A NOTE ON THE HOP DOMINATION NUMBER OF A SUBDIVISION GRAPH

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Abstract: Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. A subset \( S \subset V(G) \) is a hop dominating set of \( G \) if for every \( v \in V - S \), there exists \( u \in S \) such that \( d(u, v) = 2 \). The minimum cardinality of a hop dominating set of \( G \) is called a hop domination number of \( G \) and is denoted by \( \gamma_h(G) \). The subdivision graph \( S(G) \) of a graph \( G \) is a graph obtained by subdividing every edge of \( G \) exactly once. In this paper, we obtain an upper bound on hop domination number of subdivision graph of any connected graph \( G \) in terms of number of edges \( q \), the maximum degree \( \Delta(G) \) and domination number \( \gamma(G) \) of \( G \). We also characterize the family of connected graphs attaining this bound.

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1. Introduction

Throughout this paper, by a graph \( G = (V, E) \) we mean a connected simple graph. We denote a graph \( G \) of order \( p \) and size \( q \) by a \((p, q)\)-graph. By subdividing an edge \( e = uv \) of a graph \( G \) we mean deleting the edge \( e \) and introducing a new vertex \( x \) and the edges \( ux \) and \( xv \). For a graph \( G \), the subdivision graph \( S(G) \) is a graph obtained by subdividing every edge of \( G \) exactly once. The distance between two vertices \( u \) and \( v \) of a graph \( G \) is the...
length of the shortest path joining $u$ and $v$ in $G$ and is denoted by $d(u,v)$. A graph $G$ with exactly one cycle is called an unicyclic graph. A set $D \subseteq V$ is said to be a dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. $D$ is said to be a minimal dominating set of $G$ if no subset of it is a dominating set of $G$. The minimum cardinality of a minimal dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. In [4], Chartrand et al. studied the exact 2-step dominating sets in graphs.

Recently, Ayyaswamy and Natarajan ([2, 9]) initiated a study on a new domination parameter called hop domination number of a graph and characterized the family of trees and unicyclic graphs with equal hop domination number and total domination number. Ayyaswamy et al. ([1]) found some bounds on hop domination number of a tree. Henning et al. [8] obtained certain probabilistic bounds for this parameter. Farhadi et al. [5] discussed the complexity results of k-hop dominating set of a graph. Pabilona et al. [10, 11] studied connected hop domination and total hop domination on graphs under some binary operations. A vertex $u$ of a graph $G$ is said to hop dominate a vertex $v \in V(G)$ if $d(u,v) = 0$ or $d(u,v) = 2$. A subset $S \subseteq V(G)$ of a graph $G$ is a hop dominating set (hd-set) of $G$ if for every $v \in V - S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of $G$ is called the hop domination number of $G$ and is denoted by $\gamma_h(G)$. A path on $n$ vertices is denoted by $P_n$ and a cycle of length $n$ is denoted by $C_n$. We denote a complete graph with $n$ vertices by $K_n$ and a complete bipartite graph with a bipartition $(V_1, V_2)$ $mn$ vertices by $K_{m,n}$. For other terminologies not defined here we refer to Chartrand and Lesniak ([3]) and Haynes et al. ([6, 7]). It is easy to verify that:

(i) $\gamma_h(P_n) = \gamma_h(C_n) = \begin{cases} 2r, & \text{if } n = 6r; \\ 2r + 1, & \text{if } n = 6r + 1; \\ 2r + 2, & \text{if } n = 6r + s; 2 \leq s \leq 5. \end{cases}$

(ii) $\gamma_h(K_n) = n.$

(iii) $\gamma_h(K_{m,n}) = 2.$

(iv) $\gamma_h(W_n) = 3$, where $W_n$ is a wheel with $n - 1$ spokes.

(v) $\gamma_h(P) = 2$, where $P$ denotes the Peterson graph.
2. Main Results

Observation 1. $\gamma_h[S(P_n)] = \gamma_h(P_{2n-1})$.

