CYCLE SUPER MAGIC LABELING
OF PLANAR GRAPHS

Bin Yang¹, Muhammad Aamer Rashid²,
Sarfraz Ahmad², Muhammad Faisal Nadeem²,
Muhammad Kamran Siddiqui^{3 §}

¹ School of Artificial Intelligence and Big Data
Hefei University, Hefei 230601, CHINA

²Department of Mathematics
COMSATS University Islamabad
Lahore Campus, 54000, PAKISTAN

³ Department of Mathematics
COMSATS University Islamabad
Lahore Campus, 54000, PAKISTAN

Abstract: A simple graph $G(V, E)$ admits a H -covering, if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H . The graph G is said to be H -magic, if there exists a bijection $\psi : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G)| + |E(G)|\}$ such that for every subgraph H' of G isomorphic to H ,

$\sum_{v \in V(H')} \psi(v) + \sum_{e \in E(H')} \psi(e)$ is constant. Moreover G is said to be H -supermagic, if $\psi(V(G)) = \{1, 2, 3, \dots, |V(G)|\}$. In this paper, we study the cycle-supermagic labeling of a pumpkin graph and two classes of planar maps containing 8-sided and 4-sided faces or 6-sided and 4-sided faces, respectively.

AMS Subject Classification: 05C78

Key Words: total labeling, edge-covering, cycle-supermagic labeling

1. Introduction

An edge-covering of G is a family of subgraphs $H_1, \dots, H_r$ such that each edge

Received: June 17, 2019

© 2019 Academic Publications

[§]Correspondence author

of $E(G)$ belongs to at least one of the subgraphs H_i , $1 \leq i \leq r$. Then it is said that G admits an $(H_1, \dots, H_r)$ -(edge)-covering. If every H_i is isomorphic to a given graph H , then G admits an H -covering. Suppose G admits an H -covering. A total labeling $\psi : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G)| + |E(G)|\}$ is called an H -magic labeling of G if there exists a positive integer k (called the magic constant) such that for every subgraph H' of G isomorphic to H ,

$$\sum_{v \in V(H')} \psi(v) + \sum_{e \in E(H')} \psi(e) = k.$$

A graph that admits such a labeling is called H -magic. An H -magic labeling ψ is called an H -supermagic labeling if $\psi(V(G)) = \{1, 2, 3, \dots, |V(G)|\}$. A graph that admits an H -supermagic labeling is called H -supermagic. The sum of all vertex and edge labels on H (under a labeling ψ) is denoted by $\sum \psi(H)$.

The H -supermagic labeling was first introduced by Gutiérrez and Lladó [4]. They proved that some classes of connected graphs are H -supermagic, such as the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$ -supermagic for some h . They also proved that the paths P_n and the cycles C_n are P_h -supermagic for some h . Jeyanthi and Selvagopal [5] proved that one point union of n copies of a 2-connected graph G is H -supermagic for any positive integer n . Lladó and Moragas [9] studied some C_n -supermagic graphs. They proved that the wheels W_n , the windmill graphs $W(r, k)$ and the prisms $C_n \times P_2$ are C_h -supermagic for some h . P_h -supermagic labelings of some classes of trees such as the subdivision of stars, shrubs, and banana trees can be found in [10]. Ngurah, Salman and Susilowati [11] studied cycle-supermagic labelings of chain graphs, fans, triangle ladders, graphs obtained by joining a star $K_{1,n}$ with one isolated vertex, grids, and books.

For $H \cong K_2$, an H -supermagic graph is also called a super edge-magic graph. The notion of a super edge-magic graph was introduced by Enomoto et al. [3] as a particular type of edge-magic graph given by Kotzig and Rosa [7]. The H -magic labeling is related to a face-magic labeling of a plane graph introduced by Lih [8] (see also [1, 2, 6]). A labeling of type $(1, 1, 0)$ (i.e. a total labeling) of a plane graph is said to be face-magic if for every positive integer s , all s -sided faces have the same weight. The weight of a face under a labeling of type $(1, 1, 0)$ is the sum of labels carried by the edges and vertices surrounding that face. When a plane graph G contains only n -sided faces, then the face-magic labeling of G is also a C_n -magic labeling.

