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SOME FIXED POINT RESULTS VIA GENERALIZED CARISTI CONTRACTIONS IN PARTIAL METRIC SPACES

D. Ram Prasad[1], G.N.V. Kishore[2 §], V.S. Bhagavan[3]

[1]Research Scholar, Department of Mathematics
K L University, Vaddeswaram, Guntur - 522 502
Andhra Pradesh, INDIA

[2]Department of Engineering Mathematics
Sagi Rama Krishnam Raju Engineering College
Bhimavaram - 534 204, Andhra Pradesh, INDIA

[3] Department of Mathematics, K L University
Vaddeswaram, Guntur - 522 502
Andhra Pradesh, INDIA

Abstract: The concept of partial metric was initiated by Matthews [14] as a part of study of denotational semantics of flow networks. In fact, the partial metric plays a very important role in development of models in theory of computation and computer domain theory. In this paper we provide some common fixed point results by using generalized Caristi type contraction.

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1. Introduction

The concept of a partial metric was initiated by Matthews [14] as a part of study of denotational semantics of flow networks. In fact, the partial metric plays a very important role in development of models in theory of computation and computer domain theory (see [2, 9, 12, 17, 21, 22, 23]).

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§Correspondence author

In the year 1976, the famous mathematician Caristi [8] proved a most valuable generalized theorem of Banach Contraction result [6] and proved a fixed point theorem via a contraction condition.

Theorem 1. *Let (Ω, d) be a complete metric space and $\phi : X \to R$ be a lower semi - continuous and bounded bellow function. Let $\Im : \Omega \to \Omega$ be a Caristi type mapping on Ω dominated by ϕ (i.e., $\Im$ satisfies $d(\eta, \Im\eta) \leq \phi(\eta) - \phi(\Im\eta)$ for all $\eta \in \Omega$). Then $\Im$ has a fixed point.*

The main aim of this manuscript is to prove some Caristi type results in partial metric spaces. First, we give basic definitions and lemmas will help us to prove our main results.

Definition 2. ([14, 15]) Let Ω be non empty set. A function $p : \Omega \times \Omega \to [0, \infty)$ is said to be a partial metric on Ω, if for all $\nu, \eta, \zeta \in \Omega$,

(p_1) $\nu = \eta \Leftrightarrow p(\nu, \nu) = p(\nu, \eta) = p(\eta, \eta)$;

(p_2) $p(\nu, \nu) \leq p(\nu, \eta)$, $p(\eta, \eta) \leq p(\nu, \eta)$;

(p_3) $p(\nu, \eta) = p(\eta, \nu)$;

(p_4) $p(\nu, \eta) + p(\zeta, \zeta) \leq p(\nu, \zeta) + p(\zeta, \eta)$.

In this case, the pair (p, Ω) is termed as a partial metric space (PMS).

If p is a partial metric on Ω, then the mapping $d_p : \Omega \times \Omega \to [0, \infty)$ given by

$$d_p(\nu, \eta) = 2p(\nu, \eta) - p(\nu, \nu) - p(\eta, \eta), \tag{1}$$

is a metric on Ω.

Now, define convergence, completeness, continuity on PMS (see [1, 4, 11, 14, 15]).

Definition 3. *Let (Ω, p) be PMS and $\{\zeta_i\}$ be a sequence in Ω*

1. *$\{\zeta_i\}$ converges to ξ if and only if $p(\xi, \xi) = \lim_{i \to \infty} p(\xi, \zeta_i)$.*

2. *$\{\zeta_i\}$ is termed as a Cauchy sequence if $\lim_{i,j \to \infty} p(\zeta_i, \zeta_j)$ exists and is finite.*

3. *The PMS (Ω, p) is termed as complete if every Cauchy sequence $\{\zeta_i\}$ in X converges with respect to τ_p, to a point $\xi \in \Omega$ such that*

$$p(\xi,\xi) = \lim_{n,m\to\infty} p(\zeta_i,\zeta_j).$$

4. *A mapping $\Im : \Omega \to \Omega$ is said to be continuous at $\xi_0 \in \Omega$ if for every $\epsilon > 0$, there is $\delta > 0$ so that $\Im(B_p(\xi_0,\delta)) \subseteq B_p(\Im\xi_0,\epsilon)$.*

Lemma 4. ([14, 15])

1. *A sequence $\{\zeta_i\}$ is Cauchy in the metric space (Ω, d_p) iff it is Cauchy in the PMS (Ω, p).*

