**GENERALIZED CANAVATI TYPE
 g -FRACTIONAL POLYA TYPE INEQUALITIES**

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Abstract: We present here generalized Canavati type g -fractional Polya type inequalities. We cover also the iterated case. Our inequalities are with respect to all L_p norms: $1 \leq p \leq \infty$. We finish with applications.

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1. Introduction

We are motivated by the following famous Polya's integral inequality, see [11], [12, p. 62], [13] and [14, p. 83].

Theorem 1. *Let $f(x)$ be differentiable and not identically a constant on $[a, b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a, b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) dx. \quad (1)$$

We need the following fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$, $[\cdot]$ is the integral part, $\beta = \alpha - m$, $0 < \beta < 1$,

$f \in C([a, b])$, $[a, b] \subset \mathbb{R}$, $x \in [a, b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$(J_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (2)$$

$a \leq x \leq b$. We define the subspace $C_{a+}^{\alpha}([a, b])$ of $C^m([a, b])$:

$$C_{a+}^{\alpha}([a, b]) = \left\{ f \in C^m([a, b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a, b]) \right\}. \quad (3)$$

For $f \in C_{a+}^{\alpha}([a, b])$, we define the left generalized α -fractional derivative of f over $[a, b]$ as

$$D_{a+}^{\alpha} f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \quad (4)$$

see [1], p. 24. Canavati first in [6] introduced the above over $[0, 1]$.

Notice that $D_{a+}^{\alpha} f \in C([a, b])$.

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a, b])$, call the right Riemann-Liouville fractional integral operator by

$$(J_{b-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad (5)$$

$x \in [a, b]$, see also [7], [8], [9], [15], [2]. Define the subspace of functions

$$C_{b-}^{\alpha}([a, b]) = \left\{ f \in C^m([a, b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a, b]) \right\}. \quad (6)$$

Define the right generalized α -fractional derivative of f over $[a, b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \quad (7)$$

see [2]. We set $D_{b-}^0 f = f$. Notice that $D_{b-}^{\alpha} f \in C([a, b])$.

We are also motivated by the following fractional Polya type integral inequality without any boundary conditions.

Theorem 2. ([4], pp. 1-7) *Let $0 < \alpha < 1$, $f \in C([a, b])$. Assume $f \in C_{a+}^{\alpha}([a, \frac{a+b}{2}])$ and $f \in C_{b-}^{\alpha}([\frac{a+b}{2}, b])$. Set*

$$M(f) = \max \left\{ \|D_{a+}^{\alpha} f\|_{\infty, [a, \frac{a+b}{2}]}, \|D_{b-}^{\alpha} f\|_{\infty, [\frac{a+b}{2}, b]} \right\}. \quad (8)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq M(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2) 2^\alpha}. \quad (9)$$

Inequality (9) is sharp, namely it is attained by

$$f_*(x) = \left\{ \begin{array}{ll} (x-a)^\alpha, & x \in \left[a, \frac{a+b}{2} \right] \\ (b-x)^\alpha, & x \in \left[\frac{a+b}{2}, b \right] \end{array} \right\}, \quad 0 < \alpha < 1. \quad (10)$$

Clearly here non zero constant functions f are excluded.

In this article we present very general g -fractional Polya type inequalities.

2. Background

Here we follow [5].

Let $g : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing function. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$. Assume that $g \in C^1([a, b])$, and $g^{-1} \in C^n([g(a), g(b)])$. Call $l := f \circ g^{-1} : [g(a), g(b)] \rightarrow \mathbb{R}$. It is clear that $l, l', \dots, l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq \mathbb{R}$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$ and $n = 0$. Next we follow [1], pp. 7-9.

I) Let $h \in C([g(a), g(b)])$, we define the left Riemann-Liouville fractional integral as

$$(J_\nu^{z_0} h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^z (z-t)^{\nu-1} h(t) dt, \quad (11)$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$.

We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ ($0 < \alpha < 1$). We define the subspace $C_{g(x_0)}^\nu([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C_{g(x_0)}^\nu([g(a), g(b)]) := \left\{ h \in C^{[\nu]}([g(a), g(b)]) : \right. \\ \left. J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)]) \right\}. \quad (12)$$

So let $h \in C_{g(x_0)}^\nu([g(a), g(b)])$; we define the left g -generalized fractional derivative of h of order ν , of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^\nu h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \quad (13)$$

Clearly, for $h \in C_{g(x_0)}^\nu([g(a), g(b)])$, there exists

$$\left(D_{g(x_0)}^\nu h\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} h^{([\nu])}(t) dt, \quad (14)$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^\nu([g(a), g(b)])$ we have that

$$\left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} (f \circ g^{-1})^{([\nu])}(t) dt, \quad (15)$$

for all $g(x_0) \leq z \leq g(b)$. We have

$$D_{g(x_0)}^n (f \circ g^{-1}) = (f \circ g^{-1})^{(n)} \quad \text{and} \quad D_{g(x_0)}^0 (f \circ g^{-1}) = f \circ g^{-1}.$$

We mention and need the following left generalized g -fractional, of Canavati type, Taylor's formula:

Theorem 3. Let $f \circ g^{-1} \in C_{g(x_0)}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.

