**A VARIATIONAL APPROACH TO IMPULSIVE
STURM-LIOUVILLE DIFFERENTIAL EQUATIONS
WITH NONLINEAR DERIVATIVE DEPENDENCE**

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Abstract: We study the existence and multiplicity of solutions for a class of impulsive Sturm-Liouville differential equations with nonlinear derivative dependence. By applying a critical point theory, we give some criteria to guarantee that our impulsive problem has at least three solutions under rather different assumptions and an exact interval of parameter λ .

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1. Introduction

The aim of this paper is to study the following impulsive problem:

$$\begin{cases} -(\phi_p(u'))' = \lambda f(t, u)h(u'), & t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta J(u'(t_j)) = \lambda I_j(u(t_j)), & j = 1, 2, \dots, n, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(T) + \sigma u'(T) = 0, \end{cases} \quad (1)$$

where $p > 1$, $\phi_p(s) = |s|^{p-2}s$, $\alpha, \gamma, \beta, \sigma > 0$, $T > 0$, t_j , $j = 1, 2, \dots, n$,

are instants in which the impulses occur and $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = T$, $J(s) = \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta$, $\Delta J(u'(t_j)) = J(u'(t_j^+)) - J(u'(t_j^-))$, $u'(t_j^+) = \lim_{t \rightarrow t_j^+} u'(t)$, $u'(t_j^-) = \lim_{t \rightarrow t_j^-} u'(t)$, $I_j : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function for $j = 1, 2, \dots, n$, $f : [0, T] \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function, $h : \mathbf{R} \rightarrow [0, +\infty)$ is a bounded and continuous function with $\inf_{x \in \mathbf{R}} h(x) > 0$ and $\lambda > 0$ is a parameter.

Many biological, chemical and physical processes have been described using models based on the Sturm-Liouville equations. For the background of this equation, we refer the reader to [5], [10], [23]. In [23] Tian and Ge based on variational methods have discussed the positive solutions of a second order Sturm-Liouville boundary value problem with a p -Laplacian. Bonanno and D'Aguì in [5] have investigated the existence of infinitely many solutions to a Neumann boundary value problem for the Sturm-Liouville equation by applying multiple critical points theorems. In [10] author obtained the existence of infinitely many solutions to Sturm-Liouville problems by using critical point theory.

The existence of solutions for differential equations with impulsive conditions is studied recently by topological and variational methods (see [19], [19], [18], [2], [3], [7], [8], [1], [11]).

There have been many approaches to investigate the existence of solutions for impulsive Sturm-Liouville differential equations (see [26], [23], [24], [25], [16], [21], [9]). In [26], Zhang and Ge have ensured the existence of at least three solutions for a Sturm-Liouville boundary value problem with impulses by using critical point theorem due to Ricceri [20]. Tian and Ge in [23], [24], [25] have studied the existence of multiple solutions for some types of Sturm-Liouville impulsive problem via critical point theory, variational methods and lower and upper solutions, respectively. In [21], the authors based on variational method have obtained the existence criteria of single and multiple solutions for Sturm-Liouville boundary value problem for a class of second-order impulsive differential equations, under different assumptions. In particular, in [9] based on variational and critical point theory the existence of nontrivial solutions for the problem (1) has been investigated.

In this paper, motivated by [6], [9], we are interested to discuss the existence of at least three distinct solutions for the problem (1) under appropriate assumptions on the nonlinear terms, and we give the exact collocations for the parameter λ . In particular, f has no behavior at infinity and at zero, namely, there is a growth of the antiderivative of f is either bigger than quadratic in a suitable interval or less than quadratic in the following suitable interval. For

example see both of hypotheses (b_1) and (b_2) of Theorem 3.2, respectively.

Let

$$\Theta(\theta) = \theta \left(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}} \right)$$

for any $\theta > 0$.

We state here an application of our main result to the considered problem.

Theorem 1.1. *Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous and nonzero function and $F(\xi) = \int_0^\xi f(s)ds$ for every $\xi \in [0, +\infty)$. Suppose that*

$$\lim_{\xi \rightarrow 0^+} \frac{F(\Theta(\xi))}{\xi^p} = \lim_{\xi \rightarrow +\infty} \frac{F(\Theta(\xi))}{\xi^p} = 0 \quad (2)$$

and

$$\lim_{\xi \rightarrow 0^+} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s)ds}{\xi^p} = \lim_{\xi \rightarrow +\infty} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s)ds}{\xi^p} = 0. \quad (3)$$

Then, for each $\lambda \in]0, \lambda^*[$, where

$$\begin{aligned} \lambda^* := & \inf \left\{ \left(\frac{1}{pm} + \frac{1}{pM(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)} \right) \right. \\ & \times \frac{\nu^p(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{\int_0^T [F(\nu t) - f(0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]} : \\ & \nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p} > 0, \left(\int_0^T [F(\nu t) - f(0)\nu t]dt \right. \\ & \left. \left. + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j] \right) > 0 \right\}, \end{aligned}$$

the problem (1) admits at least two nontrivial solutions.

The rest of this paper has been organized as follows: In Section 2 we have presented some preliminary results. Our main results and their proofs have been given in Section 3.

