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# PROGRAMMING VARIATIONAL ITERATION METHOD VIA WOLFRAM-MATHEMATICA FOR SOLVING MULTI-ORDER DIFFERENTIAL EQUATIONS

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**Abstract:** In this study, we have studied the multi-order differential equations. The model we have followed agrees with initial value problem which, in its turn, has a group of linear ordinary differential equations. This paper's aim is programming a Variational Iteration Method (VIM) using Wolfram's *Mathematica*. Variational Iteration Method offers a study that introduce approximated solutions of the multi-order ordinary differential equations. Several examples of different order have been resolved.

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**Key Words:** Variational Iteration Method (VIM); multi-Order differential equations; approximated solution

## 1. Introduction

The beauty of mathematics lies in the diversity of its fields and in its close association with other sciences. From this diversity, differential equations are

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significantly prevailing and widely used. They are commonly adopted as a main use in mathematical models in order to describe existent events and situations. Such uses are best seen through out their applications which are enforced in the associated sciences. In other words, there are mathematical language can be used to turn laws and hypotheses into equations which involve derivatives. To illustrate, we can see how the derivatives appear in different forms with in different field of research: in physics, they appear in the form of velocities and accelerations, in geometry as slopes, in biology as rates of growth of populations, in psychology as rates of learning, in chemistry as reaction rates, in economics as rates of change of the cost of living, and in finance as rates of growth of investments. one part of this important equations is multi-order differential equations. Many researchers have studied this kind of differential equations by using different ways and methods, such as: generalized sine-cosine wavelets [13], Boubaker polynomials [1], homotopy analysis method [3], Legendre pseudo-spectral method [14], Chebyshev operational matrix method [4], and wavelet collocation method [5], Other related studies are on: modified HPM for solving generalized linear complex differential equations [9], solving fractional differential equations using Haar wavelet techniques [11], studies on the mathematical model of HIV infection of $CD_4^+T$ by using HPM and VIM [2], studies of general second-order partial differential equations using HPM [6], of general first-order partial differential equations using HPM [7], and solving generalized Riccati differential equation [8]. One of the most popular methods for solving equations is variational iteration (VIM), and in the present work we use this method for solving a class of multi-order ordinary differential equation.

The organization of this paper is as follows: in Section 2 we introduce the analysis (VIM) and apply it for a non linear multi-order differential equations in Section 3. Section 4 gives an algorithm of VIM, some examples are included in Section 5, followed by concluding remarks in Section 6.

## 2. Analysis of Variational Iteration Method

The analysis of Variational Iteration Method (VIM) is introduced in this section. VIM functions are giving an approximate solution to differential equation. These functions are based on the use of iteration formula in the correctional functional which, in turn, includes Lagrange multiplier. The latter is determined by the implementation of the variational theory. For illustration of the primary concept of this method, see [2].

Consider the following differential equations:

$$Au(t) + g(t) = 0, \tag{1}$$

with a differential operator $A$ acting on a function $u$, which in its turn is defined on interval $L \subseteq R$. The function $g(t)$ is defined in $L$, and it segments $A$ into its parts: linear $N$ and non linear $L$,

$$Au(t) + g(t) = Lu(t) + Nu(t) + g(t) = 0. \tag{2}$$

In equation (2), the correctional functional is then defined repeatedly as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,s)(Lu_n(s) + Nu_n(s) + g(s))ds, \tag{3}$$

when Lagrange multiplier is $\lambda(t,s)$, the approximate solution $u_n$ is the $n^{th}$, and the restricted variations are $u_n$, so that $u_n = 0$. Using the basic function $u_0$, the iterations are repeatedly performed to finally reach a converged fixed point, when fulfilling the condition

$$u_{n+1}(t) = u_n(t).$$

Then we get

$$\int_0^t \lambda(t,s)(Lu_n(s) + Nu_n(s) + g(s))ds = 0 \tag{4}$$

which is duplicate to the condition

$$Au(t) + g(t) = Lu(t) + Nu(t) + g(t) = 0. \tag{5}$$

That means $u_{n+1}(t)$ is taken as the approximate solution for the equation

$$\delta y_{n+1}(x) \quad = \quad \delta y_n(x) + \int_0^x \lambda\delta(Lu(t) + Nu(t) + g(t)). \tag{6}$$

Operating initially with $u_0$ function, the iterations are performed until converge into a fixed point. When the condition is reached, we get

$$u_{n+1}(t) = u_n(t) + \int_0^t (Lu_n(s) + Nu_n(s) + g(s))ds, \tag{7}$$

which is duplicate to the condition

$$Au(t) + g(t) = Lu(t) + Nu(t) + g(t) = 0. \tag{8}$$

That means $u_{n+1}(t)$ is taken as the approximate solution for the equation (2).

## 3. Nonlinear multi-order differential equations

Consider the following multi-order of ordinary differential equation:

$$D^n u(t) \quad = \quad \sum_{k=1}^{n-1} a_k(u(t),t) D^k u(t) + a_n(u(t),t) + f(t), \tag{9}$$

with initial conditions

$$u^{(i)}(t) = b_i, \quad i = 0, 1, \ldots, n-1, \tag{10}$$

where $b_i$ for $i = 0, 1, \ldots, n-1$, are arbitrary constants.