Observation 2. $\gamma_h[S(C_n)] = \gamma_h(C_{2n})$.

Proposition 3. $\gamma_h[S(K_n)] = \left\lceil \frac{n}{2} \right\rceil + 1, n \geq 2$.

Proof. Let $V_1 = V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $V_2 = \{w_{ij} \in S(K_n) : w_{ij}
$ is a vertex subdividing the edge $v_iv_j$ in $S(K_n)\}$. Then $|V_2| = \frac{n(n-1)}{2} = nC_2$.
Any one vertex $v_i$ is enough to hop dominate all vertices of $V_1$ and we again require exactly
$\left\lceil \frac{n}{2} \right\rceil$ vertices to hop dominate the $nC_2$ vertices of $V_2$. Thus
$\{v_i\} \cup \{w_{j(j+1)} : j \text{ odd; } 1 \leq j \leq n\}$ is a $\gamma_h$-set of $S(K_n)$ and therefore
$\gamma_h[S(K_n)] = \left\lceil \frac{n}{2} \right\rceil + 1$. □

Proposition 4. $\gamma_h[S(K_{m,n})] = 2 + \min\{m, n\}$.

Proof. Let $(V_1, V_2)$ be the bipartition of $V(K_{m,n})$ with $|V_1| = m$ and $|V_2| = n$.
Let $V_1 = \{v_i : 1 \leq i \leq m\}$ and $V_2 = \{w_j : 1 \leq j \leq n\}$.
Let $m \leq n$.
Let $V[S(K_{m,n})] = V_1 \cup V_2 \cup V_3$ where $V_3 = \{w_{ij} : w_{ij}\text{ is a vertex subdividing the edge } v_iu_j \text{ in } S(K_{m,n})\}$. As $V_1$ and $V_2$ are independent sets, any $\gamma_h$-set of $S(K_{m,n})$ contains a vertex from each $V_1$ and $V_2$. One can observe that the set $D = \{w_{kk} : 1 \leq k \leq m\}$ is a minimum hd-set of $V_3$. Therefore
$\gamma_h[S(K_{m,n})] = 2 + |D| = 2 + m = 2 + \min(m, n)$. Hence the result follows. □

Proposition 5. For the Petersen graph $P$, $\gamma_h[S(P)] = 7$.

Proof. Let us label the vertices of the outer cycle $C_5$ by $v_1, v_2, v_3, v_4, v_5$ and the inner cycle by $u_1, u_2, u_3, u_4, u_5$. Consider the three pairs $(v_i, v_j), (v_i, u_j), (u_i, u_j)$. Only one of them forms an edge in $P$. Let $w_{ij}$ be the vertex subdividing that edge. It is clear that the set $\{u_i, u_j\} \cup \{v_k\} \cup \{w_{i(i+1)} : i \text{ is odd, } 1 \leq i \leq 4\} \cup \{w_{l(l+3)} : l = 1, 2\}$ is a $\gamma_h$-set of $S(P)$ where $u_i$ and $u_j$ are non adjacent vertices and the vertex $v_k$ is adjacent to a vertex $u_k \in N(u_i) \cap N(u_j)$ in $P$. Hence $\gamma_h[S(P)] = 7$. □