2. Preliminaries

Let X be a nonempty set and $\Phi, \Psi : X \rightarrow \mathbf{R}$ be two functions. For all $r, r_1, r_2 > \inf_X \Phi, r_2 > r_1, r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(-\infty, r_1)} \sup_{v \in \Phi^{-1}[r_1, r_2]} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Our main tool in order to prove the existence results is Theorem 3.3 of [4]. We recall it as follows.

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \rightarrow \mathbf{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \rightarrow \mathbf{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$(a_1) \quad \inf_X \Phi = \Phi(0) = \Psi(0) = 0;$$

$$(a_2) \quad \text{for every } u_1, u_2 \in X \text{ such that } \Psi(u_1) \geq 0 \text{ and } \Psi(u_2) \geq 0, \text{ one has}$$

$$\inf_{s \in [0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

$$(a_3) \quad \varphi(r_1) < \beta(r_1, r_2);$$

$$(a_4) \quad \varphi(r_2) < \beta(r_1, r_2);$$

$$(a_5) \quad \gamma(r_2, r_3) < \beta(r_1, r_2).$$

Then, for each $\lambda \in]\frac{1}{\beta(r_1, r_2)}, \frac{1}{\alpha(r_1, r_2, r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}([r_1, r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

We refer to [6], [17] in which Theorem 2.1 has been employed to ensure the existence of at least three solutions for some boundary value problems.

Let $X = W^{1,p}([0, T])$ be the Sobolev space equipped with the following norm

$$\|u\| := \left(\int_0^T (|u'(t)|^p + |u(t)|^p) dt \right)^{1/p}.$$

It is well known that X is a reflexive real Banach space (see [12]).

Let

$$F(t, x) = \int_0^x f(t, \xi) d\xi \text{ for every } (t, x) \in [0, T] \times [0, +\infty),$$

$$\|v\|_{L^p} = \left(\int_0^T |v(t)|^p dt \right)^{1/p}, \text{ for } v \in L^p([0, T]),$$

and q be the conjugate of p , i.e., $1/p + 1/q = 1$. Define

$$m = \inf_{x \in \mathbf{R}} h(x), \quad M = \sup_{x \in \mathbf{R}} h(x),$$

so we have $M \geq m > 0$.

The functionals $\Phi, \Psi : X \rightarrow \mathbf{R}$ are defined by

$$\begin{aligned} \Phi(u) = \int_0^T \left(\int_0^{u'(t)} J(s) ds \right) dt + \frac{\beta}{\alpha} \int_0^{\frac{\alpha u(0)}{\beta}} J(s) ds \\ + \frac{\sigma}{\gamma} \int_0^{-\frac{\gamma u(T)}{\sigma}} J(s) ds \end{aligned} \quad (4)$$

and

$$\begin{aligned} \Psi(u) = \int_0^T [F(t, u^+(t)) - f(t, 0)u^-(t)] dt \\ + \sum_{j=1}^n \left[\int_0^{u^+(t_j)} I_j(s) ds - I_j(0)u^-(t_j) \right] \end{aligned} \quad (5)$$

for each $u \in X$.

We can obtain that

$$\begin{aligned} \frac{1}{M} \frac{1}{p} \left(\|u'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}} |u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}} |u(T)|^p \right) \\ \leq \Phi(u) \leq \frac{1}{m} \frac{1}{p} \left(\|u'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}} |u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}} |u(T)|^p \right). \end{aligned} \quad (6)$$

Definition 2.2. We say that for any $v \in X$ a function $u \in X$ is a weak solution of problem (1) if

$$\int_0^T J(u'(t))v'(t)dt + J\left(\frac{\alpha u(0)}{\beta}\right)v(0) - J\left(\frac{-\gamma u(T)}{\sigma}\right)v(T) - \int_0^T f(t, u^+(t))v(t)dt - \sum_{j=1}^n I_i(u^+(t_j))v(t_j) = 0.$$

Lemma 2.1. (see [14], [15]) *Let the functionals $\Phi, \Psi : X \rightarrow \mathbf{R}$ be defined by (4) and (5). Then,*

- (i) Φ is sequentially weakly lower semicontinuous, continuous, $\lim_{\|u\| \rightarrow \infty} \Phi(u) = \infty$, and its derivative at the point $u \in X$ is the functional $\Phi'(u)$ given by

$$\begin{aligned} \Phi'(u)(v) = & \int_0^T J(u'(t))v'(t)dt + J\left(\frac{\alpha u(0)}{\beta}\right)v(0) \\ & - J\left(\frac{-\gamma u(T)}{\sigma}\right)v(T) \end{aligned}$$

for every $v \in X$. Furthermore, $\Phi' : X \rightarrow X^*$ admits a continuous inverse on X^* .

- (ii) Ψ is sequentially weakly upper semicontinuous and its derivative at the point $u \in X$ is the functional $\Psi'(u)$ given by

$$\Psi'(u)(v) = \int_0^T f(t, u^+(t))v(t)dt + \sum_{j=1}^n I_i(u^+(t_j))v(t_j)$$

for every $v \in X$. Furthermore, $\Psi' : X \rightarrow X^*$ is compact.