(i) if $\nu \geq 1$, then

$$\begin{aligned} f(x) - f(x_0) &= \sum_{k=1}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k \\ &\quad + \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) dt, \end{aligned} \quad (16)$$

all $x \in [a, b] : x \geq x_0$;

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) dt, \quad (17)$$

all $x \in [a, b] : x \geq x_0$.

By the change of variables method, see [10], we may rewrite the remainder of (16), (17), as

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) dt \quad (18)$$

$$= \frac{1}{\Gamma(\nu)} \int_{x_0}^x (g(x) - g(s))^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right) (g(s)) g'(s) ds,$$

all $x \in [a, b] : x \geq x_0$.

We may rewrite (17) as follows: if $0 < \nu < 1$, we have

$$f(x) = \left(J_{\nu}^{g(x_0)} \left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right) \right) (g(x)), \quad (19)$$

all $x \in [a, b] : x \geq x_0$.

II) Next we follow [3], pp. 345-348.

Let $h \in C([g(a), g(b)])$, we define the right Riemann-Liouville fractional integral as

$$(J_{z_0-}^\nu h)(z) := \frac{1}{\Gamma(\nu)} \int_z^{z_0} (t-z)^{\nu-1} h(t) dt, \quad (20)$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J_{z_0-}^0 h = h$.

Let $\alpha := \nu - [\nu]$ ($0 < \alpha < 1$). We define the subspace $C_{g(x_0)-}^\nu([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C_{g(x_0)-}^\nu([g(a), g(b)]) := \left\{ h \in C^{[\nu]}([g(a), g(b)]) : J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \in C^1([g(a), g(x_0)]) \right\}. \quad (21)$$

So let $h \in C_{g(x_0)-}^\nu([g(a), g(b)])$; we define the right g -generalized fractional derivative of h of order ν , of Canavati type, over $[g(a), g(x_0)]$ as

$$D_{g(x_0)-}^\nu h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \right)'. \quad (22)$$

Clearly, for $h \in C_{g(x_0)-}^\nu([g(a), g(b)])$, there exists

$$\left(D_{g(x_0)-}^\nu h \right) (z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} h^{([\nu])}(t) dt, \quad (23)$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^\nu([g(a), g(b)])$ we have that

$$\left(D_{g(x_0)-}^\nu (f \circ g^{-1}) \right) (z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} (f \circ g^{-1})^{([\nu])}(t) dt, \quad (24)$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D_{g(x_0)-}^n (f \circ g^{-1})\right)(z) = (-1)^n (f \circ g^{-1})^{(n)}(z) \quad (25)$$

and $\left(D_{g(x_0)-}^0 (f \circ g^{-1})\right)(z) = (f \circ g^{-1})(z)$, all $z \in [g(a), g(x_0)]$.

We mention and need the following right generalized g -fractional, of Canavati type, Taylor's formula:

Theorem 4. Let $f \circ g^{-1} \in C_{g(x_0)-}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.
(i) if $\nu \geq 1$, then

$$\begin{aligned} f(x) - f(x_0) &= \sum_{k=1}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k \\ &\quad + \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu (f \circ g^{-1})\right)(t) dt, \end{aligned} \quad (26)$$

all $a \leq x \leq x_0$,

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu (f \circ g^{-1})\right)(t) dt, \quad (27)$$

all $a \leq x \leq x_0$.

By change of variable, see [10], we may rewrite the remainder of (26), (27), as

$$\begin{aligned} &\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu (f \circ g^{-1})\right)(t) dt \\ &= \frac{1}{\Gamma(\nu)} \int_x^{x_0} (g(s) - g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu (f \circ g^{-1})\right)(g(s)) g'(s) ds, \end{aligned} \quad (28)$$

all $a \leq x \leq x_0$.

We may rewrite (27) as follows:

if $0 < \nu < 1$, we have

$$f(x) = \left(J_{g(x_0)-}^\nu \left(D_{g(x_0)-}^\nu (f \circ g^{-1})\right)\right)(g(x)), \quad (29)$$

all $a \leq x \leq x_0 \leq b$.

III) Denote by

$$D_{g(x_0)}^{m\nu} = D_{g(x_0)}^\nu D_{g(x_0)}^\nu \dots D_{g(x_0)}^\nu \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (30)$$

Also denote by

$$J_{m\nu}^{g(x_0)} = J_\nu^{g(x_0)} J_\nu^{g(x_0)} \dots J_\nu^{g(x_0)} \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (31)$$

We call $D_{g(x_0)}^{m\nu}$ an iterated fractional derivative.

We mention and need the following g -modified and generalized left fractional Taylor's formula of Canavati type:

Theorem 5. *Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)}^{i\nu} (f \circ g^{-1})\right) \in C_{g(x_0)}^\nu([g(a), g(b)])$, $x_0 \in [a, b]$, for $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) \in C([g(x_0), g(b)])$. Then,*

$$\begin{aligned} f(x) &= \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right)(z) dz \\ &= \frac{1}{\Gamma((m+1)\nu)} \int_{x_0}^x (g(x) - g(s))^{(m+1)\nu-1} \\ &\quad \times \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right)(g(s)) g'(s) ds, \end{aligned} \quad (32)$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^\nu D_{g(x_0)-}^\nu \dots D_{g(x_0)-}^\nu \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (33)$$

Also denote by

$$J_{g(x_0)-}^{m\nu} = J_{g(x_0)-}^\nu J_{g(x_0)-}^\nu \dots J_{g(x_0)-}^\nu \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (34)$$

We call $D_{g(x_0)-}^{m\nu}$ an iterated fractional derivative.