2. *A PMS (Ω, p) is complete iff the metric space (Ω, d_p) is complete. Moreover,*

$$\lim_{i\to\infty} d_p(\xi,\zeta_i) = 0 \Leftrightarrow p(\xi,\xi) = \lim_{i\to\infty} p(\xi,\zeta_i) = \lim_{i,j\to\infty} p(\zeta_i,\zeta_j).$$

Lemma 5. ([1]) *Let (Ω, p) be a PMS If $\zeta_i \to \zeta$ as $i \to \infty$ with $p(\zeta,\zeta) = 0$, then $\lim\limits_{i\to\infty} p(\zeta_i,\eta) = p(\zeta,\eta)$ for each $\eta \in \Omega$.*

Lemma 6. ([1]) *Let (Ω, p) be a PMS.*

(A) If $p(\nu,\eta) = 0$, then $\nu = \eta$. The converse need not be true.

(B) If $\nu \neq \eta$, then $p(\nu,\eta) > 0$.

Definition 7. *The mappings $\Im : \Omega \to \Omega$ and $\mathscr{F} : \Omega$ are called ω -compatible if $\mathscr{F}(\Im\nu) = \Im(\mathscr{F}\nu)$, whenever $\mathscr{F}\nu = \Im\nu$.*

2. Results and discussions

Our first result is as follows.

Theorem 8. *Let $\mathscr{A}, \mathscr{B}\mathscr{F}, \mathscr{G} : \Omega \to \Omega$ be four maps on a PMS (Ω, p). Suppose that*

$$(i) p(\mathscr{A}\nu, \mathscr{B}\mu) + \alpha(\mathscr{A}\nu) + \gamma(\mathscr{B}\mu) \leq \psi(\alpha(\mathscr{F}\nu))\alpha(\mathscr{F}\nu) + \psi(\gamma(\mathscr{G}\mu))\gamma(\mathscr{G}\mu)$$

where $\alpha, \gamma : \Omega \to [0, \infty)$ *are lower semi-continuous and* $\psi : [0, \infty) \to (0, 1)$ *is continuous;*
(ii) $\mathscr{A}(\Omega) \subseteq \mathscr{G}(\Omega)$ *and* $\mathscr{B}(\Omega) \subseteq \mathscr{F}(\Omega)$*;*
(iii) $(\mathscr{A}, \mathscr{F})$ *and* $(\mathscr{B}, \mathscr{G})$ *are* ω *- compatible;*
(iv) either $\mathscr{F}(\Omega)$*, or* $\mathscr{G}(\Omega)$ *is complete.*
Then $\mathscr{A}, \mathscr{B}, \mathscr{F}$ *and* $\mathscr{G}$ *have a unique CFP.*

Proof. For arbitrary elements ν_0, η_0 in X, from condition (ii), define the sequences $\{\nu_{2n}\}$, $\{\xi_{2n}\}$ and $\{\omega_{2n}\}$ in X as

$$\begin{aligned} \xi_{2n} &= \mathscr{G}\nu_{2n+1} = \mathscr{A}\nu_{2n}, \\ \xi_{2n+1} &= \mathscr{F}\nu_{2n+2} = \mathscr{B}\nu_{2n+1}, \qquad n = 0, 1, 2, \cdots \end{aligned}$$

By (i), we have that

$$\begin{aligned} 0 &\leq p(\xi_{2n}, \xi_{2n+1}) \\ &= p(\mathscr{A}\nu_{2n}, \mathscr{B}\nu_{2n+1}) \\ &\leq \psi(\alpha(\mathscr{F}\nu_{2n}))\alpha(\mathscr{F}\nu_{2n}) - \alpha(\mathscr{A}\nu_{2n}) \\ &\quad + \psi(\gamma(\mathscr{G}\nu_{2n+1}))\gamma(\mathscr{G}\nu_{2n+1}) - \gamma(\mathscr{B}\nu_{2n+1}) \\ &\leq \psi(\alpha(\xi_{2n-1}))\alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \psi(\gamma(\xi_{2n}))\gamma(\xi_{2n}) - \gamma(\xi_{2n+1}). \end{aligned}$$

