A NORM INEQUALITY FOR FUNCTIONS OF $L^{p(\cdot)}(\Omega)$ SPACES

Yasin Kaya

Dicle University, Department of Mathematics
Diyarbakir - 21280, TURKEY

Abstract: We prove an inequality in $L^{p(\cdot)}(\Omega)$ that give a relation between norm of function and norm of function restriction to a partition of domain.

AMS Subject Classification: 46E30, 26D15

Key Words: variable Lebesgue space, Luxemburg norm

1. Introduction

Variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ are a generalization of standard Lebesgue spaces $L^p(\Omega)$ in the sense that the exponent may vary from point to point. The variable Lebesgue spaces (and variable exponent Sobolev spaces) have gained great importance in analysis and have begun to play an important role in a wide variety of applications such as fluid dynamics, image restoration, differential equations with non-standard growth conditions, fluid flow in porous media and elasticity theory (cf. [1], [4], [5], [8]).

The basic theory of $L^{p(x)}(\Omega)$ spaces can be found in [6], [9]. We refer to the monographs [2], [3] and the references therein for a detailed exposition of $L^{p(x)}(\Omega)$ spaces theory. Also see [7] for moving from local to global results in variable exponent function spaces. But we here recall some basic features of these spaces for the convenience of the reader. For distinguishing between variable and constant exponents, we denote exponent functions by $p(\cdot)$.

Given a set $\Omega \subset \mathbb{R}^n$, we denote by $P(\Omega)$ the class of all bounded Lebesgue measurable functions $p(\cdot) : \Omega \rightarrow [1, \infty)$. The elements of $P(\Omega)$ are called exponent functions on Ω . Also, we denote, for $E \subset \Omega$,

$$p_E^+ = \operatorname{ess\,sup} \{ p(x) : x \in E \}, \quad p_E^- = \operatorname{ess\,inf} \{ p(x) : x \in E \}$$

and

$$p^+ = p_\Omega^+, \quad p^- = p_\Omega^-.$$

In this paper, we always assume that $p^+ < \infty$. By $\chi_\Omega(x)$ we denote the characteristic function of a set Ω in $\mathbb{R}^n$.

As might be expected, the variable exponent Lebesgue spaces are obtained in the usual way. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the modular

$$\rho_{L^{p(\cdot)}(\Omega)}(u) = \int_\Omega |u(x)|^{p(x)} dx$$

functional is finite.

$L^{p(x)}(\Omega)$ space becomes a Banach space when furnished with the so-called Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{L^{p(\cdot)}(\Omega)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

As seen in the above formula norm is not described in the sense of standard $L^p(\Omega)$. But if $p(x) = p$ a constant, then it can be shown easily that norm $\|u\|_{L^{p(\cdot)}(\Omega)}$ reduces to the standard formula $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$ in $L^p(\Omega)$ and variable exponent Lebesgue space coincides with the standard Lebesgue space.

The modular and the norm are related through the following inequalities

$$\begin{aligned} \min \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\} &\leq \rho_{L^{p(\cdot)}(\Omega)}(u) \\ &\leq \max \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\}. \end{aligned}$$

Given $p(\cdot) \in P(\Omega)$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega.$$

Let $p(\cdot) \in P(\Omega)$, $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. Then $uv \in L^1(\Omega)$ and Hölder type inequality

$$\int_\Omega uv dx \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}$$

is valid [6].

Given Ω with $|\Omega| < \infty$ and $p(\cdot), q(\cdot) \in P(\Omega)$ with $p(x) \leq q(x)$. Then for all $u \in L^{q(\cdot)}(\Omega)$

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq (1 + |\Omega|) \|u\|_{L^{q(\cdot)}(\Omega)}$$

relation holds [2], [6].

A useful property of $L^{p(x)}(\Omega)$ spaces is that the topology induced by the norm coincides with the topology induced by modular. Equivalently,

$$\rho_{L^{p(\cdot)}(\Omega)}(u_n - u) \rightarrow 0 \text{ if and only if } \|u_n - u\|_{L^{p(\cdot)}(\Omega)} \rightarrow 0, \text{ [6].}$$

One of the most serious obstacles to the development of the $L^{p(x)}(\Omega)$ spaces arise from the lack of mean continuity property: if $p(\cdot)$ is continuous and non-constant in an open ball B , then there exists a function $u \in L^{p(x)}(B)$ such that $u(h+x) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^n$ with arbitrary small norm, [6].