The objective of this paper is to study the approximated solutions of the nonlinear multi-order (ODEs) in equation (9) according the following algorithm.

## 4. Algorithm for optimization VIM

In this section, we present the algorithm of the new modification of VIM. To better clarify the idea of VIM, we follow the algorithm below:

Start

1. Input: $m$
2. Output: $y(i+1)$
3. Globally define $i$
4. Define $t$ ,$u$
5. $\lambda \longleftarrow$ input value of lambda
6. y(0) $\longleftarrow$ initial approximation
7. for $i \longleftarrow 1 : m$
8. $f \longleftarrow$ input differential equation(9)
9. $y(i+1) \longleftarrow y(0)+$ integrate $(f)$
10. Show $y(i+1)$

11. End for

    End

Here $y(i+1) =$ is the approximate solution, $y(0) =$ initial value, $m =$ number of loops, $f =$ main function , $\lambda =$ Lagrange multiplier, $(u,t) =$ equation variable. In the optimization, the system basically focuses on solving variation iteration linear and non-linear method. The conceptual modelling of the proposed algorithm, which performs optimization to maintain the perceptual faster, provides more accuracy and less time to solve mathematical equation.

We can implement the algorithm for solving any example used in this paper. At first call the input numbering to "for-looping", and then the initial value and value of lambda. Of-course, we need to solve and process the equation of linear and non-linear VIM, so we use "for-loop" with optimization equation to process the equation and then as an alternative, the algorithm is used to give solutions to the examples in this paper.

## 5. Implementation

**Example 1.** Consider the following ODE: ([9])

$$y''(t) - y(t) + t = 0, \tag{11}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 1, \tag{12}$$

with exact solution

$$y(t) = ae^{t} + be^{-t} + t. \tag{13}$$

An alternative solution is provided according to the algorithm which uses VIM, as follows:

$$y(t) = 2 + 3t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 + \frac{1}{360}t^6 + \frac{1}{2520}t^7 \tag{14}$$

$$+ \frac{1}{20160}t^8 + \frac{1}{181440}t^9 + \frac{1}{1814400}t^{10} + \dots \tag{15}$$

**Example 2.** Consider the following ODE: ([9])

$$y'''(t) + 3y''(t) - 4y(t) = 0, \tag{16}$$

subject to the initial conditions

$$y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 1, \tag{17}$$

with exact solution

$$y(t) = ae^t + be^{-2t} + cte^{-2t}. \tag{18}$$

An alternative solution is provided according to the algorithm which uses VIM, as follows:

$$\begin{aligned} y(t) &= 2 + \frac{1}{2}t^2 + \frac{5}{6}t^3 - \frac{5}{8}t^4 + \frac{49}{120}t^5 - \frac{127}{720}t^6 \\ &+ \frac{107}{1680}t^7 - \frac{767}{40320}t^8 + \ldots \end{aligned} \tag{19}$$

**Example 3.** Consider the following ODE:

$$y(t)^{(4)} + 4y = 0, \tag{20}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 0 \text{ and } y'''(0) = \frac{-1}{3}, \tag{21}$$

with exact solution

$$y(t) = ae^t \cos(t). \tag{22}$$

An alternative solution is provided according to the algorithm which uses VIM and as follows:

$$\begin{aligned} y(t) &= 1 + t - \frac{1}{3}t^3 - \frac{1}{6}t^4 - \frac{1}{30}t^5 + \frac{1}{630}t^7 + \frac{1}{2520}t^8 \\ &+ \frac{1}{22680}t^9 + \ldots \end{aligned} \tag{23}$$

**Example 4.** Consider the following Van der Pol oscillator equation: ([12])

$$y''(t) + \mu(y^2(t) - 1)y'(t) + y(t) = 0, \tag{24}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 1, \quad \& \quad \mu = 1. \tag{25}$$

An alternative solution is provided according to the algorithm which uses VIM and as follows:

$$\begin{aligned} y(t) \;&=\; 1 + t - \frac{1}{2}t^2 - \frac{1}{2}t^3 + \frac{5\,t^4}{24} + \frac{11\,t^5}{40} - \frac{97\,t^6}{720} - \frac{269\,t^7}{1008} - \frac{2047\,t^8}{40320} \\ &+\; \frac{1883\,t^9}{17280} + \frac{1697\,t^{10}}{18900} + \frac{100549\,t^{11}}{3326400} + \frac{347\,t^{12}}{72576} + \frac{17\,t^{13}}{58968} + ... \end{aligned} \tag{26}$$

For comparison, when the last example is solved by the Homotopy Perturbation Method (HPM), we will find that the approximate solution is a nearly exactly symmetric when $\mu = 1$,[10].

## 6. Discussion and Conclusion

In this study, VIM was programmed via *Mathematica* program to solve linear and nonlinear multi-order differential equations. It has been possible to arrive at approximate solutions for this category of differential equations. It is also found that these solutions, obtained by VIM, are close to analytical solutions. Moreover, these solutions are effective and have high-precision. They are consistent with what was presented in the examples. Finally, we concluded that the proposed algorithm for VIM is highly accurate and efficient to find solutions.

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