Lemma 2.2. (see [14, Lemma 2.3]) *Suppose that, for $u \in X$, there exists $r > 0$ such that $\Phi(u) \leq r$. Then, we get*

$$\max_{t \in [0, T]} |u(t)| \leq \sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}} Mpr} + \sqrt[q]{Mpr} T^{\frac{1}{q}}. \quad (7)$$

Lemma 2.3. (see [26, Lemma 3.1]) *The function $u(t) \in X$ is a solution of the problem (1) if and only if $u(t)$ is a critical point of the Euler functional $\Phi - \lambda\Psi$.*

Remark 2.1. (see [22, Lemma 2.2]) If $u \in X$ is a critical point of the functional $\Phi - \lambda\Psi$, then u is a non-negative solution the problem (1). Further, if $f(t, 0) \neq 0$ for all $t \in [0, T]$, the solution u is positive.

3. Main results

The main result of this paper is the following.

Theorem 3.1. Assume that there exist four positive constants c_1, c_2, c_3 and ν such that

$$c_1^p < \nu^p T + \frac{\gamma^{p-1}}{\sigma^{p-1}} (\nu T)^p < \frac{m}{M} c_2^p < \frac{m}{M} c_3^p$$

and

(a₁)

$$\begin{aligned} \max \left\{ \frac{\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds}{c_1^p}, \right. \\ \frac{\int_0^T F(t, \Theta(c_2)) dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s) ds}{c_2^p}, \\ \left. \frac{\int_0^T F(t, \Theta(c_3)) dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s) ds}{c_3^p} \right\} \\ < \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)} H_1, \end{aligned}$$

where

$$\begin{aligned} H_1 = \frac{\int_0^T [F(t, \nu t) - f(t, 0) \nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0) \nu t_j]}{\nu^p} \\ - \frac{\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds}{\nu^p}. \end{aligned}$$

Then, for each

$$\lambda \in \left(\frac{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p}{pm H_1}, \right.$$

$$\min \left\{ \frac{c_1^p}{pM \left(\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds \right)}, \right. \\ \frac{c_2^p}{pM \left(\int_0^T F(t, \Theta(c_2))dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s)ds \right)}, \\ \left. \frac{c_3^p}{pM \left(\int_0^T F(t, \Theta(c_3))dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s)ds \right)} \right\},$$

the problem (1) has at least three solutions u_1 , u_2 and u_3 such that $\|u_i\|_\infty < c_3(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})$, $i = 1, 2, 3$.

Proof. For every $t \in [0, T]$, choose $w(t) = \nu t$, $r_1 := \frac{c_1^p}{pM}$, $r_2 := \frac{c_2^p}{pM}$ and $r_3 := \frac{c_3^p - c_2^p}{pM}$. By the condition

$$c_1^p < \nu^p T + \frac{\gamma^{p-1}}{\sigma^{p-1}}(\nu T)^p < \frac{m}{M} c_2^p < \frac{m}{M} c_3^p,$$

and (6), we can see that $r_1 < \Phi(w) < r_2$. From (7) it follows that

$$\Phi^{-1}(-\infty, r_2) \subseteq \left\{ u \in X; |u(t)| \leq c_2 \left(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}} \right), \right. \\ \left. \text{for all } t \in [0, T] \right\},$$

and also we obtain

$$\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u) \leq \int_0^T F(t, \Theta(c_2))dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s)ds.$$

Therefore, one has

$$\varphi(r_1) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} \\ \leq \frac{\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{\frac{c_1^p}{pM}}.$$

In a similar way, we get

$$\varphi(r_2) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}{r_2}$$

$$\leq \frac{\int_0^T F(t, \Theta(c_2))dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s)ds}{\frac{c_2^p}{pM}}$$

and

$$\begin{aligned} \gamma(r_2, r_3) &= \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2+r_3)} \Psi(u)}{r_3} \\ &\leq \frac{\int_0^T F(t, \Theta(c_3))dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s)ds}{\frac{c_3^p - c_2^p}{pM}}. \end{aligned}$$

On the other hand, one has

$$\begin{aligned} \beta(r_1, r_2) &\geq \frac{\Psi(w) - \Psi(u)}{\Phi(w) - \Phi(u)} \\ &\geq \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\frac{1}{mp}(\nu^p T + \frac{\gamma^{p-1}}{\sigma^{p-1}}(\nu T)^p)} \\ &\quad - \frac{\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{\frac{1}{mp}(\nu^p T + \frac{\gamma^{p-1}}{\sigma^{p-1}}(\nu T)^p)}. \end{aligned}$$

Assumption (a₁) yields $\alpha(r_1, r_2, r_3) < \beta(r_1, r_2)$. Now, according to Theorem 2.1, for every

$$\lambda \in \left(\frac{T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p}{pmH_1}, \right.$$

$$\left. \min \left\{ \frac{c_1^p}{pM \left(\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds \right)}, \right. \right. \\ \frac{c_2^p}{pM \left(\int_0^T F(t, \Theta(c_2))dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s)ds \right)}, \\ \left. \left. \frac{c_3^p}{pM \left(\int_0^T F(t, \Theta(c_3))dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s)ds \right)} \right\} \right),$$

the functional $\Phi - \lambda\Psi$ has three distinct critical points $u_i, i = 1, 2, 3$ in X . Moreover, $\Phi(u_i) < r_2 + r_3$, so from (7) we have $\|u_i\|_\infty < c_3(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})$, which completes the proof. $\square$