In this paper, we study cycle-supermagic labeling of a pumpkin graph and two classes of planar maps containing 8-sided and 4-sided faces or 6-sided and 4-sided faces, respectively.

2. The pumpkin graph P_a^b

Let a and b be integers, $a \geq 3$ and $b \geq 2$. Let $y_1, y_2, \dots, y_a$ be fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint paths p_i^j of length $i+1$ each, $1 \leq i \leq a-1$, $1 \leq j \leq b$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \dots, x_{i,j,i}, y_{i+1}$ be the vertices of the path p_i^j . The resulting graph embedded in the plane is denoted by P_a^b (pumpkin graph), where $V(P_a^b) = \{y_i : 1 \leq i \leq a\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^b \{x_{i,j,k} : 1 \leq k \leq i\}$, $E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{i,j,1} : 1 \leq j \leq b\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^b \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\} \cup \bigcup_{i=1}^{a-1} \{x_{i,j,i} y_{i+1} : 1 \leq j \leq b\}$, with $|V(P_a^b)| = \frac{ab(a-1)}{2} + a$, $|E(P_a^b)| = \frac{b(a-1)(a+2)}{2}$.

Figure 1: The pumpkin graph P_4^5 .

Theorem 1. Let $a, b \geq 3$ be two integers and b be odd. Then the pumpkin graph P_a^b is $C_{2(i+1)}$ -supermagic for $1 \leq i \leq a-1$.

Proof. Define a total labeling

$$\varphi : V(P_a^b) \cup E(P_a^b) \rightarrow \{1, 2, 3, \dots, |V(P_a^b)| + |E(P_a^b)|\}$$

for $1 \leq j \leq b$ as follows:

$$\varphi(y_i) = i, \quad \text{for } 1 \leq i \leq a.$$

Case 1: When i is odd, $1 \leq i \leq a-1$,

$$\varphi(x_{i,j,1}) = \begin{cases} \frac{1}{2}[2(a+1) + b(i^2 - i + 1) - j], & \text{for } j \text{ odd} \\ \frac{1}{2}[2(a+1) + b(i^2 - i + 2) - j], & \text{for } j \text{ even} \end{cases}$$

$$\begin{aligned}\varphi(x_{i,j,2}) &= \begin{cases} \frac{1}{2}[2(a+1) + b(i^2 - i + 4) - j - 1], & \text{for } j \text{ odd} \\ \frac{1}{2}[2(a+1) + b(i^2 - i + 3) - j - 1], & \text{for } j \text{ even} \end{cases} \\ \varphi(x_{i,j,k}) &= \frac{1}{2}[2(a+1-j) + b(i^2 - i + 2k)], \quad \text{for } 3 \leq k \leq i\end{aligned}$$