We mention and need the following g -modified and generalized right fractional Taylor's formula of Canavati type:

Theorem 6. Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{i\nu}(f \circ g^{-1})\right) \in C_{g(x_0)-}^\nu([g(a), g(b)])$, $x_0 \in [a, b]$, for all $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right) \in C([g(a), g(x_0)])$. Then,

$$\begin{aligned} f(x) &= \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu-1} \left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right)(z) dz \quad (35) \\ &= \frac{1}{\Gamma((m+1)\nu)} \int_x^{x_0} (g(s) - g(x))^{(m+1)\nu-1} \\ &\quad \times \left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right)(g(s)) g'(s) ds, \end{aligned}$$

all $a \leq x \leq x_0 \leq b$.

In what follows next it is based on this background.

3. Main results

We present the following generalized g -fractional Polya type integral inequalities without any boundary conditions.

Theorem 7. Let $0 < \nu < 1$, $f \in C([a, b])$, $g : [a, b] \rightarrow \mathbb{R}$ be strictly increasing and $g \in C^1([a, b])$. Assume

$$f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a) + g(b)}{2} \right] \right),$$

and

$$f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a) + g(b)}{2}, g(b) \right] \right).$$

Set

$$\begin{aligned} M(f, g) &:= \max \left\{ \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} , \right. \\ &\quad \left. \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \right\}. \quad (36) \end{aligned}$$

Then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dg \leq M(f, g) \frac{(g(b) - g(a))^{\nu+1}}{\Gamma(\nu+2) 2^\nu}. \quad (37)$$

Inequality (37) is sharp, namely it is attained by f_* such that

$$(f_* \circ g^{-1})(z) = \begin{cases} (z - g(a))^\nu, & z \in \left[g(a), \frac{g(a)+g(b)}{2} \right] \\ (g(b) - z)^\nu, & z \in \left[\frac{g(a)+g(b)}{2}, g(b) \right] \end{cases}, \quad 0 < \nu < 1. \quad (38)$$

Clearly here non zero constant functions $f \circ g^{-1}$ are excluded.

Proof. Notice that

$$\left| \int_a^b f dg \right| = \left| \int_a^b f g' dx \right| \leq \int_a^b |f| g' dx = \int_a^b |f| dg.$$

Let $x_0 \in [a, b]$ be such that $g(x_0) = \frac{g(a)+g(b)}{2}$, that is $x_0 = g^{-1} \left(\frac{g(a)+g(b)}{2} \right)$.

Let $f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)$, $0 < \nu < 1$. By Theorem 3 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (39)$$

all $x \in [a, x_0]$.

Assume also $f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)$, by Theorem 4 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (40)$$

all $x \in [x_0, b]$.

By (39) we get

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \left(\int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} dt \right) \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \\ &= \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \frac{(g(x) - g(a))^\nu}{\Gamma(\nu + 1)}. \end{aligned} \quad (41)$$

That is,

$$|f(x)| \leq \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \frac{(g(x) - g(a))^\nu}{\Gamma(\nu + 1)}, \quad (42)$$

for any $x \in [a, x_0]$.

Similarly, by (40) we get

$$|f(x)| \leq \frac{1}{\Gamma(\nu)} \left(\int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} dt \right) \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]}$$

$$= \frac{(g(b) - g(a))^\nu}{\Gamma(\nu + 1)} \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]}. \quad (43)$$

That is

$$|f(x)| \leq \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \frac{(g(b) - g(a))^\nu}{\Gamma(\nu + 1)}, \quad (44)$$

for any $x \in [x_0, b]$.

Thus, we have

$$\begin{aligned} \left| \int_a^b f dg \right| &\leq \int_a^b |f| dg = \int_a^b |f| g' dx \\ &= \int_a^{x_0} |f(x)| g'(x) dx + \int_{x_0}^b |f(x)| g'(x) dx \stackrel{(\text{by (42), (44)})}{\leq} \\ &\quad \frac{\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \int_a^{x_0} (g(x) - g(a))^\nu g'(x) dx}{\Gamma(\nu + 1)} \end{aligned} \quad (45)$$

$$\begin{aligned} &+ \frac{\left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \int_{x_0}^b (g(b) - g(x))^\nu g'(x) dx}{\Gamma(\nu + 1)} \\ &= \frac{\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} (g(x_0) - g(a))^{\nu+1}}{\Gamma(\nu + 1)} \end{aligned} \quad (46)$$

$$\begin{aligned} &+ \frac{\left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} (g(b) - g(x_0))^{\nu+1}}{\Gamma(\nu + 1)} \\ &= \frac{1}{\Gamma(\nu + 2)} \left[\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \left(\frac{g(b) - g(a)}{2} \right)^{\nu+1} \right. \\ &\quad \left. + \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \left(\frac{g(b) - g(a)}{2} \right)^{\nu+1} \right] \\ &= \frac{1}{\Gamma(\nu + 2)} \frac{(g(b) - g(a))^{\nu+1}}{2^{\nu+1}} \left[\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \right. \\ &\quad \left. + \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \right] \end{aligned} \quad (47)$$

$$\leq \frac{M(f, g)}{\Gamma(\nu + 2)} \frac{(g(b) - g(a))^{\nu+1}}{2^\nu}. \quad (48)$$

We have proved inequality (37). Next we prove sharpness of (37).