Remark 3.1. Suppose in Theorem 3.2 for some $t \in [0, T]$ we have $f(t, 0) \neq 0$, then the mentioned solution is obviously non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(t, 0) = 0$ for all $t \in [0, T]$ requiring the extra condition at zero, that there are a non-empty open set $D \subseteq (0, T)$ and $B \subseteq D$ such that

$$\limsup_{\xi \rightarrow 0^+} \frac{\inf_{t \in B} [F(t, \xi) - f(t, 0)\xi] + \sum_{j=1}^n [\int_0^{\xi_j} I_j(s) ds - I_j(0)\xi_j]}{|\xi|^p} = +\infty$$

and

$$\liminf_{\xi \rightarrow 0^+} \frac{\inf_{t \in D} [F(t, \xi) - f(t, 0)\xi] + \sum_{j=1}^n [\int_0^{\xi_j} I_j(s) ds - I_j(0)\xi_j]}{|\xi|^p} > -\infty.$$

Indeed, let

$$\bar{\lambda} \in \Lambda = \left(\frac{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p}{pmH_1}, L \right),$$

where

$$L = \min \left\{ \frac{c_1^p}{pM \left(\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds \right)}, \frac{c_2^p}{pM \left(\int_0^T F(t, \Theta(c_2)) dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s) ds \right)}, \frac{c_3^p}{pM \left(\int_0^T F(t, \Theta(c_3)) dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s) ds \right)} \right\}.$$

Then, there exists $c > 0$ such that

$$\bar{\lambda} Mp < \frac{c^p}{\int_0^T F(t, \Theta(c)) dt + \sum_{j=1}^n \int_0^{\Theta(c)} I_j(s) ds}.$$

Let Φ and Ψ be as given in (4) and (5), respectively. Due to Theorem 2.1 in [13], for every $\lambda \in (\frac{1}{H_1}, \bar{\lambda})$ there exists a critical point of $I_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda = \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda = \frac{c^p}{pM}$. In particular, u_λ is a global minimum of

the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We assert that the function u_λ cannot be trivial. Let us prove that

$$\limsup_{\|u\| \rightarrow 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \quad (8)$$

According to our assumptions at zero, we can fix a sequence $\xi_n \in \mathbf{R}^+$ converging to zero and two constants ϵ, ρ (with $\epsilon > 0$) such that for every $\xi \in [0, \epsilon]$

$$\lim_{n \rightarrow +\infty} \frac{\inf_{t \in B} [F(t, \xi_n) - f(t, 0)\xi_n] + \sum_{j=1}^n [\int_0^{\xi_{n_j}} I_j(s) ds - I_j(0)\xi_{n_j}]}{|\xi_n|^p} = +\infty,$$

and $\inf_{t \in D} [F(t, \xi) - f(t, 0)\xi] + \frac{1}{\text{meas}(D)} \sum_{j=1}^n [\int_0^{\xi_j} I_j(s) ds - I_j(0)\xi_j] \geq \rho|\xi|^p$. Now, fix a set $C \subset B$. In addition, consider a function $v \in X$ such that

$$(c_1) \quad v(t) \in [0, 1] \text{ for every } t \in [0, T],$$

$$(c_2) \quad v(t) = 1 \text{ for every } t \in C,$$

$$(c_3) \quad v(t) = 0 \text{ for every } t \in (0, T) \setminus D.$$

Finally, fix $R > 0$ and consider a real positive number η with

$$\frac{mp(\text{meas}(C)\eta|\xi_n|^p + \text{meas}(D)\rho|\xi_n v(t)|^p)}{\|\xi_n v'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}} |\xi_n v(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}} |\xi_n v(\text{meas}(D))|^p} > R.$$

Then, there is $n_0 \in N$ such that $\xi_n < \epsilon$ and

$$\inf_{t \in B} [F(t, \xi_n) - f(t, 0)\xi_n] + \frac{1}{\text{meas}(C)} \sum_{j=1}^n [\int_0^{\xi_{n_j}} I_j(s) ds - I_j(0)\xi_{n_j}] \geq \eta|\xi_n|^p$$

for every $n > n_0$. At this point, for every $n > n_0$, and bearing in mind the properties of the function v (that is $0 < \xi_n v(t) < \epsilon$ for n large enough)

$$\begin{aligned} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\int_C [F(t, \xi_n) - f(t, 0)\xi_n] dt + \sum_{j=1}^n [\int_0^{\xi_{n_j}} I_j(s) ds - I_j(0)\xi_{n_j}]}{\Phi(\xi_n v)} \\ &\quad + \frac{\int_{D \setminus C} [F(t, \xi_n v(t)) - f(t, 0)\xi_n v(t)] dt}{\Phi(\xi_n v)} \end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{j=1}^n [\int_0^{\xi_{n_j} v(t)} I_j(s) ds - I_j(0) \xi_{n_j} v(t)]}{\Phi(\xi_n v)} \\
& \geq \frac{mp(meas(C)\eta|\xi_n|^p + meas(D)\rho|\xi_n v(t)|^p)}{||\xi_n v'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|\xi_n v(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|\xi_n v(meas(D))|^p} > R.
\end{aligned}$$

Since R can be selected arbitrarily large, (8) is achieved. Hence, there exists a sequence $w_n \subset X$ converging to zero such that, for n large enough $w_n \in \Phi^{-1}(-\infty, r_\lambda)$ and $I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0$. Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$, we conclude that

$$I_\lambda(u_\lambda) < 0, \quad (9)$$

so that u_λ is not trivial.

