subseteq \delta SO(X)$.*

Proof. Obvious. $\square$

Remark 6.6. It is worth remarking that, for any space X , the families $\delta O(X)$ and $\delta SR(X)$ are independent generally. Consider the topological space (X, τ) given in 2.15, the set $\{a, b\}$ is δ -open but not δ -semiregular. On the other hand, the set $\{a, c, d\}$ is δ -semiregular but not δ -open.

Theorem 6.7. *The closure and the δ -semiclosure of a δ -semiopen subset A of a topological space (X, τ) are δ -semi-regular.*

Proof. Follows from 2.5. $\square$

Theorem 6.8. *Let A, B be subsets of a topological space (X, τ) . If $A \in RO(X)$ and $B \in \delta SR(X)$, then $A \cap B \in \delta SR(X)$.*

Proof. Since $A \in RO(X)$ and $B \in \delta SR(X)$, then $A \in aO(X)$, $B \in \delta SO(X)$ and $X \setminus B \in \delta SO(X)$. By 2.9, $A \cap B \in \delta SO(X)$. It remains to show that $X \setminus (A \cap B)$ is δ -semiclosed. By assumption $X \setminus A \in RC(X)$ which implies that $X \setminus A \in \delta SO(X)$. But $X \setminus (A \cap B) = X \setminus A \cup X \setminus B$ is δ -semiopen (it is a union of two δ -semiopens). Consequently $A \cap B \in \delta SC(X)$. Thus, $A \cap B \in \delta SR(X)$. $\square$

Theorem 6.9. *Let A, B be subsets of a topological space (X, τ) . If $A \in \delta SR(X)$ and $B \in \delta PO(X)$, then $A \cap B \in \delta SR(B)$.*

Proof. Since $A \in \delta SR(X)$, then $A \in \delta SO(X)$ and $X \setminus A \in \delta SO(X)$. Therefore, by 2.10, $A \cap B \in \delta SO(B)$ and $(X \setminus A) \cap B \in \delta SO(B)$. But $(X \setminus A) \cap B = B \setminus (A \cap B)$ and hence $B \setminus (A \cap B) \in \delta SO(B)$. This implies that $A \cap B \in \delta SC(B)$. Thus, $A \cap B \in \delta SR(B)$ $\square$

Lemma 6.10. *Let A, Y be subsets of a topological space (X, τ) with $A \subseteq Y$ and let $Y \in \delta SR(X)$. Then $A \in \delta SR(Y)$ if and only if $A \in \delta SR(X)$.*

Proof. Since $A \in \delta SR(Y)$, then $A \in \delta SO(Y)$ and $Y \setminus A \in \delta SO(Y)$. Therefore, by 2.11, $A \in \delta SO(X)$ and $Y \setminus A \in \delta SO(X)$. Since $Y \in \delta SR(X)$, then $X \setminus Y \in \delta SO(X)$. Therefore, by (Theorem 3, [4]), $(Y \setminus A) \cup (X \setminus Y) = X \setminus A \in$

$\delta SO(X)$ and consequently $A \in \delta SC(X)$. Thus $A \in \delta SR(X)$.

The converse can be followed from part one of the proof of 2.11 and 2.4. $\square$

7. δ -semi- θ -open sets

Definition 7.1. A subset A of a space X is said to be δ -semi- θ -open if for each $x \in A$, there is a δ -semiopen set U in X such that $x \in U \subseteq \delta\text{-sCl}(U) \subseteq A$. The complement of a δ -semi- θ -open set is δ -semi- θ -closed. The family of all δ -semi- θ -open (resp. δ -semi- θ -closed) sets in X is denoted by $\delta S\theta O(X)$ (resp. $\delta S\theta C(X)$).

Theorem 7.2. A subset A of a space X is δ -semi- θ -open if and only if for each $x \in A$, there is a δ -semi-regular set U in X such that $x \in U \subseteq A$, or equivalently, A is δ -semi- θ -open if it is a union of δ -semi-regular sets.

Proof. Since $\delta\text{-sCl}(U) \in \delta SR(X)$ for each $U \in \delta SO(X)$, by 2.5, the claim follows. $\square$

It can be easily shown that

$$\delta\text{-semi-regular} \implies \delta\text{-semi-}\theta\text{-open} \implies \delta\text{-semi-open}.$$

But the converse need not be true in general as shown in the following examples:

Example 7.3. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, we can easily find that $\delta S\theta O(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Therefore, $\{a, b\} \in \delta S\theta O(X)$, but $\{a, b\} \notin \delta SR(X)$.