Case 2: When i is even, $2 \leq i \leq a-1$,

$$\begin{aligned}\varphi(x_{i,j,1}) &= \begin{cases} \frac{1}{2}[2a + b(i^2 - i) + j + 1], & \text{for } j \text{ odd} \\ \frac{1}{2}[2a + b(i^2 - i + 1) + j + 1], & \text{for } j \text{ even} \end{cases} \\ \varphi(x_{i,j,2}) &= \begin{cases} \frac{1}{2}[2a + b(i^2 - i + 3) + j], & \text{for } j \text{ odd} \\ \frac{1}{2}[2a + b(i^2 - i + 2) + j], & \text{for } j \text{ even} \end{cases} \\ \varphi(x_{i,j,k}) &= \frac{1}{2}[2(a+j) + b(i^2 - i + 2k - 2)], \quad \text{for } 3 \leq k \leq i \\ \varphi(y_i x_{i,j,1}) &= \begin{cases} \frac{1}{2}[2(a+j) + ab(a-1)] + & \text{for } i \text{ odd} \\ + \frac{1}{2}[b(i^2 + i - 2)], & 1 \leq i \leq a-1 \\ \frac{1}{2}[2(a+1-j) + ab(a-1)] + & \text{for } i \text{ even} \\ + \frac{1}{2}[b(i^2 + i)], & 2 \leq i \leq a-1 \end{cases} \\ \varphi(x_{1,j,1} y_2) &= \begin{cases} \frac{1}{2}[a(ab - b + 2) + 4b + 1 - j], & \text{for } j \text{ odd} \\ \frac{1}{2}[a(ab - b + 2) + 3b + 1 - j], & \text{for } j \text{ even} \end{cases} \\ \varphi(x_{i,j,i} y_{i+1}) &= \begin{cases} \frac{1}{2}[2(a+1-j) + ab(a-1)] + & \text{for } i \text{ odd,} \\ + \frac{1}{2}[b(i^2 + 3i)], & 3 \leq i \leq a-1 \\ \frac{1}{2}[2(a+j) + ab(a-1)] + & \text{for } i \text{ even,} \\ + \frac{1}{2}[b(i^2 + 3i - 2)], & 2 \leq i \leq a-1 \end{cases}\end{aligned}$$

For $1 \leq k \leq i-1$,

$$\varphi(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{1}{2}[2(a+j) + ab(a-1)] + & \text{for } i \text{ odd,} \\ + \frac{1}{2}[b(i^2 + i + 2k - 2)], & \text{and } 1 \leq i \leq a-1 \\ \frac{1}{2}[2(a+1-j) + ab(a-1)] & \text{for } i \text{ even,} \\ + \frac{1}{2}[b(i^2 + i + 2k)], & \text{and } 2 \leq i \leq a-1. \end{cases}$$

Let $C_{2(i+1)}$, for $1 \leq i \leq a-1$, $1 \leq j, j' \leq b$ and $j \neq j'$, be a cycle in the pumpkin graph with vertices P_a^b with $V(C_{2(i+1)}) = \{y_i; y_{i+1}; x_{i,j,k}; x_{i,j',k} : 1 \leq k \leq i\}$ and edges

$$E(C_{2(i+1)}) = \{y_i x_{i,j,1}; y_i x_{i,j',1}; x_{i,j,i} y_{i+1}; x_{i,j',i} y_{i+1}; x_{i,j,k} x_{i,j,k+1}; x_{i,j',k} x_{i,j',k+1} : 1 \leq k \leq i-1\}.$$

It can be checked out that the supermagic constant

$$\begin{aligned} \sum \varphi(C_{2(i+1)}) &= \varphi(y_i) + \varphi(y_{i+1}) + \sum_{k=1}^i [\varphi(x_{i,j,k}) + \varphi(x_{i,j',k})] \\ &+ \varphi(y_i x_{i,j,1}) + \varphi(y_i x_{i,j',1}) + \varphi(x_{i,j,i} y_{i+1}) \\ &+ \sum_{k=1}^{i-1} [\varphi(x_{i,j,k} x_{i,j,k+1}) + \varphi(x_{i,j',k} x_{i,j',k+1})] \\ &+ \varphi(x_{i,j',i} y_{i+1}) \end{aligned}$$

admits the following value:

For $i = 1$,

$$\sum \varphi(C_{2(i+1)}) = 2a(ab - b + 3) + 5b + 6.$$

For $2 \leq i \leq a - 1$,

$$\sum \varphi(C_{2(i+1)}) = (4i + 2)(a + 1) + (i + 1)ab(a - 1) + b(2i^3 + 3i^2 + i - 1).$$

□

Theorem 2. Let $b \geq 2$, $a \geq 3$ be two integers and b be even. Then the pumpkin graph P_a^b is $C_{2(i+1)}$ -supermagic for $1 \leq i \leq a - 1$.