Here $[\nu] = 0$ and $\alpha = \nu$. We see that

$$\begin{aligned} \left(J_{1-\alpha}^{g(a)} (f_* \circ g^{-1}) \right) (z) &= \frac{1}{\Gamma(1-\alpha)} \int_{g(a)}^z (z-t)^{-\alpha} (f_* \circ g^{-1})(t) dt \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{g(a)}^z (z-t)^{(1-\alpha)-1} (t-g(a))^{(\nu+1)-1} dt \end{aligned} \quad (49)$$

(by [16], p. 256)

$$= \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha) \Gamma(\nu+1)}{\Gamma(2)} (z-g(a)) = \Gamma(\nu+1) (z-g(a)).$$

Therefore,

$$\left(D_{g(a)}^\nu (f_* \circ g^{-1}) \right) (z) = \Gamma(\nu+1), \text{ for all } z \in \left[g(a), \frac{g(a)+g(b)}{2} \right]. \quad (50)$$

That is,

$$\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{\infty, \left[g(a), \frac{g(a)+g(b)}{2} \right]} = \Gamma(\nu+1). \quad (51)$$

Furthermore we have

$$\begin{aligned} - \left(J_{g(b)-}^{1-\alpha} (f_* \circ g^{-1}) \right) (z) &= \frac{-1}{\Gamma(1-\alpha)} \int_z^{g(b)} (t-z)^{-\alpha} (g(b)-t)^\nu dt \\ &= - \frac{1}{\Gamma(1-\alpha)} \int_z^{g(b)} (g(b)-t)^{(\nu+1)-1} (t-z)^{(1-\alpha)-1} dt \end{aligned}$$

(by [16], p. 256)

$$= - \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(\nu+1) \Gamma(1-\alpha)}{\Gamma(2)} (g(b)-z) = \Gamma(\nu+1) (z-g(b)), \quad (52)$$

for all $z \in \left[\frac{g(a)+g(b)}{2}, g(b) \right]$.

Hence

$$\left(D_{g(b)-}^\nu (f_* \circ g^{-1}) \right) (z) = \Gamma(\nu+1), \quad (53)$$

and

$$\left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b) \right]} = \Gamma(\nu+1). \quad (54)$$

Consequently we get that

$$M(f_*, g) = \Gamma(\nu + 1). \quad (55)$$

Applying f_* into (37) we obtain:

$$\text{R.H.S (37) for } f_* = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \frac{(g(b) - g(a))^{\nu+1}}{2^\nu} = \frac{(g(b) - g(a))^{\nu+1}}{(\nu + 1) 2^\nu}. \quad (56)$$

We get the same result when we apply f_* in the

$$\begin{aligned} \text{L.H.S (37)} &= \left| \int_a^b f_*(x) dg(x) \right| \\ &= \left| \int_a^{x_0} f_*(x) dg(x) + \int_{x_0}^b f_*(x) dg(x) \right| \\ &= \left| \int_a^{x_0} (f_* \circ g^{-1})(g(x)) dg(x) + \int_{x_0}^b (f_* \circ g^{-1})(g(x)) dg(x) \right| \\ &= \left| \int_{g(a)}^{g(x_0)} (f_* \circ g^{-1})(z) dz + \int_{g(x_0)}^{g(b)} (f_* \circ g^{-1})(z) dz \right| \quad (57) \\ &= \left| \int_{g(a)}^{g(x_0)} (z - g(a))^\nu dz + \int_{g(x_0)}^{g(b)} (g(b) - z)^\nu dz \right| \\ &= \frac{(g(x_0) - g(a))^{\nu+1}}{\nu + 1} + \frac{(g(b) - g(x_0))^{\nu+1}}{\nu + 1} \\ &= \frac{2}{\nu + 1} \frac{(g(b) - g(a))^{\nu+1}}{2^{\nu+1}} = \frac{(g(b) - g(a))^{\nu+1}}{(\nu + 1) 2^\nu}. \quad (58) \end{aligned}$$

So equality holds in (37) for f_* . Furthermore we notice that

$$\begin{aligned} (f_* \circ g^{-1}) \left(\left(\frac{g(a) + g(b)}{2} \right)_- \right) &= (f_* \circ g^{-1}) \left(\left(\frac{g(a) + g(b)}{2} \right)_+ \right) \quad (59) \\ &= \left(\frac{g(b) - g(a)}{2} \right)^\nu, \end{aligned}$$

thus $f_* \circ g^{-1}$ is continuous on $[g(a), g(b)]$ and hence $f_* \in C([a, b])$.