Therefore,

$$p(\xi_{2n}, \xi_{2n+1}) < \alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \gamma(\xi_{2n}) - \gamma(\xi_{2n+1}) \tag{2}$$

and

$$\begin{aligned} \alpha(\xi_{2n}) &+ \gamma(\xi_{2n+1}) \\ &\leq \psi(\alpha(\xi_{2n-1}))\alpha(\xi_{2n-1}) + \psi(\gamma(\xi_{2n}))\gamma(\xi_{2n}) \\ &\leq \max\{\psi(\alpha(\xi_{2n-1})), \psi(\gamma(\xi_{2n}))\}(\alpha(\xi_{2n-1}) + \gamma(\xi_{2n})) \\ &< (\alpha(\xi_{2n-1}) + \gamma(\xi_{2n})). \end{aligned} \tag{3}$$

Take $t_n = \alpha(\xi_n) + \gamma(\xi_{n+1})$. From the precedent inequality,

$$t_{2n} = \alpha(\xi_{2n}) + \gamma(\xi_{2n+1}) < t_{2n-1} = \alpha(\xi_{2n-1}) + \gamma(\xi_{2n}).$$

Similarly, $t_{2n-1} < t_{2n-2}$ and so on.

This shows that the sequence $\{t_n\}$ is a strictly decreasing, bounded below sequence and so it converges to some $l \geq 0$.

Suppose $l > 0$. Letting $n \to \infty$ in Equation (3),we have

$$\begin{aligned} l &\leq \lim_{n\to\infty} \max\{\psi(\alpha(\xi_{2n-1})), \psi(\gamma(\xi_{2n}))\}\, l \\ &= \max\left\{\psi\left(\lim_{n\to\infty}\alpha(\xi_{2n-1})\right), \psi\left(\lim_{n\to\infty}\gamma(\xi_{2n})\right)\right\} l \\ &< l, \qquad \text{since } \psi \text{ is continuous and } \psi : [0,\infty) \to (0,1)\ . \end{aligned}$$

It is a contradiction. Consequently,

$$\lim_{n\to\infty} [\alpha(\xi_{2n}) + \gamma(\xi_{2n+1})] = 0.$$

Thus, we have

$$\lim_{n\to\infty} \alpha(\xi_{2n}) = \lim_{n\to\infty} \gamma(\xi_{2n+1}) = 0. \tag{4}$$

Also, from (2), $\lim_{n\to\infty} p(\xi_{2n}, \xi_{2n+1}) = 0$. Now for any $n, m > 0$ and from (2), we have that

$$\begin{aligned} p\ &(\xi_{2n}, \xi_{2m+1}) \\ &\leq p(\xi_{2n}, \xi_{2n+1}) + p(\xi_{2n+1}, \xi_{2n+2}) + \cdots + p(\xi_{2m-1}, \xi_{2m}) \\ &\quad + p(\xi_{2m}, \xi_{2m+1}) - p(\xi_{2n+1}, \xi_{2n+1}) - p(\xi_{2n+2}, \xi_{2n+2}) - \cdots \\ &\quad - p(\xi_{2m-1}, \xi_{2m-1}) - p(\xi_{2m}, \xi_{2m}) \\ &\leq \alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \gamma(\xi_{2n}) - \gamma(\xi_{2n+1}) + \alpha(\xi_{2n}) - \alpha(\xi_{2n+1}) \\ &\quad + \gamma(\xi_{2n+1}) - \gamma(\xi_{2n+2}) + \alpha(\xi_{2n+1}) - \alpha(\xi_{2n+2}) + \gamma(\xi_{2n+2}) \\ &\quad - \gamma(\xi_{2n+3}) + \cdots + \alpha(\xi_{2m-2}) - \alpha(\xi_{2m-1}) + \gamma(\xi_{2m-1}) - \gamma(\xi_{2m}) \\ &\quad + \alpha(\xi_{2m-1}) - \alpha(\xi_{2m}) + \gamma(\xi_{2m}) - \gamma(\xi_{2m+1}) \\ &= \alpha(\xi_{2n-1}) - \alpha(\xi_{2m}) + \gamma(\xi_{2n}) - \gamma(\xi_{2m+1}) \\ &\to 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

Now from p_4, we have that

$$\begin{aligned} p(\xi_{2n}, \xi_{2m}) &\leq p(\xi_{2n}, \xi_{2m+1}) + p(\xi_{2m+1}, \xi_{2m}) - p(\xi_{2m+1}, \xi_{2m+1}) \\ &\to 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