Although the norm is defined by an infimum, if u is non-trivial, then the infimum is always attained. If $u \equiv 0$, then the infimum is zero and is not attained.

Lemma 1. ([6]) *Given Ω and $p(\cdot) \in P(\Omega)$, if $u \in L^{p(x)}(\Omega)$ and $\|u\|_{L^{p(\cdot)}(\Omega)} > 0$, then $\rho_{L^{p(\cdot)}(\Omega)}\left(\frac{u}{\|u\|_{L^{p(\cdot)}(\Omega)}}\right) = 1$ for all non-trivial $u \in L^{p(x)}(\Omega)$.*

2. Main results

Before we state our theorem, we first note the following result.

Assume Lebesgue measurable sets Ω_1 and Ω_2 satisfy $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ conditions. Then, we have the following inequality

$$\begin{aligned} \|u\|_{L^p(\Omega)} &= \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_{\Omega_1} |u(x)|^p dx + \int_{\Omega_2} |u(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left[2 \max \left\{ \int_{\Omega_1} |u(x)|^p dx, \int_{\Omega_2} |u(x)|^p dx \right\} \right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}} \left[\left(\int_{\Omega_1} |u(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega_2} |u(x)|^p dx \right)^{\frac{1}{p}} \right] \end{aligned}$$

$$= 2^{\frac{1}{p}} \left(\|u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_2)} \right).$$

Also, it is not difficult to see that for any finite collection of pairwise disjoint measurable sets a similar result holds:

$$\|u\|_{L^p(\Omega)} \leq m^{\frac{1}{p}} \sum_{i=1}^m \|u\|_{L^p(\Omega_i)}.$$

Since

$$\left(\int_{\Omega_1} |u(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}}$$

and

$$\left(\int_{\Omega_2} |u(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}},$$

we have

$$\|u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_2)} \leq 2 \|u\|_{L^p(\Omega)}.$$

More generally we have that

$$\sum_{i=1}^m \|u\|_{L^p(\Omega_i)} \leq m \|u\|_{L^p(\Omega)}.$$

Now we prove that a similar result holds for $L^{p(\cdot)}(\Omega)$ space.

Theorem 2. *Let Lebesgue measurable sets Ω_1 and Ω_2 satisfy the following partition conditions*

$$\Omega = \Omega_1 \cup \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset.$$

Then we have

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq 2 \left(\|u\|_{L^{p(\cdot)}(\Omega_1)} + \|u\|_{L^{p(\cdot)}(\Omega_2)} \right). \quad (1)$$

Proof. It is easy to see that if $|\Omega_1| = 0$ or $|\Omega_2| = 0$ then (1) inequality holds. So, we assume none of the $|\Omega_1|$, $|\Omega_2|$ is zero. Let us assign a value to each norm such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \alpha, \|u\|_{L^{p(\cdot)}(\Omega_1)} = \beta \quad \text{and} \quad \|u\|_{L^{p(\cdot)}(\Omega_2)} = \gamma.$$

By Lemma 1, we can write that

$$\int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx = 1$$

and

$$\int_{\Omega_1} \left| \frac{u(x)}{\beta} \right|^{p(x)} dx = 1$$

and

$$\int_{\Omega_2} \left| \frac{u(x)}{\gamma} \right|^{p(x)} dx = 1.$$

From these equalities we have the following

$$\begin{aligned} & \int_{\Omega_1} \left| \frac{u(x)}{\beta + \gamma} \right|^{p(x)} dx + \int_{\Omega_2} \left| \frac{u(x)}{\beta + \gamma} \right|^{p(x)} dx \\ & \leq \int_{\Omega_1} \left| \frac{u(x)}{\beta} \right|^{p(x)} dx + \int_{\Omega_2} \left| \frac{u(x)}{\gamma} \right|^{p(x)} dx = 2. \end{aligned}$$

We now multiply both sides of this last inequality by $\frac{1}{2}$ then we have

$$\int_{\Omega_1} \frac{1}{2} \left| \frac{u(x)}{\beta + \gamma} \right|^{p(x)} dx + \int_{\Omega_2} \frac{1}{2} \left| \frac{u(x)}{\beta + \gamma} \right|^{p(x)} dx \leq 1.$$