Sharpness of (37) has been proved. The theorem is proved completely. $\square$

Remark 8. When $\nu \geq 1$, thus $n = [\nu] \geq 1$, and by assuming that $(f \circ g^{-1})^{(k)}(g(a)) = (f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \dots, n-1$, we can prove inequality (37) again. Here also $f \in C^n([a, b])$ and $g^{-1} \in C^n([g(a), g(b)])$.

The function $(f \circ g^{-1})^{(n)}$ cannot be a constant different than zero, equivalently, $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree n .

We continue with a fractional iterated Polya type inequality:

Theorem 9. Let $0 < \nu < 1$, $f \in C([a, b])$, $g : [a, b] \rightarrow \mathbb{R}$ be strictly increasing and $g \in C^1([a, b])$. Assume that

$$\left(D_{g(a)}^{i\nu}(f \circ g^{-1})\right) \in C_{g(a)}^\nu\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right),$$

for $i = 0, 1, \dots, m \in \mathbb{N}$, and $\left(D_{g(a)}^{(m+1)\nu}(f \circ g^{-1})\right) \in C\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)$. Also assume that

$$\left(D_{g(b)-}^{i\nu}(f \circ g^{-1})\right) \in C_{g(b)-}^\nu\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right),$$

for $i = 0, 1, \dots, m \in \mathbb{N}$, and

$$\left(D_{g(b)-}^{(m+1)\nu}(f \circ g^{-1})\right) \in C\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right).$$

Set

$$M^*(f, g) := \max \left\{ \left\| D_{g(a)}^{(m+1)\nu}(f \circ g^{-1}) \right\|_{\infty, \left[g(a), \frac{g(a)+g(b)}{2}\right]}, \right. \\ \left. \left\| D_{g(b)-}^{(m+1)\nu}(f \circ g^{-1}) \right\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]} \right\}. \quad (60)$$

Then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dg \leq M^*(f, g) \frac{(g(b) - g(a))^{(m+1)\nu+1}}{\Gamma((m+1)\nu + 2) 2^{(m+1)\nu}}. \quad (61)$$

Proof. By Theorem 5 we have

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(a)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \\ \times \left(D_{g(a)}^{(m+1)\nu}(f \circ g^{-1})\right)(z) dz, \quad (62)$$

all $x \in [a, x_0]$, where $x_0 := g^{-1} \left(\frac{g(a)+g(b)}{2} \right)$.

By Theorem 6 we have

$$\begin{aligned} f(x) &= \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(b)} (z - g(x))^{(m+1)\nu-1} \\ &\quad \times \left(D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (z) dz, \end{aligned} \quad (63)$$

all $x \in [x_0, b]$.

By (62) we get

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma((m+1)\nu)} \\ &\quad \left(\int_{g(a)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} dz \right) \left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[g(a), \frac{g(a)+g(b)}{2} \right]} \\ &= \frac{(g(x) - g(a))^{(m+1)\nu}}{\Gamma((m+1)\nu + 1)} \left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[g(a), \frac{g(a)+g(b)}{2} \right]}, \end{aligned} \quad (64)$$

for all $x \in [a, x_0]$.

Similarly, by (63), we obtain

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma((m+1)\nu)} \\ &\quad \left(\int_{g(x)}^{g(b)} (z - g(x))^{(m+1)\nu-1} dz \right) \left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b) \right]} \\ &= \frac{(g(b) - g(x))^{(m+1)\nu}}{\Gamma((m+1)\nu + 1)} \left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b) \right]}, \end{aligned} \quad (65)$$

for all $x \in [x_0, b]$.

Thus, we have

$$\begin{aligned} \left| \int_a^b f dg \right| &\leq \int_a^b |f| dg = \int_a^b |f| g' dx \\ &= \int_a^{x_0} |f(x)| g'(x) dx + \int_{x_0}^b |f(x)| g'(x) dx \quad (\text{by (64), (65)}) \\ &\leq \frac{\left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[g(a), \frac{g(a)+g(b)}{2} \right]}}{\Gamma((m+1)\nu + 1)} \int_a^{x_0} (g(x) - g(a))^{(m+1)\nu} g'(x) dx \\ &\quad + \frac{\left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b) \right]}}{\Gamma((m+1)\nu + 1)} \int_{x_0}^b (g(b) - g(x))^{(m+1)\nu} g'(x) dx \end{aligned} \quad (66)$$

$$\begin{aligned}
&= \frac{1}{\Gamma((m+1)\nu+2)} \\
&\times \left[\left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} (g(x_0) - g(a))^{(m+1)\nu+1} \right. \\
&\quad \left. + \left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} (g(b) - g(x_0))^{(m+1)\nu+1} \right] \\
&(\text{by } g(x_0) = \frac{g(a)+g(b)}{2}) \\
&= \frac{1}{\Gamma((m+1)\nu+2)} \frac{(g(b) - g(a))^{(m+1)\nu+1}}{2^{(m+1)\nu+1}} \\
&\quad \left[\left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, [g(a), \frac{g(a)+g(b)}{2}]} \right. \\
&\quad \left. + \left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{\infty, [\frac{g(a)+g(b)}{2}, g(b)]} \right] \tag{67} \\
&\leq \frac{M^*(f, g)}{\Gamma((m+1)\nu+2)} \frac{(g(b) - g(a))^{(m+1)\nu+1}}{2^{(m+1)\nu}}, \tag{68}
\end{aligned}$$

proving inequality (61). $\square$

Next we give an L_1 generalized g -fractional Polya inequality:

Theorem 10. Let $\nu \geq 1$, $n = [\nu]$, and $f \in C^n([a, b])$, $g : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing function. Assume that $g \in C^1([a, b])$ and $g^{-1} \in C^n([g(a), g(b)])$. Also assume $f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)$, and $f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)$, and that $(f \circ g^{-1})^{(k)}(g(a)) = (f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \dots, n-1$. Set

$$\begin{aligned}
M_1(f, g) := \max \left\{ \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_1 \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \right. \\
\left. \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_1 \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)} \right\}. \tag{69}
\end{aligned}$$

Then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dg \leq M_1(f, g) \frac{(g(b) - g(a))^\nu}{\Gamma(\nu+1) 2^{\nu-1}}. \tag{70}$$

Here $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree n .

Proof. Let $f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)$, $\nu \geq 1$, and $(f \circ g^{-1})^{(k)}(g(a)) = 0$, for $k = 0, 1, \dots, n-1$, $n = [\nu]$. By Theorem 3 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (71)$$

all $x \in [a, x_0]$, where $x_0 := g^{-1} \left(\frac{g(a)+g(b)}{2} \right)$.

Let $f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)$, and $(f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \dots, n-1$. By Theorem 4 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (72)$$

all $x \in [x_0, b]$.

By (71) we have

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left| \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{(g(x) - g(a))^{\nu-1}}{\Gamma(\nu)} \int_{g(a)}^{g(x)} \left| \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{(g(x) - g(a))^{\nu-1}}{\Gamma(\nu)} \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_1 \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \end{aligned} \quad (73)$$

for all $x \in [a, x_0]$.

Similarly, by (72) we get

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left| \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{(g(b) - g(x))^{\nu-1}}{\Gamma(\nu)} \int_{g(x)}^{g(b)} \left| \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{(g(b) - g(x))^{\nu-1}}{\Gamma(\nu)} \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_1 \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)}, \end{aligned} \quad (74)$$

for all $x \in [x_0, b]$.

Thus, we have

$$\begin{aligned} \left| \int_a^b f dg \right| &\leq \int_a^b |f| dg = \int_a^b |f| g' dx \\ &= \int_a^{x_0} |f(x)| g'(x) dx + \int_{x_0}^b |f(x)| g'(x) dx \stackrel{(\text{by (73), (74)})}{\leq} \end{aligned}$$

$$\begin{aligned}
& \frac{\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)}}{\Gamma(\nu)} \int_a^{x_0} (g(x) - g(a))^{\nu-1} g'(x) dx \\
& + \frac{\left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)}}{\Gamma(\nu)} \int_{x_0}^b (g(b) - g(x))^{\nu-1} g'(x) dx \\
& = \frac{\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)}}{\Gamma(\nu+1)} (g(x_0) - g(a))^\nu \\
& + \frac{\left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)}}{\Gamma(\nu+1)} (g(b) - g(x_0))^\nu \quad (75) \\
& = \frac{1}{\Gamma(\nu+1)} \frac{(g(b) - g(a))^\nu}{2^\nu} \left[\left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)} \right. \\
& \quad \left. + \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_1\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)} \right] \\
& \leq \frac{M_1(f, g)}{\Gamma(\nu+1)} \frac{(g(b) - g(a))^\nu}{2^{\nu-1}}, \quad (76)
\end{aligned}$$

proving the claim. $\square$

We continue with a L_q generalized g -fractional Polya type inequality:

Theorem 11. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = [\nu]$, $f \in C^n([a, b])$, $g : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing function. Assume that $g \in C^1([a, b])$ and $g^{-1} \in C^n([g(a), g(b)])$. Also assume $f \circ g^{-1} \in C_{g(a)}^\nu\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)$, and $f \circ g^{-1} \in C_{g(b)-}^\nu\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)$, and that $(f \circ g^{-1})^{(k)}(g(a)) = (f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \dots, n-1$. When $\frac{1}{q} < \nu < 1$, the last boundary conditions are void. Set

$$\begin{aligned}
M_q(f, g) := \max \left\{ \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_q\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)}, \right. \\
\left. \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_q\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)} \right\}. \quad (77)
\end{aligned}$$

Then

$$\begin{aligned} \left| \int_a^b f dg \right| &\leq \int_a^b |f| dg \\ &\leq \frac{M_q(f, g)}{2^{\nu-\frac{1}{q}} \Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (g(b) - g(a))^{\nu+\frac{1}{p}}. \end{aligned} \quad (78)$$

Again here $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree n .

Proof. Let $f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)$, $\nu > 0$, and $(f \circ g^{-1})^{(k)}(g(a)) = 0$, for $k = 0, 1, \dots, n-1$, $n = [\nu]$. By Theorem 3 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (79)$$

all $x \in [a, x_0]$, where $x_0 := g^{-1} \left(\frac{g(a)+g(b)}{2} \right)$.

