**CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL
EQUATION WITH NONLOCAL CONDITIONS
IN BANACH SPACE**

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Abstract: The aim of the present paper is to prove the existence of solutions of the initial value problem for a nonlinear integro-differential equation of fractional order $\alpha \in (0, 1)$ with nonlocal conditions in Banach spaces. The fractional differential operator is taken in the Caputo sense. The presented analysis relies on the Krasnoselskii's fixed point theorem. At the end, an illustrative example will be introduced to justify our result.

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1. Introduction

Fractional differential equations are joined with intensive applications such as continuum phenomena mechanics, electrochemistry, biophysics, biotechnology engineering etc. For a lot of details see studies of Kilbas et al. [13], Miller and Ross [14], Samko et al. [16], and plenty of different references. The existence and uniqueness of solutions to fractional differential equations have attracted

the attention of many scientists and researchers, as an example, (see [2, 3, 4, 10, 11, 12]). In recent analysis work, several researchers of mathematics community studied fractional integro-differential equations because of their applications in different fields of science and engineering, for a lot of data see associate, [1, 5, 6, 7, 8, 9, 15]. Nonlocal conditions come up once values of the function on the boundary are connected to values within the domain. It is found to be a lot of plausible than the standard initial conditions for the formulation of some physical phenomena in certain problems of thermodynamics, elasticity and wave propagation. In passing, we have a tendency to noticed that nonlocal condition $u(0) = \sum_{k=1}^m c_k u(\tau_k)$ can be applied in physical problems yields better effect than the initial conditions $u(0) = u_0$.

The purpose of this paper is to prove the existence of solutions of the following nonlinear fractional integro-differential equations in Banach space

$${}^c D_{0+}^\alpha u(t) = a(t)u(t) + h(t) + \int_0^t K(t, \sigma) f(u(\sigma)) d\sigma, \quad t \in [0, b], \quad (1)$$

with nonlocal conditions

$$u(0) = \sum_{k=1}^m c_k u(\tau_k), \quad \tau_k \in (0, b). \quad (2)$$

where $0 < \alpha < 1$, ${}^c D_{0+}^\alpha$ denotes the Caputo fractional derivative of order α , and $f : \mathcal{X} \rightarrow \mathcal{X}$, $K : \{(t, \sigma) : 0 \leq \sigma \leq t \leq b\} \rightarrow \mathbb{R}^+$, $a, h : [0, b] \rightarrow \mathcal{X}$ are appropriate functions satisfying some conditions which will be stated later, τ_k , $k = 1, \dots, m$ are prefixed points satisfying $0 < \tau_1 \leq \dots \leq \tau_m < b$ and c_k is real numbers.

The organization of this paper is as follows. In Section 2, we state some known notations and definitions and we also list the hypotheses which are used throughout this paper. Section 3 provides the proof of the existence of solution to the problem (1)-(2) in Banach space. Finally, an illustrative example is presented in Section 4.

2. Preliminaries

In this section, we present some essential notations, definitions and lemmas concerning fractional calculus and fixed point theorem. Let $J = [0, b]$ and $(\mathcal{X}, \|\cdot\|)$ is a Banach space and $C(J, \mathcal{X})$ denotes the Banach space of all continuous functions from $J \rightarrow \mathcal{X}$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$. Moreover, $C^n(J, \mathcal{X})$ we denote the Banach space of all continuously differentiable functions up to order $(n - 1)$ on J .

Definition 1. ([13]) Let $\alpha > 0$ and $g : J \rightarrow \mathcal{X}$. The left sided Riemann–Liouville fractional integral of order α of a function g is defined as

$$I_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t \in J,$$

where $\Gamma(\cdot)$ is the Euler gamma function and $I_{0+}^{\alpha}g$ exists for all $\alpha > 0$. Moreover, $I_{0+}^0g(t) = g(t)$.

Definition 2. ([13]) Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $g \in C^n(J, \mathcal{X})$. The left sided Caputo fractional derivative of order α of a function g is defined as

$${}^cD_{0+}^{\alpha}g(t) = I_{0+}^{n-\alpha} \frac{d^n}{dt^n}g(t), \quad t \in J,$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α .

Lemma 3. ([13, 16]) Let $\alpha, \beta > 0$ and g, p be appropriate functions, then for $t \in J$, we have:

1. $I_{0+}^{\alpha}I_{0+}^{\beta}g(t) = I_{0+}^{\alpha+\beta}g(t) = I_{0+}^{\beta}I_{0+}^{\alpha}g(t)$.
2. $I_{0+}^{\alpha}(g(t) + p(t)) = I_{0+}^{\alpha}g(t) + I_{0+}^{\alpha}p(t)$.
3. ${}^cD_{0+}^{\alpha}(g(t) + p(t)) = {}^cD_{0+}^{\alpha}g(t) + {}^cD_{0+}^{\alpha}p(t)$.
4. $I_{0+}^{\alpha}{}^cD_{0+}^{\alpha}g(t) = g(t) - g(0)$, $0 < \alpha < 1$.
5. ${}^cD_{0+}^{\alpha}I_{0+}^{\alpha}g(t) = g(t)$.
6. ${}^cD_{0+}^{\alpha}c = 0$, where c is a constant.