Remark 3.2. For non-trivial u_λ we can show that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the map

$$\lambda \rightarrow I_\lambda(u_\lambda)$$

is strictly decreasing in Λ . Indeed, bearing in mind that Φ is coercive and for every $\lambda \in \Lambda$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$, one has that there exists a positive constant N such that $\|u_\lambda\| \leq N$ for every $\lambda \in \Lambda$. Then, there exists positive constant M such that for every $\lambda \in \Lambda$,

$$\left| \int_0^T f(t, u_\lambda^+(t)) u_\lambda(t) dt + \sum_{j=1}^n I_i(u_\lambda^+(t_j)) u_\lambda(t_j) \right| \leq M \|u\| \leq MN. \quad (10)$$

Since u_λ is a critical point of I_λ , we have $I'_\lambda(u_\lambda)(v) < 0$ for every $\lambda \in \Lambda$ and every $v \in X$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$ for every $\lambda \in \Lambda$. Then, it follows

$$\begin{aligned}
0 & \leq \int_0^T J(u'_\lambda(t)) u'_\lambda(t) dt + J\left(\frac{\alpha u_\lambda(0)}{\beta}\right) u_\lambda(0) - J\left(\frac{-\gamma u_\lambda(T)}{\sigma}\right) u_\lambda(T) \\
& = \Phi'(u_\lambda)(u_\lambda) = \lambda \Psi'(u_\lambda)(u_\lambda) \\
& = \lambda \left(\int_0^T f(t, u_\lambda^+(t)) u_\lambda(t) dt + \sum_{j=1}^n I_i(u_\lambda^+(t_j)) u_\lambda(t_j) \right)
\end{aligned}$$

for every $\lambda \in \Lambda$. Letting $\lambda \rightarrow 0^+$, by (10), we obtain

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Then, we have the desired conclusion. Finally, we have to prove that the map

$$\lambda \rightarrow I_\lambda(u_\lambda)$$

is strictly decreasing in Λ . We see that for any $v \in X$, one has

$$I_\lambda(u_\lambda) = \lambda \left(\frac{\Phi(u_\lambda)}{\lambda} - \Psi(u_\lambda) \right) \quad (11)$$

Now, fix $0 < \lambda_1 < \lambda_2 < L$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}(-\infty, r_{\lambda_i})$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) = \inf_{v \in \Phi^{-1}(-\infty, r_{\lambda_i})} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for every $i = 1, 2$. By (9) we observe that the map

$$\Lambda \ni \lambda \rightarrow I_\lambda(u_\lambda) \quad (12)$$

where

$$\begin{aligned} I(u_\lambda) &= \int_0^T \left(\int_0^{u'_\lambda(t)} J(s) ds \right) dt + \frac{\beta}{\alpha} \int_0^{\frac{\alpha u_\lambda(0)}{\beta}} J(s) ds \\ &+ \frac{\sigma}{\gamma} \int_0^{-\frac{\gamma u_\lambda(T)}{\sigma}} J(s) ds - \lambda \left(\int_0^T [F(t, u_\lambda^+(t)) - f(t, 0)u_\lambda^-(t)] dt \right. \\ &\quad \left. + \sum_{j=1}^n \left[\int_0^{u_\lambda^+(t_j)} I_j(s) ds - I_j(0)u_\lambda^-(t_j) \right] \right) \end{aligned}$$

is negative in the open interval Λ . By (12), (11) and the positivity of λ we have

$$m_{\lambda_i} < 0 \quad \text{for } i = 1, 2.$$

Furthermore, due to the fact that $0 < \lambda_1 < \lambda_2$,

$$m_{\lambda_2} < m_{\lambda_1}. \quad (13)$$

Then, by (11), (13) and again by the fact that $0 < \lambda_1 < \lambda_2$, we obtain

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \rightarrow I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \Lambda$.

Theorem 3.2. Assume that there exist three positive constants c_1, c_4 and ν such that

$$c_1^p < \nu^p T + \frac{\gamma^{p-1}}{\sigma^{p-1}} (\nu T)^p < \frac{m}{2M} c_4^p$$

and

$$(b_1) \quad \frac{\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds}{c_1^p} < \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p) + m} H_2,$$

$$(b_2) \quad \frac{\int_0^T F(t, \Theta(c_4)) dt + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s) ds}{c_4^p} < \frac{m}{2(M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p) + m)} H_2,$$

where

$$H_2 := \frac{\int_0^T [F(t, \nu t) - f(t, 0) \nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0) \nu t_j]}{\nu^p}.$$

Then, for each

$$\lambda \in \left(\left(\frac{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p}{pm} + \frac{1}{pM} \right) \frac{1}{H_2}, \right. \\ \left. \min \left\{ \frac{c_1^p}{pM \left(\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds \right)}, \right. \right. \\ \left. \left. \frac{c_4^p}{pM \left(\int_0^T F(t, \Theta(c_4)) dt + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s) ds \right)} \right\} \right),$$

the problem (1) admits at least three solutions $u_1, u_2, u_3 \in X$ such that $\|u_i\|_\infty < c_4 \left(\sqrt[p]{\frac{\beta p - 1}{\alpha p - 1}} + T^{\frac{1}{q}} \right)$, $i = 1, 2, 3$.