Let $f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)$, and $(f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \dots, n-1$. By Theorem 4 we have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (80)$$

all $x \in [x_0, b]$.

By (79) we have

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left| \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{1}{\Gamma(\nu)} \left(\int_{g(a)}^{g(x)} (g(x) - t)^{p(\nu-1)} dt \right)^{\frac{1}{p}} \left(\int_{g(a)}^{g(x)} \left| \left(D_{g(a)}^\nu (f \circ g^{-1}) \right)(t) \right|^q dt \right)^{\frac{1}{q}} \\ &\leq \frac{1}{\Gamma(\nu)} \frac{(g(x) - g(a))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_q \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \end{aligned} \quad (81)$$

for all $x \in [a, x_0]$.

Similarly, by (80) we get

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left| \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) \right| dt \leq \\ &\frac{1}{\Gamma(\nu)} \left(\int_{g(x)}^{g(b)} (t - g(x))^{p(\nu-1)} dt \right)^{\frac{1}{p}} \left(\int_{g(x)}^{g(b)} \left| \left(D_{g(b)-}^\nu (f \circ g^{-1}) \right)(t) \right|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \frac{1}{\Gamma(\nu)} \frac{(g(b) - g(x))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{g(b)-}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)}, \quad (82)$$

for all $x \in [x_0, b]$.

Thus, we have

$$\begin{aligned} & \left| \int_a^b f dg \right| \leq \int_a^b |f| dg = \\ & \int_a^{x_0} |f(x)| g'(x) dx + \int_{x_0}^b |f(x)| g'(x) dx \stackrel{(\text{by (81), (82)})}{\leq} \frac{1}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}} \\ & \left[\left\| D_{g(a)}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)} \int_a^{x_0} (g(x) - g(a))^{\nu-1+\frac{1}{p}} g'(x) dx \right. \\ & \left. + \left\| D_{g(b)-}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)} \int_{x_0}^b (g(b) - g(x))^{\nu-1+\frac{1}{p}} g'(x) dx \right] \quad (83) \\ & = \frac{1}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} \\ & \times \left[\left\| D_{g(a)}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)} (g(x_0) - g(a))^{\nu+\frac{1}{p}} \right. \\ & \left. + \left\| D_{g(b)-}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)} (g(b) - g(x_0))^{\nu+\frac{1}{p}} \right] \\ & = \frac{1}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} \frac{(g(b) - g(a))^{\nu+\frac{1}{p}}}{2^{\nu+\frac{1}{p}}} \times \\ & \left[\left\| D_{g(a)}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[g(a), \frac{g(a)+g(b)}{2}\right]\right)} + \left\| D_{g(b)-}^{\nu} (f \circ g^{-1}) \right\|_{L_q\left(\left[\frac{g(a)+g(b)}{2}, g(b)\right]\right)} \right] \\ & \leq \frac{M_q(f, g)}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} \frac{(g(b) - g(a))^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \quad (84) \end{aligned}$$

proving the claim. $\square$

Combining facts from Theorems 7, 10, 11 and Remark 8, we get:

Theorem 12. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu \geq 1$, $n = [\nu]$, $f \in C^n([a, b])$, $g : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing function. Assume that $g \in C^1([a, b])$ and $g^{-1} \in C^n([g(a), g(b)])$. Also assume $f \circ g^{-1} \in C_{g(a)}^\nu \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)$, and $f \circ g^{-1} \in C_{g(b)-}^\nu \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)$, and that

$$(f \circ g^{-1})^{(k)}(g(a)) = (f \circ g^{-1})^{(k)}(g(b)) = 0,$$

for $k = 0, 1, \dots, n-1$. Then

$$\begin{aligned} & \left| \int_a^b f dg \right| \leq \int_a^b |f| dg \\ & \leq \min \left\{ M(f, g) \frac{(g(b) - g(a))^{\nu+1}}{\Gamma(\nu+2) 2^\nu}, M_1(f, g) \frac{(g(b) - g(a))^\nu}{\Gamma(\nu+1) 2^{\nu-1}}, \right. \\ & \quad \left. \frac{M_q(f, g)}{2^{\nu-\frac{1}{q}} \Gamma(\nu) (p(\nu-1) + 1)^{\frac{1}{p}} \left(\nu + \frac{1}{p} \right)} (g(b) - g(a))^{\nu+\frac{1}{p}} \right\}, \end{aligned} \quad (85)$$

where $M(f, g)$ as in (36), $M_1(f, g)$ as in (69) and $M_q(f, g)$ as in (77).

Above $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree n .