Lemma 4. ([1]) Let $\alpha > 0$ and $g \in C(J, \mathcal{X})$. Then, $I_{0+}^{\alpha}g \in C(J, \mathcal{X})$ and $I_{0+}^{\alpha}g(0) = \lim_{t \rightarrow 0^+} I_{0+}^{\alpha}g(t) = 0$.

Lemma 5. ([17], Contraction mapping theorem) Let (U, d) be a metric space. Then a mapping $T : U \rightarrow U$ is called a contraction mapping on U if there exists a constant $k \in [0, 1)$ such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in U$.

Lemma 6. ([17], Krasnoselskii fixed point theorem) Let $\mathcal{E}$ be bounded, closed and convex subset of a Banach space $\mathcal{X}$. Let $T_1, T_2 : \mathcal{E} \rightarrow \mathcal{E}$ be operators such that (i) $T_1u + T_2v \in \mathcal{E}$, for every $u, v \in \mathcal{E}$; (ii) T_1 is contraction; (iii) T_2 is compact and continuous. Then, there exists $z \in \mathcal{E}$ such that $z = T_1z + T_2z$.

3. Main results

In this section, we shall demonstrate the existence result of the problem (1)–(2). For reader's comfort, a list of hypotheses is supplied as follows:

(A₁) $a(t)$ and $h(t)$ are bounded and continuous on J .

(A₂) $f : \mathcal{X} \rightarrow \mathcal{X}$ is continuous function and there exists $L > 0$ such that

$$\|f(u(t)) - f(v(t))\| \leq L \|u - v\|, \quad t \in J, \quad u, v \in \mathcal{X}.$$

(A₃) $K : D \rightarrow \mathbb{R}^+$ is continuous on D with $K_0 = \max\{|K(t, \sigma)| : (t, \sigma) \in D\}$, where $D = \{(t, \sigma) : 0 \leq \sigma \leq t \leq b\}$.

First, we will state the following lemma.

Lemma 7. *Let $0 < \alpha < 1$, h, f and K are continuous functions. If $u \in C(J, \mathcal{X})$, then u satisfies the problem (1) – (2) if and only if u satisfies the integral equation*

$$u(t) = B \sum_{k=1}^m \frac{c_k}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} H_u(s) ds,$$

where $H_u(s) := [a(s)u(s) + h(s) + \int_0^s K(s, \sigma)f(u(\sigma))d\sigma]$ and

$$B = \frac{1}{1 - \sum_{k=1}^m c_k}.$$

Proof. Let $u \in C(J, \mathcal{X})$ be a solution of (1) – (2). By using Lemma 3, we get

$$I_{0+}^{\alpha} {}^c D_{0+}^{\alpha} u(t) = u(t) - u(0). \quad (3)$$

Applying the operator I_{0+}^{α} on both sides of Eq. (1) and comparing result with Eq. (3) we obtain

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} H_u(s) ds. \quad (4)$$

Now, we substitute $t = \tau_k$ in Eq. (4), and use nonlocal conditions Eq. (2) to get

$$u(0) = \sum_{k=1}^m c_k u(\tau_k) = u(0) \sum_{k=1}^m c_k + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^m c_k \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds$$

$$= B \sum_{k=1}^m \frac{c_k}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds. \quad (5)$$

Substituting Eq. (5) in Eq. (4), we get

$$u(t) = B \sum_{k=1}^m \frac{c_k}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} H_u(s) ds. \quad (6)$$

On the other hand, by applying the fractional derivative operator ${}^c D_{0+}^\alpha$ on both sides of Eq.(6) and using Lemma 3, we have

$$\begin{aligned} {}^c D_{0+}^\alpha u(t) &= {}^c D_{0+}^\alpha I_{0+}^\alpha H_u(t) \\ &= a(t)u(t) + h(t) + \int_0^t K(t, \sigma) f(u(\sigma)) d\sigma, \end{aligned} \quad (7)$$

which means that, Eq. (1) holds. Finally, we take $t \rightarrow 0$ in Eq. (6), and use Lemma 4, and can conclude that $u(0) = \sum_{k=1}^m c_k u(\tau_k)$. This completes the proof. $\square$

Next, we will prove the existence of solution for the problem (1)-(2) in the space $C(J, \mathcal{X})$ by means of Krasnoselskii's fixed point theorem.