Example 7.4. Clearly the subset $A = [0, 1] \setminus \{\frac{1}{n}\}$, where $n = 1, 2, \dots$, of the usual topological space $(\mathbf{R}, U)$ is δ -semiopen (because $\delta\text{-Int}(A) = [0, 1] \setminus \{\frac{1}{n}\} \cup \{0\}$). But it is not δ -semi- θ -open, for the point 0, there exists no δ -semi-regular F such that $0 \in F \subseteq A$.

Theorem 7.5. Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of δ -semi- θ -open sets in a space X . Then $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is also δ -semi- θ -open.

Proof. Let $x \in \bigcup\{A_\alpha : \alpha \in \Delta\}$. Since A_α is δ -semi- θ -open set for each $\alpha \in \Delta$, there exists $U \in \delta SR(X)$ such that $x \in U \subset A_\alpha \subset \bigcup\{A_\alpha : \alpha \in \Delta\}$.

Therefore $x \in U \subset \bigcup\{A_\alpha : \alpha \in \Delta\}$ and thus by 7.2, $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is δ -semi- θ -open. $\square$

The intersection of two δ -semi- θ -open sets need not be δ -semi- θ -open in general as shown in the following example:

Example 7.6. Consider the space given in 7.3, so $\{a, c\}, \{b, c\} \in \delta\text{SO}(X)$, but $\{a, c\} \cap \{b, c\} = \{c\} \notin \delta\text{SO}(X)$.

Definition 7.7. The union of all δ -semi- θ -open sets of X contained in A is called the δ -semi- θ -interior of A and is denoted by $\delta\text{-sInt}_\theta(A)$.

Definition 7.8. A point x in a space X is said to be δ -semi- θ -adherent of a subset A of X if $A \cap \delta\text{-sCl}(U) \neq \phi$, for every $U \in \delta\text{SO}(X)$. The set of all δ -semi- θ -adherent points of A is called the δ -semi- θ -closure of A and is denoted by $\delta\text{-sCl}_\theta(A)$.

Theorem 7.9. For any subset A of a topological space (X, τ) , $\delta\text{-sCl}_\theta(A) = \bigcap\{V : A \subset V \text{ and } V \in \delta\text{SR}(X)\}$.

Proof. Let $x \notin \delta\text{-sCl}_\theta(A)$. Then there exists a δ -semiopen set U containing x such that $A \cap \delta\text{-sCl}(U) = \phi$. Therefore $A \subset X \setminus \delta\text{-sCl}(U) = V$ (say). Thus $V \in \delta\text{SR}(X)$ such that $x \notin V$. Hence $x \notin \text{R.H.S.}$

On the other hand, if $x \notin \text{R.H.S.}$, then there exists a δ -semiregular set V such that $x \notin V$. Therefore $(X \setminus V = U)$ (say) is a δ -semiopen set containing x such that $V \cap \delta\text{-sCl}(U) = \phi$. This shows that $A \cap \delta\text{-sCl}(U) = \phi$ and so $x \notin \text{L.H.S.}$ $\square$

Theorem 7.10. Let A and B be subsets of a topological space (X, τ) . Then the following are true:

1. A is δ -semi- θ -closed if and only if $A = \delta\text{-sCl}_\theta(A)$.
2. $\delta\text{-sCl}_\theta(A)$ is δ -semi- θ -closed in X .
3. $\delta\text{-sCl}_\theta(A) \subseteq \delta\text{-sCl}_\theta(B)$ whenever $A \subseteq B$.
4. $\delta\text{-sCl}_\theta(\delta\text{-sCl}_\theta(A)) = \delta\text{-sCl}_\theta(A)$.
5. $\delta\text{-sCl}_\theta(A \cap B) \subseteq \delta\text{-sCl}_\theta(A) \cap \delta\text{-sCl}_\theta(B)$.

$$6. \delta\text{-sCl}_\theta(A) \cup \delta\text{-sCl}_\theta(B) \subseteq \delta\text{-sCl}_\theta(A \cup B).$$

Proof. Straightforward. $\square$

Theorem 7.11. *Let A, B be subsets of a topological space (X, τ) . If $A \in RO(X)$ and $B \in \delta SO(X)$, then $A \cap B \in \delta SO(X)$.*

Proof. Applying 6.8 and 7.2, the proof will be followed. $\square$

Theorem 7.12. *Let A, B be subsets of a space (X, τ) . If $A \in \delta SO(X)$ and $B \in \delta PO(X)$, then $A \cap B \in \delta SO(X)$.*

Proof. Follows from 6.9 and 7.2. $\square$

Theorem 7.13. *Let A, Y be subsets of a topological space (X, τ) such that $A \subseteq Y \in aO(X)$. Then $A \in \delta SO(X)$ if and only if $A \in \delta SO(Y)$.*

Proof. Assume that $A \in \delta SO(X)$. Then for each $x \in A$, there is a δ -semiopen set U in X such that $x \in U \subseteq \delta\text{-sCl}(U) \subseteq A$. Since $Y \in aO(X)$ and $A \subseteq Y$, so Y is δ -semiopen, by 2.11, $U \cap Y = U \in \delta SO(Y)$. Therefore, by 2.14, $\delta\text{-sCl}(A) \cap Y = \delta\text{-sCl}_Y(A)$ and hence $x \in U \subseteq \delta\text{-sCl}_Y(U) \subseteq A$. Thus $A \in \delta SO(Y)$.