Corollary 13. Here all as in Theorem 12. Then

$$\begin{aligned} & \left| \frac{1}{g(b) - g(a)} \int_a^b f dg \right| \leq \frac{1}{g(b) - g(a)} \int_a^b |f| dg \\ & \leq \min \left\{ M(f, g) \frac{(g(b) - g(a))^\nu}{\Gamma(\nu+2) 2^\nu}, M_1(f, g) \frac{(g(b) - g(a))^{\nu-1}}{\Gamma(\nu+1) 2^{\nu-1}}, \right. \\ & \quad \left. \frac{M_q(f, g)}{2^{\nu-\frac{1}{q}} \Gamma(\nu) (p(\nu-1) + 1)^{\frac{1}{p}} \left(\nu + \frac{1}{p} \right)} (g(b) - g(a))^{\nu-\frac{1}{q}} \right\}. \end{aligned} \quad (86)$$

We continue with an L_1 iterated fractional Polya type inequality:

Theorem 14. All as in Theorem 9, with $\frac{1}{m+1} \leq \nu < 1$. Set

$$M_1^*(f, g) := \max \left\{ \left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_1 \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \right.$$

$$\left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_1 \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)} \Bigg\}. \quad (87)$$

Then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dg \leq M_1^* (f, g) \frac{(g(b) - g(a))^{(m+1)\nu}}{\Gamma((m+1)\nu + 1) 2^{(m+1)\nu-1}}. \quad (88)$$

Proof. Similar to Theorem 10. $\square$

We continue with an L_p iterated fractional Polya type inequality:

Theorem 15. All as in Theorem 9. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, such that $\frac{1}{(m+1)q} < \nu < 1$. Set

$$M_q^* (f, g) := \max \left\{ \left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_q \left(\left[g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \right. \\ \left. \left\| D_{g(b)-}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_q \left(\left[\frac{g(a)+g(b)}{2}, g(b) \right] \right)} \right\}. \quad (89)$$

Then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dg \leq \\ \frac{M_q^* (f, g) (g(b) - g(a))^{(m+1)\nu + \frac{1}{p}}}{2^{(m+1)\nu - \frac{1}{q}} \Gamma((m+1)\nu) (p((m+1)\nu - 1) + 1)^{\frac{1}{p}} \left((m+1)\nu + \frac{1}{p} \right)}. \quad (90)$$

Proof. Similar to Theorem 11. $\square$

Applications follow:

Proposition 16. Let $0 < \nu < 1$, $f \in C([a, b])$. Assume $f \circ \ln x \in C_{e^a}^\nu \left(\left[e^a, \frac{e^a+e^b}{2} \right] \right)$, and $f \circ \ln x \in C_{e^b-}^\nu \left(\left[\frac{e^a+e^b}{2}, e^b \right] \right)$. Set

$$M(f, e^x) := \max \left\{ \|D_{e^a}^\nu (f \circ \ln)\|_{\infty, [e^a, \frac{e^a+e^b}{2}]} , \right. \\ \left. \|D_{e^b-}^\nu (f \circ \ln)\|_{\infty, [\frac{e^a+e^b}{2}, e^b]} \right\}. \quad (91)$$

Then

$$\left| \int_a^b f(x) e^x dx \right| \leq \int_a^b |f(x)| e^x dx \leq M(f, e^x) \frac{(e^b - e^a)^{\nu+1}}{\Gamma(\nu + 2) 2^\nu}. \quad (92)$$

Inequality (92) is sharp, namely it is attained by f_* such that

$$(f_* \circ \ln x)(z) = \begin{cases} (z - e^a)^\nu, & z \in \left[e^a, \frac{e^a + e^b}{2} \right] \\ (e^b - z)^\nu, & z \in \left[\frac{e^a + e^b}{2}, e^b \right] \end{cases}, \quad 0 < \nu < 1. \quad (93)$$

Clearly here non zero constant functions $f \circ \ln x$ are excluded.

Proof. By Theorem 7. □

We continue with

Proposition 17. Here $[a, b] \subset (0, +\infty)$, $0 < \nu < 1$, $f \in C([a, b])$. Assume that $(D_{\ln a}^{i\nu}(f \circ e^x)) \in C_{\ln a}^\nu\left(\left[\ln a, \frac{\ln(ab)}{2}\right]\right)$, for $i = 0, 1, \dots, m \in \mathbb{N}$, and $(D_{\ln a}^{(m+1)\nu}(f \circ e^x)) \in C\left(\left[\ln a, \left(\frac{\ln(ab)}{2}\right)\right]\right)$. Also assume that $(D_{\ln b-}^{i\nu}(f \circ e^x)) \in C_{\ln b-}^\nu\left(\left[\frac{\ln(ab)}{2}, \ln b\right]\right)$, for $i = 0, 1, \dots, m \in \mathbb{N}$, and $(D_{\ln b-}^{(m+1)\nu}(f \circ e^x)) \in C\left(\left[\frac{\ln(ab)}{2}, \ln b\right]\right)$. Set

$$M^*(f, \ln x) := \max \left\{ \left\| D_{\ln a}^{(m+1)\nu}(f \circ e^x) \right\|_{\infty, \left[\ln a, \frac{\ln(ab)}{2}\right]}, \left\| D_{\ln b-}^{(m+1)\nu}(f \circ e^x) \right\|_{\infty, \left[\frac{\ln(ab)}{2}, \ln b\right]} \right\}. \quad (94)$$

Then

$$\left| \int_a^b \frac{f(x)}{x} dx \right| \leq \int_a^b \frac{|f(x)|}{x} dx \leq M^*(f, \ln x) \frac{\left(\ln \frac{b}{a}\right)^{(m+1)\nu+1}}{\Gamma((m+1)\nu+2) 2^{(m+1)\nu}}. \quad (95)$$

Proof. By Theorem 9. □

We can have many other interesting applications but we stop here.

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