Theorem 8. Assume that the hypotheses $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, and $(\mathbf{A}_3)$ hold. If

$$B \sum_{k=1}^m c_k [\rho + bK_0L] \frac{(\tau_k)^\alpha}{\Gamma(\alpha + 1)} < 1 \quad (8)$$

and

$$\left[B \sum_{k=1}^m c_k (\tau_k)^\alpha + b^\alpha \right] \frac{(\rho + bK_0L)}{\Gamma(\alpha + 1)} < \frac{1}{2}, \quad (9)$$

then the fractional integro-differential equation (1) – (2) has a solution in $C(J, \mathcal{X})$ on J .

Proof. Define the operator $\Upsilon : C(J, \mathcal{X}) \rightarrow C(J, \mathcal{X})$ by

$$\Upsilon u(t) = B \sum_{k=1}^m c_k \frac{1}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} H_u(s) ds.$$

We need to analyze the operator Υ into sum two operators $P+Q$ as follows

$$Pu(t) = B \sum_{k=1}^m c_k \frac{1}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} H_u(s) ds \quad (10)$$

and

$$Qu(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H_u(s) ds. \quad (11)$$

Now, we apply Lemma 6 in several steps:

Step(1): We prove that $Pu + Qu^* \in S_r \subset C(J, \mathcal{X})$, for every $u, u^* \in S_r$.

Let $\mu_0 = \|f(0)\|$, $\eta = \sup_{t \in J} \|h(t)\|$, $\rho = \sup_{t \in J} \|a(t)\|$ and choosing

$$r \geq 2 \left[B \sum_{k=1}^m c_k (\tau_k)^\alpha + b^\alpha \right] \frac{(\eta + bK_0\mu_0)}{\Gamma(\alpha + 1)}$$

such that $S_r = \{u \in C(J, \mathcal{X}) : \|u\| \leq r\}$.

For the operator P : By the previous assumptions, for $u \in S_r$ and $t \in J$, we have

$$\|Pu(t)\| \leq B \sum_{k=1}^m \frac{c_k}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} \|H_u(s)\| ds.$$

Note that

$$\begin{aligned} \|H_u(s)\| &\leq \|a\| \|u\| + \|h\| + \int_0^s \|K(s, \sigma)\| [\|f(u) - f(0)\| + \|f(0)\|] d\sigma \\ &\leq (\rho + bK_0L) \|u\| + \eta + bK_0\mu_0. \end{aligned}$$

Consequently,

$$\begin{aligned} \|Pu(t)\| &\leq B \sum_{k=1}^m \frac{c_k}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} [(\rho + bK_0L) \|u\| + \eta + bK_0\mu_0] ds \\ &\leq B \sum_{k=1}^m c_k \left[\frac{(\rho + bK_0L) r}{\Gamma(\alpha + 1)} + \frac{(\eta + bK_0\mu_0)}{\Gamma(\alpha + 1)} \right] (\tau_k)^\alpha. \end{aligned} \quad (12)$$

For the operator Q : By the previous assumptions, for $u^* \in S_r$ and $t \in J$, we have

$$\begin{aligned} \|Qu^*(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [(\rho + bK_0L) \|u^*\| + \eta + bK_0\mu_0] ds \\ &\leq \left[\frac{(\rho + bK_0L) r}{\Gamma(\alpha + 1)} + \frac{(\eta + bK_0\mu_0)}{\Gamma(\alpha + 1)} \right] b^\alpha. \end{aligned} \quad (13)$$

From Eq. (12) and Eq. (13), we get

$$\|Pu + Qu^*\| \leq \|Pu\| + \|Qu^*\|$$

$$\begin{aligned}
&\leq \left[B \sum_{k=1}^m c_k (\tau_k)^\alpha + b^\alpha \right] \frac{(\rho + bK_0L) r}{\Gamma(\alpha + 1)} \\
&\quad + \left[B \sum_{k=1}^m c_k (\tau_k)^\alpha + b^\alpha \right] \frac{(\eta + bK_0\mu_0)}{\Gamma(\alpha + 1)} \\
&\leq \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}$$

This means that, $Pu + Qu^* \in S_r$.

Step(2): We prove that operator P is a contraction map on S_r .

Let us make S_r as in step (1), by the preceding assumptions, then for $u, u^* \in S_r$ and for $t \in J$, we have

$$\begin{aligned}
&\|Pu(t) - Pu^*(t)\| \\
&\leq B \sum_{k=1}^m c_k \frac{1}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} \|H_u(s) - H_{u^*}(s)\| \\
&\leq B \sum_{k=1}^m c_k \frac{1}{\Gamma(\alpha)} \int_0^{\tau_k} (\tau_k - s)^{\alpha-1} [\rho \|u(s) - u^*(s)\| + bK_0L \|u(s) - u^*(s)\|] ds \\
&\leq B \sum_{k=1}^m c_k [\rho + bK_0L] \frac{(\tau_k)^\alpha}{\Gamma(\alpha + 1)} \|u - u^*\|.
\end{aligned}$$

By Eq. (8), we conclude that P is contraction map on S_r .