Proof. Define a total labeling

$$\varphi : V(P_a^b) \cup E(P_a^b) \rightarrow \{1, 2, 3, \dots, |V(P_a^b)| + |E(P_a^b)|\}$$

for $1 \leq j \leq b$ in the following way:

$$\varphi(y_i) = \begin{cases} 1, & \text{for } i = 1, \\ \frac{1}{2}[2i + b(i^2 - 3i + 3)], & \text{for } 2 \leq i \leq a. \end{cases}$$

Case 1: When i is odd, $1 \leq i \leq a - 1$,

$$\begin{aligned} \varphi(x_{i,j,1}) &= \begin{cases} \frac{1}{2}[3 + 2i + b(i^2 - i + 2) - j], & \text{for } j \text{ odd}, \\ \frac{1}{2}[2(i + 1) + b(i^2 - i + 1) - j], & \text{for } j \text{ even}, \end{cases} \\ \varphi(x_{i,j,k}) &= \frac{1}{2}[2(i - j + 2) + b(i^2 - i + 2k)], \quad \text{for } 2 \leq k \leq i. \end{aligned}$$

Case 2: when i is even, $2 \leq i \leq a - 1$,

$$\varphi(x_{i,j,1}) = \begin{cases} \frac{1}{2}[2i + 1 + b(i^2 - i) + j], & \text{for } j \text{ odd}, \\ \frac{1}{2}[2(i + 1) + b(i^2 - i + 1) + j], & \text{for } j \text{ even}, \end{cases}$$

$$\varphi(x_{i,j,k}) = \frac{1}{2}[2(i+j+1) + b(i^2 - i + 2k - 2)], \quad \text{for } 2 \leq k \leq i,$$

$$\varphi(x_{i,j,i}y_{i+1}) = \begin{cases} \frac{1}{2}[2(a+j) + ab(a-1)] + & \text{for } i \text{ odd,} \\ + \frac{1}{2}[b(i^2 + 3i - 2)], 1 \leq i \leq a-1 \\ \frac{1}{2}[2(a+1-j) + ab(a-1)] + & \text{for } i \text{ even,} \\ + \frac{1}{2}[b(i^2 + 3i)], 2 \leq i \leq a-1. \end{cases}$$

For $1 \leq k \leq i-1$,

$$\varphi(x_{i,j,k}x_{i,j,k+1}) = \begin{cases} \frac{1}{2}[2(a+j)] + & \text{for } i \text{ odd,} \\ \frac{1}{2}[ab(a-1)] + \\ + \frac{1}{2}[b(i^2 + i + 2k - 2)], & \text{and } 1 \leq i \leq a-1 \\ \frac{1}{2}[2(a+1-j)] + & \text{for } i \text{ even,} \\ \frac{1}{2}[ab(a-1)] + \\ + \frac{1}{2}[b(i^2 + i + 2k)], & \text{and } 2 \leq i \leq a-1. \end{cases}$$

$$\varphi(y_i x_{i,j,1}) = \begin{cases} \frac{1}{2}[2(a+1)] + & \text{for } i \text{ odd} \\ \frac{1}{2}[ab(a-1) - j - 1] & \text{and, } 1 \leq i \leq a-1, \\ + \frac{1}{2}[b(i^2 + i - 1)], & \text{and } j \text{ odd} \\ \frac{1}{2}[2a + 1 + j] + & \text{for } i \text{ even,} \\ \frac{1}{2}[ab(a-1)] + & \text{and } 2 \leq i \leq a-1, \\ + \frac{1}{2}[b(i^2 + i - 1)], & \text{and } j \text{ odd.} \end{cases}$$

$$\varphi(y_i x_{i,j,1}) = \begin{cases} \frac{1}{2}[2(a+1) - j] + & \text{for } i \text{ odd,} \\ \frac{1}{2}[ab(a-1)] + & \text{for } 1 \leq i \leq a-1, \\ + \frac{1}{2}[b(i^2 + i)], & \text{and } j \text{ even} \\ \frac{1}{2}[2a + j] + & \text{for } i \text{ even,} \\ \frac{1}{2}[ab(a-1)] + & \text{for } 2 \leq i \leq a-1, \\ + \frac{1}{2}[b(i^2 + i - 2)], & \text{and } j \text{ even.} \end{cases}$$