Proof. The conclusion follows from Theorem 3.1 by picking $c_2 = \frac{1}{\sqrt[p]{2}} c_4$ and $c_3 = c_4$. On the other hand using (b₂) we get

$$\begin{aligned} & \frac{\int_0^T F(t, \Theta(c_2)) dt + \sum_{j=1}^n \int_0^{\Theta(c_2)} I_j(s) ds}{c_2^p} \\ &= \frac{\int_0^T F(t, \Theta(\frac{1}{\sqrt[p]{2}} c_4)) dt + \sum_{j=1}^n \int_0^{\Theta(\frac{1}{\sqrt[p]{2}} c_4)} I_j(s) ds}{\frac{1}{2} c_4^p} \\ &< 2 \frac{\int_0^T F(t, \Theta(c_4)) dt + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s) ds}{c_4^p} \end{aligned}$$

$$\begin{aligned}
&< \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m} \\
&\times \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\int_0^T F(t, \Theta(c_3))dt + \sum_{j=1}^n \int_0^{\Theta(c_3)} I_j(s)ds}{c_3^p} \\
&= \frac{\int_0^T F(t, \Theta(c_4))dt + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s)ds}{c_4^p} \\
&< \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m} \\
&\times \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p} \\
&- \frac{\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{\nu^p} \\
&> \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p} \\
&- \frac{\int_0^T F(t, \Theta(c_1))dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{c_1^p} \\
&> \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p} \\
&- \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m} \\
&\times \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p} \\
&= \frac{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m} \\
&\times \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p}.
\end{aligned}$$

So,

$$\begin{aligned}
 & \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)} \\
 & \times \left(\frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j]}{\nu^p} \right. \\
 & \left. - \frac{\int_0^T F(t, \Theta(c_1)) dt + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds}{\nu^p} \right) \\
 & > \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p) + m} \\
 & \times \frac{\int_0^T [F(t, \nu t) - f(t, 0)\nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j]}{\nu^p}.
 \end{aligned}$$

Hence, using the above inequalities, the hypotheses (a_1) of Theorem 3.1 is fulfilled. $\square$

The example below illustrates one application of Theorem 3.2.

Example 3.1. Put $p = 3$, $T = 1$, $\alpha = \gamma = \beta = \sigma = 1$, $n = 2$, $t_2 = \frac{1}{4}$ and $t_1 = \frac{1}{2}$. Consider the problem

$$\begin{cases}
 -(\phi_3(u'))' = \lambda(t \sin(x))h(u'), & t \neq t_1, t \neq t_2, \text{ a.e. } t \in [0, 1], \\
 \Delta J(u'(\frac{1}{2})) = \frac{34\lambda}{5} \sin(u(\frac{1}{2})) \cos(u(\frac{1}{2})), \\
 \Delta J(u'(\frac{1}{4})) = \lambda \sec(u(\frac{1}{4})) \tan(u(\frac{1}{4})), \\
 u(0) = u'(0), \quad u(1) + u'(1) = 0.
 \end{cases} \quad (14)$$

Let $h(x) = \begin{cases} 1, & x < 0, \\ x + 1, & 0 \leq x \leq 1, \\ 2, & x > 1, \end{cases}$ we obtain that $J(s) = \frac{1}{2}s^2 + \frac{3}{2} - 2 \ln 2$ for all $s \in R$, $m = 1$ and $M = 2$. By choosing, for instance $\nu = c_1 = 1$ and $c_4 = \sqrt[3]{8.7}$, we have

$$\begin{aligned}
 H_2 &= \int_0^1 [F(t, t) - f(t, 0)t] dt + \sum_{j=1}^2 [\int_0^{t_j} I_j(s) ds - I_j(0)t_j] \\
 &= \frac{1}{2} - \cos(1) - \sin(1) + \frac{34}{10} \sin^2(\frac{1}{2}) + \sec(\frac{1}{4}) = 1.4318,
 \end{aligned}$$

$$\begin{aligned}
& \frac{\int_0^1 F(t, \Theta(1))dt + \sum_{j=1}^2 \int_0^{\Theta(1)} I_j(s)ds}{c_1^3} \\
&= -\frac{1}{2} \cos(\Theta(1)) + \frac{34}{10} \sin^2(\Theta(1)) + \sec(\Theta(1)) - \frac{1}{2} = 0.11627 \\
&< \frac{\frac{1}{2} - \cos(1) - \sin(1) + \frac{34}{10} \sin^2(\frac{1}{2}) + \sec(\frac{1}{4})}{5} = \frac{H_2}{5} = 0.28636
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\int_0^1 F(t, \Theta(\sqrt[3]{8.7}))dt + \sum_{j=1}^2 \int_0^{\Theta(\sqrt[3]{8.7})} I_j(s)ds}{c_4^3} = \\
& \frac{-\frac{1}{2} \cos(\Theta(\sqrt[3]{8.7})) + \frac{34}{10} \sin^2(\Theta(\sqrt[3]{8.7})) + \sec(\Theta(\sqrt[3]{8.7})) - \frac{1}{2}}{8.7} \\
&= 0.037639 < \frac{\frac{1}{2} - \cos(1) - \sin(1) + \frac{34}{10} \sin^2(\frac{1}{2}) + \sec(\frac{1}{4})}{10} \\
&= \frac{H_2}{10} = 0.14318.
\end{aligned}$$

So, all the assumptions of Theorem 3.2 are satisfied. Hence, it follows that for each $\lambda \in (0.58202, 1.4334)$, problem (14) has at least three positive solutions u_i ($i = 1, 2, 3$) by taking into account Remark 2.1 and also $\|u_i\|_\infty < 4.1$, $i = 1, 2, 3$.