Step(3): We show that the operator Q is completely continuous on S_r .

For this end, we consider S_r defined as in step (1) and we prove that (QS_r) is uniformly bounded, $(\overline{QS_r})$ is equicontinuous and $Q : S_r \rightarrow S_r$ is continuous. Firstly, we show that (QS_r) is uniformly bounded. By step (1), then for any $u \in S_r$ and for each $t \in J$, we have

$$\|Qu\| \leq \left[\frac{(\rho + bK_0L) r}{\Gamma(\alpha + 1)} + \frac{(\eta + bK_0\mu_0)}{\Gamma(\alpha + 1)} \right] b^\alpha := \ell.$$

This means that $QS_r \subset S_\ell$, for any $u \in S_r$, i.e. the set $\{Qu : u \in S_r\}$ is uniformly bounded. Next, we will prove that $(\overline{QS_r})$ is equicontinuous. Let $u \in S_r$ and for $t_1, t_2 \in J$ with $t_1 \leq t_2$, we have

$$\begin{aligned}
\|Qu(t_2) - Qu(t_1)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|H_u(s)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \|H_u(s)\| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\rho + bK_0 Lr}{\Gamma(\alpha)} \left[\int_0^{t_2-t_1} ((t_2 - t_1) - s)^{\alpha-1} ds \right. \\
&\quad \left. + \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) ds \right] \\
&\quad + \frac{(\eta + bK_0 \mu_0)}{\Gamma(\alpha)} \left[\int_0^{t_2-t_1} ((t_2 - t_1) - s)^{\alpha-1} ds \right. \\
&\quad \left. + \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) ds \right] \\
&\leq 2 \left[\frac{\rho + bK_0 Lr + \eta + bK_0 \mu_0}{\Gamma(\alpha + 1)} \right] (t_2 - t_1)^\alpha,
\end{aligned}$$

where η, μ_0 and ρ are defined as in step (1). Observe that, $\|Qu(t_2) - Qu(t_1)\| \rightarrow 0$ when $|t_2 - t_1| \rightarrow 0$, which means that $(\overline{QS_r})$ is equicontinuous. Finally, from the continuity of h, f and K , we can directly reach that the operator $Q : S_r \rightarrow S_r$. So, Q is relatively compact on S_r . Hence, by the Arzela-Ascoli theorem, the operator Q is compact on S_r . An application of Lemma 6 shows that operator $\Upsilon = P + Q$ has a fixed point on S_r . So the fractional integro-differential equation (1) – (2) has a solution $u(t) \in C(J, \mathcal{X})$. This proves the required. $\square$

4. An example

The present section contains an example to point up the key result established in Section 3. Consider the following nonlocal fractional integro-differential equation

$${}^c D_{0+}^{\frac{1}{2}} u(t) = \frac{1}{6} \cos(t) u(t) + \frac{1}{3} \sin(t) + \int_0^t \frac{e^{-(\sigma-t)} u(s)}{4(3 + e^t)} ds, \quad (14)$$

$$u(0) = \frac{1}{9} u\left(\frac{2}{3}\right). \quad (15)$$

Here, $c_1 = \frac{1}{9}$, $\tau_1 = \frac{2}{3}$, $\alpha = \frac{1}{2}$, $a(t) = \frac{1}{6} \cos(t)$, $h(t) = \frac{1}{3} \sin(t)$, $f(u(t)) = \frac{u(t)}{3+e^t}$, and $K(t, \sigma) = \frac{e^{-(\sigma-t)}}{4}$. For $u, v \in \mathcal{X} = \mathbb{R}^+$ and $t \in [0, 1]$. We can see that $\|f(u) - f(v)\| \leq \frac{1}{4} \|u - v\|$, and $K_0 = \|K(t, \sigma)\| = \frac{e-1}{4}$.

So, the conditions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$ are satisfied with $L = \frac{1}{4}$, $\mu_0 = 0$, $\eta = \frac{1}{3}$ and $\rho = \frac{1}{6}$. Further, it is easy to check that Eq. (9) and Eq. (8) hold too. In fact, $B = \frac{1}{1-c_1} = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$, a simple computation shows that

$$c_1 B [\rho + bK_0 L] \frac{(\tau_1)^\alpha}{\Gamma(\alpha + 1)} \approx 0.03 < 1$$

and

$$[c_1 B(\tau_1)^\alpha + b^\alpha] \frac{(\rho + bK_0 L)}{\Gamma(\alpha + 1)} \approx 0.34 < \frac{1}{2}.$$

An application of Theorem 8 implies that problem (14)-(15) has a solution on $[0, 1]$.

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