Again it is not difficult to check out that the supermagic constant admits

the value

$$\sum \varphi(C_{2(i+1)}) = \begin{cases} 4a + 9 + 2ab(a-1) + & \text{for } i = 1, \\ + \frac{11b}{2}, \\ 2i(i+3) + 1 + & \text{for } 2 \leq i \leq a-1, \\ + (i+1)ab(a-1) + \\ + b(2i^3 + 4i^2 - i + 1) + \\ + 2a(i+1). \end{cases}$$

□

3. Two classes of planar maps

For $n, m \geq 2$ we denote by O_n^m the planar map labeled as in Figure 2 with m rows and n columns of octagons and 4-sided faces between them. The symbols $V(O_n^m)$ and $E(O_n^m)$ will denote the vertex set and the edge set of O_n^m , where $V(O_n^m) = \{a_i^j, b_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m+1\} \cup \{c_i^j, d_i^j; 1 \leq i \leq n+1 \text{ and } 1 \leq j \leq m\}$, $E(O_n^m) = \{b_i^{j+1}d_{i+1}^j, a_i^jc_i^j, b_i^jc_{i+1}^j, a_i^{j+1}d_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \cup \{c_i^jd_i^j; 1 \leq i \leq n+1 \text{ and } 1 \leq j \leq m\} \cup \{a_i^jb_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m+1\}$, with $|V(O_n^m)| = (4m+2)n + 2m$ and $|E(O_n^m)| = (6m+1)n + m$.

Theorem 3. *Let $m, n \geq 2$ be two integers. Then the planar map O_n^m is C_8 -supermagic.*

Proof. Define a total labeling

$$\varphi : V(O_n^m) \cup E(O_n^m) \rightarrow \{1, 2, 3, \dots, |V(O_n^m)| + |E(O_n^m)|\}$$

as follows:

For $1 \leq j \leq m$ and $1 \leq i \leq n+1$:

$$\varphi(c_i^j) = (n+1)(j-1) + i,$$

$$\varphi(d_i^j) = (n+1)(2m-j) + i.$$

For $1 \leq j \leq m+1$ and $1 \leq i \leq n$:

$$\varphi(a_i^j) = 2m(n+1) + 2n(m+1-j) + 2i-1,$$

$$\varphi(b_i^j) = 2m(n+1) + 2nj + 2 - 2i.$$

Figure 2: The planar map O_6^3

For $1 \leq j \leq m$ and $1 \leq i \leq n$:

$$\varphi(b_i^{j+1}d_{i+1}^j) = 2m(2n+1) + 2n + nj + 1 - i,$$

$$\varphi(a_i^jc_i^j) = 2m(3n+1) + 3n - nj + 1 - i,$$

$$\varphi(b_i^jc_{i+1}^j) = 2m(3n+1) + 2n + nj + 1 - i,$$

$$\varphi(a_i^{j+1}d_i^j) = 7mn + 2n + 2m + nj + 1 - i.$$

For $1 \leq j \leq m+1$ and $1 \leq i \leq n$:

$$\varphi(a_i^jb_i^j) = 3n(3m+1) + 2m - nj + i.$$

If n is odd and $1 \leq j \leq m$, i , even, $2 \leq i \leq n+1$,

$$\varphi(c_i^jd_i^j) = 10mn + 4n + 3m - j(n+1) + 2 - i.$$

If n is odd and $1 \leq j \leq m$, i , odd, $1 \leq i \leq n+1$,

$$\varphi(c_i^jd_i^j) = 9mn + 2m + 3n + (n+1)j + 1 - i.$$

If n is even and $1 \leq j \leq m$, i , odd, $1 \leq i \leq n+1$,

$$\varphi(c_i^jd_i^j) = 10mn + 4n + 3m - j(n+1) + 2 - i.$$

If n is even and $1 \leq j \leq m$, i , even, $2 \leq i \leq n+1$,

$$\varphi(c_i^j d_i^j) = 9mn + 2m + 3n + (n+1)j + 1 - i.$$

The plane map O_n^m is covered by the cycles $C_{8,i}^j$, for $1 \leq j \leq m$, $1 \leq i \leq n$, with the following vertex sets and edge sets:

$$\begin{aligned} V(C_{8,i}^j) &= \{a_i^j, c_i^j, d_i^j, a_i^{j+1}; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \\ &\cup \{b_i^{j+1}, d_{i+1}^j, c_{i+1}^j, b_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}, \end{aligned}$$

$$\begin{aligned} E(C_{8,i}^j) &= \{a_i^j c_i^j, c_i^j d_i^j, d_i^j a_i^{j+1}; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \\ &\cup \{d_{i+1}^j c_{i+1}^j, b_i^j c_{i+1}^j, a_i^j b_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \\ &\cup \{a_i^{j+1} b_i^{j+1}, b_i^{j+1} d_{i+1}^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}. \end{aligned}$$

It is not difficult to verify by a routine procedure that the super magic constant under the labeling φ , for every $1 \leq i \leq n$, $1 \leq j \leq m$, admits the following value:

$$\begin{aligned} \sum \varphi(C_{8,i}^j) &= \varphi(a_i^j) + \varphi(c_i^j) + \varphi(d_i^j) + \varphi(a_i^{j+1}) + \varphi(b_i^{j+1}) \\ &+ \varphi(c_{i+1}^j) + \varphi(b_i^j) + \varphi(a_i^j c_i^j) + \varphi(d_{i+1}^j) \\ &+ \varphi(b_i^{j+1} d_{i+1}^j) + \varphi(d_{i+1}^j c_{i+1}^j) + \varphi(b_i^j c_{i+1}^j) + \\ &+ \varphi(a_i^{j+1} b_i^{j+1}) + \varphi(d_i^j a_i^{j+1}) + \varphi(a_i^j b_i^j) + \varphi(c_i^j d_i^j) \\ &= 76mn + 29m + 23n + 8. \end{aligned}$$

□

Now, we consider a planar map H_n^m , $n, m \geq 2$, labeled as in Figure 4 with m rows and n columns of hexagons and 4-sided faces between them. Let $V(H_n^m) = \{u_i^j, v_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m+1\} \cup \{w_i^j; 1 \leq i \leq n+1 \text{ and } 1 \leq j \leq m\}$ and $E(H_n^m) = \{v_i^{j+1} w_{i+1}^j, u_i^j w_i^j, v_i^j w_{i+1}^j, w_i^j u_i^{j+1}; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \cup \{u_i^j v_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m+1\}$ be the vertex set and the edge set of H_n^m with $|V(H_n^m)| = 3mn + 2n + m$ and $|E(H_n^m)| = 5mn + n$.

Theorem 4. *Let $m, n \geq 2$ be two integers. Then the planar map H_n^m is C_6 -supermagic.*

Figure 3: C_8 -supermagic labeling of O_2^2 and O_3^2

Figure 4: The planar map H_6^3

Proof. Define a total labeling

$$\varphi : V(H_n^m) \cup E(H_n^m) \rightarrow \{1, 2, 3, \dots, |V(H_n^m)| + |E(H_n^m)|\}$$

as follows:

For $1 \leq j \leq m + 1$ and $1 \leq i \leq n$:

$$\begin{aligned} \varphi(u_i^j) &= 2n(j - 1) + 2i - 1, \\ \varphi(v_i^j) &= 2n(m + 2 - j) + 2 - 2i \\ \varphi(u_i^j v_i^j) &= 8mn + 3n + m - jn + i. \end{aligned}$$

For $1 \leq j \leq m$ and $1 \leq i \leq n+1$:

$$\varphi(w_i^j) = \begin{cases} 2n(m+1) + (n+1)(m-j) + i, & \text{if } i \text{ is odd,} \\ 2n(m+1) + (n+1)(j-1) + i, & \text{if } i \text{ is even.} \end{cases}$$