From this we have the following inequality

$$\begin{aligned} & \int_{\Omega_1} \left| \frac{u(x)}{2(\beta + \gamma)} \right|^{p(x)} dx + \int_{\Omega_2} \left| \frac{u(x)}{2(\beta + \gamma)} \right|^{p(x)} dx \\ & = \int_{\Omega} \left| \frac{u(x)}{2(\beta + \gamma)} \right|^{p(x)} dx \leq 1. \end{aligned}$$

Clearly this means that

$$\alpha \leq 2(\beta + \gamma), \text{ i.e. } \|u\|_{L^{p(\cdot)}(\Omega)} \leq 2 \left(\|u\|_{L^{p(\cdot)}(\Omega_1)} + \|u\|_{L^{p(\cdot)}(\Omega_2)} \right).$$

□

We can in fact apply the same approach to any finite collection of pairwise disjoint measurable sets to have that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq m \sum_{i=1}^m \|u\|_{L^{p(\cdot)}(\Omega_i)}.$$

It follows from Lemma 1 and definition of norm that

$$\|u\|_{L^{p(\cdot)}(\Omega_i)} \leq \|u\|_{L^{p(\cdot)}(\Omega)}.$$

Hence, we have

$$\sum_{i=1}^m \|u\|_{L^{p(\cdot)}(\Omega_i)} \leq m \|u\|_{L^{p(\cdot)}(\Omega)}.$$

Minkowski's inequality provides us an alternative way of proving Theorem 1

$$\begin{aligned} \|u\|_{L^{p(\cdot)}(\Omega)} &= \|u\chi_{\Omega_1} + u\chi_{\Omega_2}\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \|u\chi_{\Omega_1}\|_{L^{p(\cdot)}(\Omega)} + \|u\chi_{\Omega_2}\|_{L^{p(\cdot)}(\Omega)} \\ &= \|u\|_{L^{p(\cdot)}(\Omega_1)} + \|u\|_{L^{p(\cdot)}(\Omega_2)}. \end{aligned}$$

If $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and Ω_1, Ω_2 and Ω_3 are pairwise disjoint measurable sets, then

$$\begin{aligned} \|u\|_{L^{p(\cdot)}(\Omega)} &= \|u\chi_{\Omega_1 \cup \Omega_2} + u\chi_{\Omega_3}\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \|u\chi_{\Omega_1 \cup \Omega_2}\|_{L^{p(\cdot)}(\Omega)} + \|u\chi_{\Omega_3}\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \|u\chi_{\Omega_1}\|_{L^{p(\cdot)}(\Omega)} + \|u\chi_{\Omega_2}\|_{L^{p(\cdot)}(\Omega)} + \|u\chi_{\Omega_3}\|_{L^{p(\cdot)}(\Omega)} \\ &= \|u\|_{L^{p(\cdot)}(\Omega_1)} + \|u\|_{L^{p(\cdot)}(\Omega_2)} + \|u\|_{L^{p(\cdot)}(\Omega_3)}. \end{aligned}$$

Again, by generalizing, we have that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq \sum_{i=1}^m \|u\|_{L^{p(\cdot)}(\Omega_i)}.$$

References

- [1] B. Amaziane, L. Pankratov, A. Piatnitski, Nonlinear flow through double porosity media in variable exponent Sobolev spaces, *Nonlinear Anal. Real World Appl.*, **10**, No 4 (2009), 2521-2530.
- [2] D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Birkhäuser (2013).
- [3] D. Lars, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer (2011).
- [4] L. Diening, M. Růžička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics, *J. Reine Angew. Math.*, **563** (2003), 197-220.

- [5] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin (2000).
- [6] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k, p(x)}$, *Czechoslovak Math. J.*, **41**, No 16 (1991), 592-618.
- [7] P. Hästö. Local-to-global results in variable exponent spaces, *Math. Res. Letters*, **16**, No 2 (2009), 263-278.
- [8] S. Levine, J. Stanich, Y. Chen, Image restoration via nonstandard diffusion, Dept. Math. Computer Science, Duquesne University, *Technical Report* 04-01 (2004).
- [9] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m, p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263**, No 2 (2001), 424-446.