Proposition 6. For a wheel graph $W_{n+1}$ with $n+1$ vertices, $\gamma_h[S(W_{n+1})] = \left\lceil \frac{n}{3} \right\rceil + 2$. 
Proof. Let the centre of $W_{n+1}$ be $v$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the outer cycle $C_n$ of $W_{n+1}$. Let the vertex which is adjacent to $v$ and $v_i$ in $S(W_{n+1})$ be denoted by $u_i$; $1 \leq i \leq n$ and let the vertex subdividing the edge $v_i v_j$ in $S(W_{n+1})$ be denoted by $w_{ij}$. One can easily observe that the centre vertex $v$ of $W_{n+1}$ hop dominates the vertices $\{v_i : 1 \leq i \leq n\}$ and the vertex $u_2$ hop dominates the vertices $\{u_i : 1 \leq i \leq n\}$ and also the vertices $w_{12}$ and $w_{23}$. Furthermore, the set $D \setminus \{w_{12}\}$ hop dominates the remaining $n - 2$ vertices in $S(W_{n+1})$ where $D = \{w_{i(i+1)} : i \equiv 1 \pmod{3}\}$. Therefore, $\{v\} \cup \{u_j\} \cup D$ is a $\gamma_h$-set of $S(W_{n+1})$. Thus $\gamma_h[S(W_{n+1})] = \left\lceil \frac{n}{3} \right\rceil + 2$. \qed

**Theorem 7.** For any graph $G$, $\gamma(G) < \gamma_h[S(G)]$.

Proof. Let $D$ be a hop dominating set of $S(G)$. Let $D_1 = D \cap V(G)$ and $D_2 = D \setminus D_1$. Clearly $D_1$ is the only set hop dominating $V(G)$ since every vertex in $D_2$ is at odd distance from any vertex of $V(G)$. This shows that $D_1 \neq \emptyset$. Similarly $D_2 \neq \emptyset$. Further, if a vertex $v$ is hop dominated by a vertex $u$ in $D_1$, then $d(u, v) = 2$ in $S(G)$. This implies there is a path $uvw$ in $S(G)$ and $uv \in E(G)$. Thus $v$ is dominated by $u$ and so $\gamma(G) \leq |D_1| < |D| = \gamma_h[S(G)]$. \qed

**Theorem 8.** Let $G$ be a $(p, q)$-graph. Let $u$ be a vertex of maximum degree $\Delta(G)$ and $v$ be a vertex in $N(u)$ such that $\deg(v) = \max_{y \in N(u)} \{\deg(y)\}$. Then $\gamma_h[S(G)] \leq \gamma(G) + q - \Delta(G) - \deg(v) + 2$.

Proof. Let $w_0$ be the new vertex subdividing the edge $uv$ and $S'$ be a $\gamma$-set of $G$. Let $E'$ be the set of all edges incident with $u$ or $v$. Let $D_1 = \{w : w \in V[S(G)] \setminus V(G) \text{ is a vertex subdividing the edge } v v' \in E(G) \setminus E'\}$. Then $D = S' \cup \{w_0\} \cup D_1$ is a hop dominating set of $S(G)$ and hence $\gamma_h[S(G)] \leq |D| = \gamma(G) + 1 + (q - \Delta(G) - (\deg(v) - 1)) = \gamma(G) + 2 - \deg(v) + q - \Delta(G)$. \qed

**Theorem 9.** Let $T$ be a tree with $q$ edges. Let $u$ be a vertex of maximum degree $\Delta(T)$ and let $v \in N(u)$ be a vertex in $T$ such that $\deg(v) = \max_{y \in N(u)} \{\deg(y)\}$. Then $\gamma_h[S(T)] = \gamma(T) + q + 2 - (\Delta(T) + \deg(v))$ if and only if the following conditions hold:

(i) every vertex $w \in N(u) \cup N(v) \setminus \{u, v\}$ is either a leaf or a weak support.

(ii) both $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ cannot contain weak support vertices.

Proof. Assume that $\gamma_h[S(T)] = \gamma(T) + q + 2 - (\Delta(T) + \deg(v))$. Let $w_0$ be
Thus deg of T hop dominate all vertices of D. Let E be the set of vertices subdividing the edges in E1. In what follows hereafter, we call S' a γ-set of T.