This shows that $\{\xi_{2n}\}$ is a Cauchy sequence in (Ω, p).
From Lemma 4, $\{\xi_{2n}\}$ is Cauchy sequences in metric space (Ω, d_p).
Hence $\{\xi_n\}$ is Cauchy sequences in metric space (Ω, d_p).
Hence, we have that

$$\lim_{n,m\to\infty} d_p(\xi_n, \xi_m) = 0.$$

By the definition of d_p, we have

$$\lim_{n,m\to\infty} p(\xi_n, \xi_m) = 0. \tag{5}$$

Suppose $\mathscr{F}(\Omega)$ is a complete. Since $\{\xi_{2n+1}\} \subseteq \mathscr{F}(\Omega)$ is Cauchy sequences in complete metric space $(\mathscr{F}(\Omega), d_p)$. It follows that $\{\xi_{2n+1}\}$ is convergent in $(\mathscr{F}(\Omega), d_p)$. Thus

$$\lim_{n\to\infty} d_p(\xi_{2n+1}, \mu) = 0, \quad \text{for some} \quad \mu \in \mathscr{F}(\Omega).$$

Since $\{\xi_n\}$ is Cauchy sequences in $(\mathscr{F}(\Omega), d_p)$, $\xi_{2n+1} \to \mu$, it follows that $\xi_{2n} \to \mu$. From Lemma 4,
$p(\mu, \mu) = \lim_{n\to\infty} p(\xi_{2n}, \mu) = \lim_{n\to\infty} p(\xi_{2n+1}, \mu) = \lim_{m,n\to\infty} p(\xi_{2n}, \xi_{2m}) = 0.$

Since α and γ are lower semi-continuous functions, $\xi_{2n} \to \mu$ and as $n \to \infty$ from (4) we have $\alpha(\mu) = \gamma(\mu) = 0$.

Since $\mathscr{F} : \Omega \to \Omega$ and $\mu \in \mathscr{F}(\Omega)$, there exist $s \in \Omega$ such that $\mathscr{F}s = \mu$. From (i), we have

$$\begin{aligned}
p(\mathscr{A}s, \mu) &= p(\mathscr{A}s, \xi_{2n+1}) + p(\xi_{2n+1}, \mu) - p(\xi_{2n+1}, \xi_{2n+1}) \\
&= p(\mathscr{A}s, \mathscr{B}\nu_{2n+1}) + p(\xi_{2n+1}, \mu) - p(\xi_{2n+1}, \xi_{2n+1}) \\
&\leq \psi(\alpha(\mathscr{F}s))\alpha(\mathscr{F}s) - \alpha(\mathscr{A}s) \\
&\quad + \psi(\gamma(\mathscr{G}\nu_{2n+1}))\gamma(\mathscr{G}\nu_{2n+1}) - \gamma(\mathscr{B}\nu_{2n+1}) \\
&\quad + p(\xi_{2n+1}, \mu) - p(\xi_{2n+1}, \xi_{2n+1}) \\
&< \alpha(\mathscr{F}s) - \alpha(\mathscr{A}s) + \gamma(\mathscr{G}\nu_{2n+1}) - \gamma(\mathscr{B}\nu_{2n+1}) \\
&\quad + p(\xi_{2n+1}, \mu) - p(\xi_{2n+1}, \xi_{2n+1}).
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}
&p(\mathscr{A}s, \mu) \leq \alpha(\mu) - \alpha(\mathscr{A}s) + \gamma(\mu) - \gamma(\mu) + p(\mu, \mu) - p(\mu, \mu) \\
&= \alpha(\mu) - \alpha(\mathscr{A}s, t) \leq \alpha(\mu) = 0.
\end{aligned}$$

Therefore $p(\mathscr{A}s, \mu) = 0$, so we have $\mathscr{A}s = \mu = \mathscr{F}s$.
Since $(\mathscr{A}, \mathscr{F})$ are ω - compatible, we have that $\mathscr{A}\mu = \mathscr{F}\mu$.
We have

$$
\begin{aligned}
&p(\mathscr{F}\mu,\xi_{2n}) = p(\mathscr{A}\mu,\mathscr{B}\nu_{2n})\\
&\le \psi(\alpha(\mathscr{F}\mu))\alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \psi(\gamma(\mathscr{G}\nu_{2n}))(\gamma(\mathscr{G}\nu_{2n}) - \gamma(\mathscr{B}\nu_{2n})\\
&< \alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}(\mu,\vartheta)) + \gamma(\mathscr{G}\nu_{2n}) - \gamma(\mathscr{B}(\nu_{2n},\eta_{2n}))\\
&< \alpha(\mathscr{F}\mu) - \alpha(\mathscr{F}\mu) + \gamma(\xi_{2n-1}) - \gamma(\xi_{2n})\\
&< \gamma(\xi_{2n-1}) - \gamma(\xi_{2n}).
\end{aligned}
$$

Letting $n \to \infty$, we have that

$$p(\mathscr{F}\mu,\mu) \to \gamma(\mu) - \gamma(\mu) = 0.$$

Therefore, $\mathscr{F}\mu = \mu$.