Remark 3.3. When f does not depend on t , assumptions become the following simpler form

$$\begin{aligned}
(b'_1) \quad & \frac{F(\Theta(c_1))T + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{c_1^p} < \frac{m}{M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m} H, \\
(b'_2) \quad & \frac{F(\Theta(c_4))T + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s)ds}{c_4^p} < \frac{m}{2(M(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p) + m)} H,
\end{aligned}$$

where

$$H := \frac{\int_0^T [F(\nu t) - f(0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]}{\nu^p}$$

and the interval becomes

$$\begin{aligned}
& \lambda \in \left(\frac{1}{pm} + \frac{1}{pM(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)} \right) \\
& \times \frac{\nu^p(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{\int_0^T [F(\nu t) - f(0)\nu t]dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s)ds - I_j(0)\nu t_j]},
\end{aligned}$$

$$\min \left\{ \frac{c_1^p}{pM \left(F(\Theta(c_1))T + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds \right)}, \right. \\ \left. \frac{c_4^p}{pM \left(F(\Theta(c_4))T + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s) ds \right)} \right\}.$$

Now, we present the proof of Theorem 1.1.

Proof. For fixed $\lambda > \lambda^*$, let $\nu > 0$ such that

$$\int_0^T [F(\nu t) - f(0)\nu t] dt + \sum_{j=1}^n \left[\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j \right] > 0$$

and

$$\left(\frac{1}{pm} + \frac{1}{pM(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)} \right) \\ \times \frac{\nu^p(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)}{\int_0^T [F(\nu t) - f(0)\nu t] dt + \sum_{j=1}^n \left[\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j \right]} < \lambda.$$

From (2) and (3) we have $c_1 > 0$ such that $c_1 < \nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p}$ and

$$\frac{F(\Theta(c_1))T + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s) ds}{c_1^p} < \frac{pM}{\lambda},$$

also there is $c_4 > 0$ such that $\nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p} < \sqrt[p]{\frac{m}{2M}} c_4$ and

$$\frac{F(\Theta(c_4))T + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s) ds}{c_4^p} < \frac{pM}{\lambda}.$$

Therefore, Theorem 3.2 and Remark 3.3 ensure the conclusion. $\square$

A consequence of Theorem 1.1 is the following existence result.

Corollary 3.1. *Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous and nonzero function, and denote $sf(s) > 0$ for all $s \neq 0$, also*

$$\lim_{\xi \rightarrow 0^+} \frac{F(\Theta(\xi))}{\xi^p} = \lim_{\xi \rightarrow +\infty} \frac{F(\Theta(\xi))}{\xi^p} = 0 \quad (15)$$

and

$$\lim_{\xi \rightarrow 0^+} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s) ds}{\xi^p} = \lim_{\xi \rightarrow +\infty} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s) ds}{\xi^p} = 0. \quad (16)$$

Then, for each $\lambda > \bar{\lambda}$ where

$$\bar{\lambda} := \left(\frac{(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)}{mp} + \frac{1}{pM} \right)$$

$$\times \max \left\{ \inf_{\nu > 0} \frac{\nu^p}{\int_0^T [F(\nu t) - f(0)\nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j]}, \right. \\ \left. \inf_{\nu < 0} \frac{\nu^p}{\int_0^T [F(\nu t) - f(0)\nu t] dt + \sum_{j=1}^n [\int_0^{\nu t_j} I_j(s) ds - I_j(0)\nu t_j]} \right\},$$

the problem

$$\begin{cases} -(\phi_p(u'))' = \lambda f(u)h(u'), \\ \Delta J(u'(t_j)) = \lambda I_j(u(t_j)), & j = 1, 2, \dots, n, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(T) + \sigma u'(T) = 0, \end{cases}$$

has at least four nontrivial classical solutions.

Proof. Choose

$$f_1(x) := \begin{cases} 0, & \text{if } x < 0, \\ f(x), & \text{if } x \geq 0, \end{cases}$$

and

$$f_2(x) := \begin{cases} 0, & \text{if } x < 0, \\ -f(-x), & \text{if } x \geq 0. \end{cases}$$

Due to Remark 3.3 and applying Theorem 1.1 to f_1 and f_2 , the proof is completed. $\square$

The following example deals with the previous corollary.

Example 3.2. Put $T = 1$, $\alpha = \beta = \gamma = \sigma = 1$, $p = 3$, $n = 1$ and $t_1 = \frac{1}{2}$. Consider the problem

$$\begin{cases} -(\phi_3(u'))' = \lambda f(u)h(u'), \\ \Delta J(u'(\frac{1}{2})) = \lambda I(u(\frac{1}{2})), \\ u(0) = u'(0), \quad u(1) + u'(1) = 0, \end{cases} \quad (17)$$

where $f(x) = \begin{cases} x^3, & 0 \leq x \leq 1, \\ 1, & x > 1, \end{cases}$ and

$$I(s) = \begin{cases} (4)s^3 \cos s^4, & 0 \leq s \leq \frac{\nu}{2}, \\ 0, & s > \frac{\nu}{2}. \end{cases}$$