(i) Let w ∈ N(u) ∪ N(v) \ {u, v}. Suppose w ∈ N(u) \ {v} be a vertex of degree r ≥ 3. Let w1 be a vertex subdividing one of the edges incident at w except the edge uw, say ww'. Let Ew denote the set of edges incident at w except the edge uw and let Dw be the set of vertices subdividing the edges in Ew. Then w1 hop dominates all vertices of Dw. Therefore S' ∪ {w0, w1} ∪ (D1 \ Dw) is a hd-set of S(T) and so,

$$\gamma_h[S(T)] \leq \gamma(T) + q + 2 - (\Delta(T) + \text{deg}(v) - 1) - r + 1$$

$$= \gamma(T) + q + 2 - (\Delta(T) + \text{deg}(v)) + r + 2$$

$$= \gamma(T) + q + 4 - (\Delta(T) + \text{deg}(v)) + r$$

$$\leq \gamma(T) + q + 4 - (\Delta(T) + \text{deg}(v)) - 3, \text{ since } r \geq 3$$

$$= \gamma(T) + q + 1 - (\Delta(T) + \text{deg}(v))$$

$$< \gamma(T) + q + 2 - (\Delta(T) + \text{deg}(v)), \text{ a contradiction.}$$

Hence deg(w) ≤ 2 for every w ∈ N(u) \ {v}.

If deg(w) = 1, then nothing to prove. So, let deg(w) = 2.

Now we show that w is a weak support vertex in T. Let y ∈ N(w) \ {u, v} be vertex such that deg(y) ≥ 2.

Suppose y ∈ N(w) \ {u, v} is vertex such that deg(y) ≥ 2.

Let Eu be the set of edge incident with u except the edge uw and Du be the set of vertices subdividing the edges in Eu. Let Ev be the set of edges which are incident at v except the edge uw and Dv be the set of vertices subdividing the edges in Ev. Let E2 be the set of edge which are not incident at u. Let D2 be the set of vertices subdividing the edges in E2. Let Ey be the set of edges incident at y except the edge wy and Dy be the set of vertices subdividing the edges in Ey. Then the vertex w2 which subdivides the edge wy in S(T) will hop dominate all vertices of Dy and the vertex w0 hop dominates all vertices in Du ∪ Dv. Therefore, S' ∪ {w0} ∪ (D2 \ (Dv ∪ Dy)) is a hd-set of S(T). Hence,

$$\gamma_h[S(T)] \leq \gamma(T) + q + 1 - (\Delta(T) + \text{deg}(v) - 1) - \text{deg}(y) + 1$$

$$= \gamma(T) + q + 3 - (\Delta(T) + \text{deg}(v)) - \text{deg}(y)$$

$$\leq \gamma(T) + q + 1 - (\Delta(T) + \text{deg}(v)), \text{ since } \text{deg}(y) \geq 2$$

$$< \gamma(T) + q + 2 - (\Delta(T) + \text{deg}(v)), \text{ a contradiction.}$$

Thus deg(y) = 1 for all y ∈ N(w) \ {u, v}. That is, w is a weak support vertex of T.
Similarly, one can prove that every vertex \( w \in N(v) \setminus \{u\} \) is either a leaf of a weak support of \( T \).

(ii) Suppose both \( N(u) \setminus \{v\} \) and \( N(v) \setminus \{u\} \) have weak support vertices in \( T \).

Let \( N'(u) = \{ w \in N(u) \setminus \{v\} : \deg(w) = 2 \} \) and \( N'(v) = \{ w \in N(v) \setminus \{u\} : \deg(w) = 2 \} \). Let \( N''(u) = \{ w' : w' \) is the vertex subdividing the edge \( uw \) where \( w \in N'(u) \} \) and \( N''(v) = \{ w' : w' \) is the vertex subdividing the edge \( vw \) where \( v \in N'(v) \} \).

By our assumption \( N'(u) \neq \emptyset \) and \( N'(v) \neq \emptyset \). Clearly, \( |N''(u) \cup N''(v)| = q - \Delta(T) - (\deg(v) - 1) \) and so \( S' \cup N''(u) \cup N''(v) \) is a \( \gamma \)-set of \( S(T) \). Therefore, \( \gamma_h[S(T)] \leq \gamma(T) + |S(T)| < \gamma(T) + q - (|S(T)| - 1) = \gamma(T) + q + 2 - (\Delta(T) + \deg(v)) \), a contradiction.