Therefore, $\mathscr{A}\mu = \mathscr{F}\mu = \mu$.

Since $\mathscr{A}(\Omega) \subseteq \mathscr{G}(\Omega)$, there exist $a \in X$ such that $\mathscr{A}\mu = \mathscr{G}a$. Therefore, $\mu = \mathscr{A}\mu = \mathscr{G}a$,

$$
\begin{aligned}
&p(\mu,\mathscr{B}a) = p(\mathscr{A}\mu,\mathscr{B}a)\\
&\le \psi(\alpha(\mathscr{F}\mu))\alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \psi(\gamma(\mathscr{G}a))\gamma(\mathscr{G}a) - \gamma(\mathscr{B}a)\\
&\le \alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \gamma(\mathscr{G}a) - \gamma(\mathscr{B}a)\\
&\\
&= \alpha(\mu) - \alpha(\mu) + \gamma(\mu) - \gamma(\mathscr{B}a) \le \gamma(\mu) = 0.
\end{aligned}
$$

Therefore, $p(\mu,\mathscr{B}a) = 0$, that is, $\mu = \mathscr{B}a$.

Since $(\mathscr{B},\mathscr{G})$ are ω - compatible, we have $\mathscr{B}\mu = \mathscr{G}\mu$. We have

$$
\begin{aligned}
&p(\mu,\mathscr{G}\mu) = p(\mathscr{A}\mu,\mathscr{B}\mu)\\
&\le \psi(\alpha(\mathscr{F}\mu))\alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \psi(\gamma(\mathscr{G}\mu))\gamma(\mathscr{G}\mu) - \gamma(\mathscr{B}\mu)\\
&= \alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \gamma(\mathscr{G}\mu) - \gamma(\mathscr{B}\mu)\\
&\le \alpha(\mu) - \alpha(\mu) + \gamma(\mathscr{G}\mu) - \gamma(\mathscr{G}\mu) = 0.
\end{aligned}
$$

Therefore, $p(\mu,\mathscr{G}\mu) = 0$, so $\mu = \mathscr{G}\mu$.

We obtained that $\mu = \mathscr{G}\mu = \mathscr{B}\mu$.

This shows that μ is a CFP of $\mathscr{A},\mathscr{B},\mathscr{F}$ and $\mathscr{G}$. Suppose μ^* is an another common fixed point of $\mathscr{A},\mathscr{B},\mathscr{F}$ and $\mathscr{G}$. One writes

$$
\begin{aligned}
&p(\mu,\mu^*) = p(\mathscr{A}\mu,\mathscr{B}\mu^*)\\
&\le \psi(\alpha(\mathscr{F}\mu))\alpha(\mathscr{F}\mu) - \alpha(\mathscr{A}\mu) + \psi(\gamma(\mathscr{G}\mu^*))\gamma(\mathscr{G}\mu^*) - \gamma(\mathscr{B}\mu^*)\\
&< \alpha(\mu) - \alpha(\mu) + \gamma(\mu^*) - \gamma(\mu^*) \le 0.
\end{aligned}
$$

Therefore, $\mu = \mu^*$. This shows that μ is the unique CFP of $\mathscr{A},\mathscr{B},\mathscr{F}$ and $\mathscr{G}$. $\square$

Corollary 9. *Let (Ω, p) be a partial metric space. Let $\mathscr{A}$, $\mathscr{F}$, $\mathscr{G} : \Omega \to \Omega$ be mappings such that*

$$(a) \quad p(\mathscr{A}\nu, \mathscr{A}\mu) \leq \psi(\alpha(\mathscr{F}\nu))\alpha(\mathscr{F}\nu) - \alpha(\mathscr{A}\nu) + \psi(\gamma(\mathscr{B}\mu))\gamma(\mathscr{B}\mu) - \gamma(\mathscr{A}\mu),$$

where $\alpha, \gamma : \Omega \to [0, \infty)$ are lower semi-continuous and $\psi : [0, \infty) \to (0, 1)$ is continuous. Suppose that
$(b)\mathscr{A}(\Omega) \subseteq \mathscr{G}(\Omega)$ and $\mathscr{A}(\Omega) \subseteq \mathscr{F}(\Omega)$;
(c) Either $(\mathscr{A}, \mathscr{F})$, or $(\mathscr{A}, \mathscr{G})$ are ω - Compatible;
(d) Either $\mathscr{F}(\Omega)$, or $\mathscr{G}(\Omega)$ is complete.
Then $\mathscr{A}, \mathscr{F}$ and $\mathscr{G}$ have a unique CFP of the form μ.