Suppose that $A \in \delta SO(Y)$. Then for each $x \in A$, there is a δ -semiopen set U in Y such that $x \in U \subseteq \delta\text{-sCl}_Y(U) \subseteq A$. This implies that $x \in U \subseteq \delta\text{-sCl}(U) \subseteq A$ and $U \in \delta SO(X)$, by 2.11. This shows that $A \in \delta SO(X)$. $\square$

Theorem 7.14. *Let A, Y be subsets of a topological space (X, τ) such that $A \subseteq Y$. If Y is a -open, then $\delta\text{-sCl}_\theta(A) \cap Y = \delta\text{-sCl}_{\theta Y}(A)$.*

Proof. Let $x \in \delta\text{-sCl}_\theta(A) \cap Y$. Then $x \in \delta\text{-sCl}_\theta(A)$ and $x \in Y$. Let $U \in \delta SO(Y)$ containing x . Since $Y \in aO(X)$, which implies that $Y \in \delta SO(X)$, by 2.11, $U \in \delta SO(X)$ containing x . By assumption $\delta\text{-sCl}(U) \cap A \neq \emptyset$. Therefore, by 2.14, $\delta\text{-sCl}_Y(U) \cap A = (\delta\text{-sCl}(U) \cap Y) \cap A \neq \emptyset$ and so $x \in \delta\text{-sCl}_{\theta Y}(A)$. Thus, $\delta\text{-sCl}_\theta(A) \cap Y \subseteq \delta\text{-sCl}_{\theta Y}(A)$.

On the other hand, let $x \in \delta\text{-sCl}_{\theta Y}(A)$, so obviously $x \in Y$. Let $U \in \delta SO(X)$ containing x . Since $Y \in aO(X)$ and $aO(X) \subseteq \delta SO(X)$, by 2.10, $x \in V = U \cap Y \in \delta SO(Y)$. By assumption $\emptyset \neq A \cap \delta\text{-sCl}_Y(V) \subseteq A \cap \delta\text{-sCl}(U)$, so $x \in \delta\text{-sCl}_\theta(A)$ which implies that $x \in \delta\text{-sCl}_\theta(A) \cap Y$. Therefore $\delta\text{-sCl}_{\theta Y}(A) \subseteq \delta\text{-sCl}_\theta(A) \cap Y$.

$\delta\text{-sCl}_\theta(A) \cap Y$. Consequently $\delta\text{-sCl}_\theta(A) \cap Y = \delta\text{-sCl}_{\theta Y}(A)$. $\square$

Theorem 7.15. *For any subset A of a space (X, τ) , the following statements are equivalent:*

1. A is regular open.
2. A is δ -open and δ -semiregular.
3. A is δ -open and δ -semi- θ -closed.
4. A is δ -open and δ -semiclosed.
5. A is a -open and δ -semiclosed.
6. A is δ -preopen and δ -semiclosed.
7. A is a -open and e^* -closed.

Proof. We only prove (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (1), the proofs of the others are straightforward.

(6) $\Rightarrow$ (7) Let A be δ -preopen and δ -semiclosed. Then $A \subset \text{Int}(\delta\text{-Cl}(A))$ and $\text{Int}(\delta\text{-Cl}(A)) \subset A$. Therefore $A = \text{Int}(\delta\text{-Cl}(A)) = \text{Int}(\text{Cl}(A))$ by 1.5 and hence A is regular open. This shows that A is a -open. Since $\delta\text{-SC}(X) \subset e^*\text{C}(X)$, it is clear that A is e^* -closed.

(7) $\Rightarrow$ (1) Let A be a -open and e^* -closed. Then $A = \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ and so $\text{Int}(\text{Cl}(A)) = \text{Int}(\text{Cl}(\text{Int}(\text{Cl}(\delta\text{-Int}(A))))) = \text{Int}(\text{Cl}(\delta\text{-Int}(A))) = A$. Therefore, A is regular open. $\square$

We end this section by the following diagram that shows the relation between all the sets dealt in present article.

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