The converse is obvious. \( \square \)

**Theorem 10.** Let \( G \) be a connected \((p,q)\)-graph having at least one cycle and let \( u \) and \( v \) be vertices as in Theorem 8. Then \( \gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + \deg(v)) \) if and only if the following conditions hold:

(i) Every cycle \( C \) in \( G \) contains \( u \) or \( v \) or the edge \( uv \) and the length of \( C \) is at most 5.

(ii) If the longest cycle containing the edge \( uv \) in \( G \) is \( C_3 \), then

\( \textbf{(a)} \) every vertex \( w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u,v\}) \) is a leaf or weak support of degree 2 or a vertex of degree 2 in another cycle \( C_3 \) of \( G \).

\( \textbf{(b)} \) both \( N(u) \setminus \{v\} \) and \( N(v) \setminus \{u\} \) cannot contain weak support vertices in \( G \).

(iii) If the longest cycle containing the edge \( uv \) in \( G \) is \( C = C_4 \), then every vertex \( w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u,v\}) \) is a leaf or a vertex of degree 2 in \( C \).

(iv) If the longest cycle in \( G \) is \( C = C_5 \), then

\( \textbf{(a)} \) the edge \( uv \) is a chord of \( C \)

\( \textbf{(b)} \) every vertex \( w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u,v\}) \) is a leaf or a vertex of degree 2 in \( C \).

(v) Every vertex \( w \in N(u) \cap N(v) \) is of degree at most 3 and if \( w \in N(u) \cap N(v) \) is of degree 3 in \( G \), then there exists at most one edge not adjacent to \( uv \) in \( G \).
Proof. Assume that $\gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + \deg(v))$.
Let $E_1, D_1$ and $w_0$ be as in Theorem 9.
Throughout this proof, $S'$ denotes a $\gamma$-set of $G$.
(i) Suppose there exists a cycle $C$ in $G$ not containing $u$ and $v$.
Let $V(C) = \{v_1, v_2, \cdots, v_k\}$. Let $w_i, w_{i-1}$ and $w_{i+1}$ be the vertices in $D_1$ subdividing the edges $v_{i-1}v_i$, $v_{i-1}v_{i-2}$ and $v_iv_{i+1}$ in $C$, respectively. Then clearly the vertex $w_i$ hop dominates the vertices $w_{i-1}$ and $w_{i+1}$ in $S(G)$. Therefore $S' \cup \{w_0\} \cup (D_1 \setminus \{w_{i-1}, w_{i+1}\})$ is a hd-set of $S(G)$. Hence

$$\gamma_h[S(G)] \leq \gamma(G) + 1 + q - 2 - (\Delta(G) + \deg(v) - 1)$$

$$= \gamma(G) + q - (\Delta(G) + \deg(v))$$

$$< \gamma(G) + 2 + q - (\Delta(G) + \deg(v)),$$ a contradiction.

Claim: Every cycle $C$ in $G$ is of length at most 5.
Suppose there exists a cycle $C$ containing the edge $uv$ in $G$ such that the length $k$ of $C \geq 6$. Let $V(C) = \{u = v_1, v = v_2, v_3, \cdots, v_k\}$. Then $C$ contains at least three edges not incident at $u$ or $v$. Let $v_{i-1}v_i$, $v_{i-2}v_{i-1}$ and $v_iv_{i+1}$ be three edges in $C$ not incident at $u$ or $v$ and let $w_i, w_{i-1}$ and $w_{i+1}$ be the vertices in $S(G)$ subdividing these edges respectively. Then $w_i$ hop dominates $w_{i-1}$ and $w_{i+1}$. Therefore $S' \cup \{w_0\} \cup (D_1 \setminus \{w_{i-1}, w_{i+1}\})$ is a hd-set of $S(G)$ so that

$$\gamma_h[S(G)] \leq \gamma(G) + 1 + q - 2 - (\Delta(G) + \deg(v) - 1)$$

$$= \gamma(G) + q - (\Delta(G) + \deg(v))$$

$$< \gamma(G) + 2 + q - (\Delta(G) + \deg(v)),$$ a contradiction.