For $1 \leq j \leq m$ and $1 \leq i \leq n$:

$$\begin{aligned} \varphi(v_i^{j+1}w_{i+1}^j) &= 3mn + 2n + m + jn + 1 - i, \\ \varphi(u_i^jw_i^j) &= 5mn + 3n + m - jn + 1 - i, \\ \varphi(v_i^jw_{i+1}^j) &= 5mn + 2n + m + jn + 1 - i, \\ \varphi(w_i^ju_i^{j+1}) &= 6mn + 2n + m + jn + 1 - i. \end{aligned}$$

The planar map H_n^m is covered by the cycles $C_{6,i}^j$, for $1 \leq j \leq m$, $1 \leq i \leq n$, with the following vertex sets and edge sets:

$$V(C_{6,i}^j) = \{u_i^j, w_i^j, u_i^{j+1}, v_i^{j+1}, w_{i+1}^j, v_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\},$$

$$E(C_{6,i}^j) = \{u_i^ju_i^j, w_i^ju_i^{j+1}, u_i^{j+1}v_i^{j+1}, v_i^{j+1}w_{i+1}^j, w_{i+1}^jv_i^j, v_i^ju_i^j; 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

We observe that under the labeling φ , for every $1 \leq i \leq n$ and $1 \leq j \leq m$, the supermagic constant attains the value

$$\begin{aligned} \sum \varphi(C_{6,i}^j) &= \varphi(u_i^j) + \varphi(w_i^j) + \varphi(u_i^{j+1}) + \varphi(v_i^{j+1}) + \varphi(w_{i+1}^j) \\ &\quad + \varphi(v_i^j) + \varphi(u_i^ju_i^j) + \varphi(w_i^ju_i^{j+1}) + \varphi(u_i^{j+1}v_i^{j+1}) \\ &\quad + \varphi(v_i^{j+1}w_{i+1}^j) + \varphi(w_{i+1}^jv_i^j) + \varphi(u_i^ju_i^j) \\ &= 44mn + 7m + 21n + 6. \end{aligned}$$

□

4. Acknowledgments

This work was supported by Major University Science Research Project of Anhui Province (KJ2016A605), Science Research Talents Foundation of Hefei University (13RC02) and the Natural Science Foundation of the Education Department of Anhui Province (KJ2018A0598).

Figure 5: C_6 -supermagic labeling of H_2^2 and H_3^2

References

- [1] M. Bača, On magic labelings of grid graphs, *Ars Combin.*, **33** (1992), 295–299.
- [2] M. Bača, On magic labelings of honeycomb, *Discrete Math.*, **105** (1992), 305–311.
- [3] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge magic graphs, *SUT J. Math.*, **34** (1998), 105–109.
- [4] A. Gutiérrez and A. Lladó, Magic coverings, *J. Combin. Math. Combin. Comput.*, **55** (2005), 43–56.
- [5] P. Jeyanthi and P. Selvagopal, More classes of H -supermagic graphs, *Int. J. Algor. Comp and Math.*, **3** (2010), 93–108.
- [6] K. Kathiresan and S. Gokulakrishnan, On magic labelings of type $(1, 1, 1)$ for the special classes of plane graphs, *Utilitas Math.*, **63** (2003), 25–32.
- [7] A. Kotzig and A. Rosa, Magic valuation of finite graphs, *Canad. Math. Bull.*, **13**, No 4 (1970), 451–461.
- [8] K.W. Lih, On magic and consecutive labelings of plane graphs, *Utilitas Math.*, **24** (1983), 165–197.
- [9] A. Lladó and J. Moragas, Cycle-magic graphs, *Discrete Math.*, **307**, No 23 (2007), 2925–2933.

- [10] T.K. Maryati, E.T. Baskoro and A.N.M. Salman, P_h -supermagic labelings of some trees, *J. Combin. Math. Combin. Comput.*, **65** (2008), 197–204.
- [11] A.A.G. Ngurah, A.N.M. Salman and L. Susilowati, H -supermagic labeling of graphs, *Discrete Math.*, **310** (2010), 1293–1300.