Let $h(x) = \frac{1}{2+\cos(x)}$ for all $x \in R$, we get $J(s) = 2s + \sin(s)$ for all $s \in R$, $m = \frac{1}{3}$ and $M = 1$. Also,

$$\lim_{\xi \rightarrow 0^+} \frac{F(\Theta(\xi))}{\xi^3} = \lim_{\xi \rightarrow 0^+} \frac{\xi^4}{4\xi^3} = 0,$$

$$\lim_{\xi \rightarrow +\infty} \frac{F(\Theta(\xi))}{\xi^3} = \lim_{\xi \rightarrow +\infty} \frac{1}{\xi^3} = 0,$$

$$\lim_{\xi \rightarrow 0^+} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s) ds}{\xi^3} = \lim_{\xi \rightarrow 0^+} \frac{\sin((\Theta(\xi))^4)}{\xi^3} = \lim_{\xi \rightarrow 0^+} \frac{\sin(16\xi^4)}{\xi^3} = 0$$

and

$$\lim_{\xi \rightarrow +\infty} \frac{\sum_{j=1}^n \int_0^{\Theta(\xi)} I_j(s) ds}{\xi^3} = \lim_{\xi \rightarrow +\infty} \frac{0}{\xi^3} = 0.$$

Since all the assumptions of Corollary 3.1 are satisfied, it follows that for every $\lambda > \frac{7}{3} \inf_{\nu > 0} \frac{4\nu^3}{2\nu-3+\sin(\frac{\nu}{2})^4}$ the problem (17) has at least four nontrivial solutions.

In our main result actually no asymptotic condition on f is requested. In the following we observe a special case of Theorem 3.2, in which the problem (1) for $\lambda = 1$ is investigated.

Theorem 3.3. *Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function and denote $F(\xi) = \int_0^\xi f(s) ds$ for every $\xi \in [0, +\infty)$. Assume there exist three positive constants c_1 , ν and c_4 with*

$$c_1 < \nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p} < \sqrt[p]{\frac{m}{2M}} c_4$$

such that

$$f(s)T + \sum_{j=1}^n I_j(s) < \frac{c_1^{p-1}}{pM(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})}; \text{ for all } s \in [0, \Theta(c_1)];$$

$$F(\nu\xi) - f(0)\nu\xi > \left(\frac{(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{pm} + \frac{1}{pM}\right)\frac{\nu^p}{T}; \text{ for all } \xi \in [0, T];$$

$$f(s)T + \sum_{j=1}^n I_j(s) < \frac{c_4^{p-1}}{pM(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})}; \text{ for all } s \in [0, \Theta(c_4)].$$

In addition, $\sum_{j=1}^n [\int_0^{\nu\xi_j} I_j(s)ds - I_j(0)\nu\xi_j] > 0$. Then, the problem

$$\begin{cases} -(\phi_p(u'))' = f(u)h(u'), \\ \Delta J(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, n, \\ \alpha u(0) - \beta u'(0) = 0, & \gamma u(1) + \sigma u'(1) = 0 \end{cases}$$

admits at least three classical solutions.

Proof. From our assumptions we obtain

$$\int_0^{\Theta(c_1)} [f(s)T + \sum_{j=1}^n I_j(s)]ds < \int_0^{\Theta(c_1)} \left[\frac{c_1^{p-1}}{pM(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})} \right] ds,$$

$$\int_0^T ([F(\nu\xi) - f(0)\nu\xi])d\xi > \int_0^T \left(\frac{(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{pm} + \frac{1}{pM} \right) \frac{\nu^p}{T} d\xi,$$

$$\int_0^{\Theta(c_4)} [f(s)T + \sum_{j=1}^n I_j(s)]ds < \int_0^{\Theta(c_4)} \left[\frac{c_4^{p-1}}{pM(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})} \right] ds.$$

In another words,

$$\begin{aligned} \frac{\int_0^T F(\Theta(c_1))d\xi + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds}{c_1^p} &< \frac{1}{pM}; \\ \frac{\int_0^T [F(\nu\xi) - f(0)\nu\xi]d\xi + \sum_{j=1}^n [\int_0^{\nu\xi_j} I_j(s)ds - I_j(0)\nu\xi_j]}{\nu^p} \\ &> \frac{\int_0^T [F(\nu\xi) - f(0)\nu\xi]d\xi}{\nu^p} \\ &> \frac{(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{pm} + \frac{1}{pM}; \end{aligned}$$

$$\frac{\int_0^T F(\Theta(c_4))d\xi + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s)ds}{c_4^p} < \frac{1}{pM}.$$

Therefore, the conditions of Remark 3.3 are achieved, and

$$1 \in \left(\left(\frac{(T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)}{pm} + \frac{1}{pM} \right) \times \frac{\nu^p}{\int_0^T [F(\nu\xi) - f(0)\nu\xi]d\xi + \sum_{j=1}^n [\int_0^{\nu\xi_j} I_j(s)ds - I_j(0)\nu\xi_j]}, \right. \\ \left. \min \left\{ \frac{c_1^p}{pM \left(\int_0^T F(\Theta(c_1))d\xi + \sum_{j=1}^n \int_0^{\Theta(c_1)} I_j(s)ds \right)}, \right. \right. \\ \left. \left. \frac{c_4^p}{pM \left(\int_0^T F(\Theta(c_4))d\xi + \sum_{j=1}^n \int_0^{\Theta(c_4)} I_j(s)ds \right)} \right\} \right).$$

Thus, taking Remark 3.3 again into account, Theorem 3.2 ensures the result. $\square$

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