Applying a similar argument given in Theorem 9 one can easily prove the conditions $(ii - a)$ and $(ii - b)$.

(iii) Let $C = \langle u, v, x, y \rangle$ be a longest cycle of length 4 containing the edge $uv$ in $G$.

Claim: Every vertex $w \in N(u) \cup N(v) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$ is a leaf or a vertex of degree 2 in $C$. Let $w \in N(u) \cup N(v) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$.

Then either $w \in N(u) \setminus (N(u) \cap N(v) \cup \{u, v\})$ or $w \in N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$.

Case 1: Let $w \in N(u) \setminus (N(u) \cap N(v) \cup \{u, v\})$. Then as discussed in Theorem 9, $\deg(w) \leq 2$. If $\deg(w) = 1$, then clearly $w$ is a leaf. So assume that $\deg(w) \neq 1$.

We claim that $w$ is neither a weak support vertex of degree 2 in $G$ nor a vertex of degree 2 in any other cycle of length 3 or 4 or 5.
Suppose $w$ is a weak support vertex of degree 2 in $G$. Let $z$ be the leaf adjacent to $w$ in $G$. Let $v_{uw}$ and $v_{wz}$ be the vertices subdividing the edges $uw$ and $wz$ in $S(G)$. Then the vertex $v_{uw}$ hop dominates all the vertices in $D_u$ and the vertex $v_{wz}$ in $S(G)$. Let $w_1$ be the vertex subdividing the edge $vy$ in $S(G)$. Then $w_1$ hop dominates all the vertices in $D_v$ and the vertex $v_{xy}$ that subdivides the edge $xy$ in $S(G)$.

Therefore $S' \cup \{w_1, v_{wz}\} \cup D_2 \setminus (D_v \cup \{v_{xy}, v_{wz}\})$ is clearly a hd-set of $S(G)$. Hence

$$\gamma_h[S(G)] \leq \gamma(G) + 2 + q - \Delta(G) - \deg(v) - 1 - 2 = \gamma(G) + q - \Delta(G) - \deg(v) + 1 < \gamma(G) + 2 + q - (\Delta(G) + \deg(v)), \text{ a contradiction.}$$

Thus the vertex $w$ cannot be a weak support of degree 2 in $G$. The other cases follow similarly.

Similarly, Case 2 can be argued for $w \in N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$.

Next we prove condition (iv). Let $C = C_5$ be a longest cycle of length 5 in $G$.

**Claim:** The edge $uv$ is a chord of $C$ in $G$.

Suppose the edge $uv$ is not a chord of $C$.

**Case 1:** $u \in V(C)$ and $v \notin V(C)$.

Let $V(C) = \{u, w, x, y, z\}$. Then clearly the edges $wx, xy$ and $yz$ are in $E_1$. Let $w_1, w_2$ and $w_3$ be the vertices subdividing the edges $wx, xy$ and $yz$ respectively. Then $S' \cup \{w_0\} \cup D_1 \setminus \{w_1, w_3\}$ is a hd-set of $S(G)$. Hence

$$\gamma_h[S(G)] \leq \gamma(G) + 1 + q - 2 - (\Delta(G) + \deg(v) - 1) = \gamma(G) + q - 1 - \Delta(G) - \deg(v) + 1 = \gamma(G) + q - \Delta(G) - \deg(v) < \gamma(G) + 2 + q - (\Delta(G) + \deg(v)), \text{ a contradiction.}$$

Similarly we can prove that $uv$ is not an edge in $C_5$.

One can prove the condition (b) of (iv) with similar arguments given in the proof of (iii).

(v) Suppose there exist two edges $xw_1$ and $yw_2$ in $G$ such that $w_1, w_2 \in N(u) \cap N(v)$ and $\deg(w_1) = \deg(w_2) = 3$. Let $E_u$ and $E_v$ be the set of edges incident at $u$ and $v$ respectively except the edge $uv$. Let $D_u$ be the set of vertices subdividing the edges in $E_u$ and $D_v$ be the set of vertices subdividing the edges in $E_v$. Let $w'_1$ and $w'_2$ be the vertices subdividing the edges $uw_1$ and $vw_2$ respectively. Let $x'$ and $y'$ be the vertices subdividing the edges $xw_1$ and $yw_2$ respectively.
$yw_2$ respectively. Then $w'_1$ hop dominates all vertices in $D_u$ and the vertex $x'$. Similarly, $w'_2$ hop dominates all vertices in $D_v$ and the vertex $y'$ in $S(G)$.

Therefore $S' \cup \{w'_1, w'_2\} \cup (D_1 \setminus \{x', y'\})$ is a hd-set of $S(G)$. Hence,

$$\gamma_h[S(G)] \leq \gamma(G) + 2 + q - (\Delta(G) + \deg(v) - 1)$$

$$= \gamma(G) + q - \Delta(G) - \deg(v) + 1$$

$$< \gamma(G) + 2 + q - (\Delta(G) + \deg(v)), \text{ a contradiction.}$$

Conversely, assume that the conditions (i) to (v) hold good.

Let $w_0$ be a vertex subdividing the edge $uv$. Then $w_0$ hop dominates all vertices subdividing the edges incident at $u$ and $v$. As $\deg(u) = \Delta(G)$ and by the choice of $v$, every hd-set of $S(G)$ contains $w_0$. Furthermore, as any two vertices of $V(G)$ in $S(G)$ are of even distance, every vertex $v \in V(G)$ can be hop dominated only by a vertex of $V(G)$ in $S(G)$. Therefore every $\gamma_h$-set of $S(G)$ contains $\gamma(G)$ vertices of $V(G)$. Let $e = wx \in E_1$. If $w \in N(u)$ and $x \notin N(v)$, then by condition (ii-a), $w$ is a weak support vertex of degree 2 in $G$.

If $w \in N(u)$ and $x \in N(v)$, then by condition (iii), $wx$ is an edge in $C_4$ that contains the edge $uv$. Thus in both cases either the vertex subdividing the edge $uw$ or the vertex subdividing the edge $wx$ is in every $\gamma_h$-set of $S(G)$.

If $w \in N(u) \cap N(v)$, then by condition (v) $wx$ is the only edge not adjacent to $uv$ in $G$. Therefore, one of the vertices subdividing the edges $uv, vw$ and $wx$ is in any $\gamma_h$-set of $S(G)$. If $w \notin N(u)$, then by condition (ii-b) the vertex $w$ is a weak support vertex of degree 2 in $N(v) \setminus \{u\}$. Then the vertex subdividing the edge $wx$ or $vw$ is in every $\gamma_h$-set of $S(G)$.

Thus in all cases we see that for every edge in $E_1$ there corresponds a subdividing vertex in every $\gamma_h$-set of $S(G)$. Therefore every $\gamma_h$-set of $S(G)$ contains at least $\gamma(G) + 1 + |E_1|$ vertices. This implies

$$\gamma_h[S(G)] \geq \gamma(G) + 1 + |E_1|$$

$$= \gamma(G) + 1 + q - (\Delta(G) + \deg(v) - 1)$$

$$= \gamma(G) + 2 + q - (\Delta(G) + \deg(v) - 1).$$

But by Theorem 8, $\gamma_h[S(G)] \leq \gamma(G) + 2 + q - (\Delta(G) + \deg(v))$.

Thus $\gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + \deg(v))$. \hfill \Box

References