Theorem 10. *Let (Ω, p) be PMS. Let $\mathscr{A}, \mathscr{B}, \mathscr{F}, \mathscr{G} : \Omega \to \Omega$ be such that*
$(i) \quad p(\mathscr{A}\nu, \mathscr{B}\mu) \leq \alpha(\psi(\mathscr{F}\nu, \mathscr{G}\mu))\psi(\mathscr{F}\nu, \mathscr{G}\mu) - \psi(\mathscr{A}\nu, \mathscr{B}\mu)$
where $\psi : \Omega \times \Omega \to [0, \infty)$ is lower semi-continuous and $\alpha : [0, \infty) \to (0, 1)$ is continuous. Suppose that
(ii) $\mathscr{A}(\Omega) \subseteq \mathscr{G}(\Omega)$ and $\mathscr{B}(\Omega) \subseteq \mathscr{F}(\Omega)$;
(iii) $(\mathscr{A}, \mathscr{F})$ and $(\mathscr{B}, \mathscr{G})$ are ω-compatible;
(iv) either $\mathscr{F}(\Omega)$, or $\mathscr{G}(\Omega)$ is complete.
Then $\mathscr{A}, \mathscr{B}, \mathscr{F}$ and $\mathscr{G}$ have a unique CFP of the form μ.

Proof. Consider ν_0, η_0 in X. By (ii), define $\{\nu_{2n}\}$ and $\{\zeta_{2n}\}$ as follows

$$\begin{aligned} \zeta_{2n} &= \mathcal{G}\nu_{2n+1} = \mathcal{A}\nu_{2n}, \\ \zeta_{2n+1} &= \mathcal{F}\nu_{2n+2} = \mathcal{B}\nu_{2n+1}, \qquad n = 0, 1, 2, \cdots \end{aligned}$$

Now,

$$\begin{aligned} 0 &\leq p(\zeta_{2n}, \zeta_{2n+1}) \\ &= p(\mathcal{A}\nu_{2n}, \mathcal{B}\nu_{2n+1}) \\ &\leq \alpha(\psi(\mathcal{F}\nu_{2n}, \mathcal{G}\nu_{2n-1}))\psi(\mathcal{F}\nu_{2n}, \mathcal{G}\nu_{2n-1}) - \psi(\mathcal{A}\nu_{2n}, \mathcal{B}\nu_{2n+1}) \\ &\leq \alpha(\psi(\zeta_{2n-1}, \zeta_{2n}))\psi(\zeta_{2n-1}, \zeta_{2n}) - \psi(\zeta_{2n}, \zeta_{2n+1}). \end{aligned}$$

Therefore,

$$p(\zeta_{2n}, \zeta_{2n+1}) \leq \psi(\zeta_{2n-1}, \zeta_{2n}) - \psi(\zeta_{2n}, \zeta_{2n+1}). \tag{6}$$

and

$$\psi(\zeta_{2n}, \zeta_{2n+1}) \leq \alpha(\psi(\zeta_{2n-1}, \zeta_{2n}))\psi(\zeta_{2n-1}, \zeta_{2n}) \tag{7}$$

$$< \quad \psi(\zeta_{2n-1}, \zeta_{2n}).$$

Thus, $\{\psi(\zeta_{2n}, \zeta_{2n+1})\}$ is non-increasing, so it converge to $k \geq 0$. Suppose that $k > 0$. Letting $n \to \infty$ in equation (7), we get a contradiction. Therefore,

$$\lim_{n\to\infty} \psi(\zeta_{2n}, \zeta_{2n+1}) = 0.$$

Now, from (p_4) and Equation (6),

$$\begin{aligned}
p \quad & (\zeta_{2n}, \zeta_{2m+1}) \\
& \leq p(\zeta_{2n}, \zeta_{2n+1}) + p(\zeta_{2n+1}, \zeta_{2n+2}) + \cdots + p(\zeta_{2m-1}, \zeta_{2m}) \\
& \quad + p(\zeta_{2m}, \zeta_{2m+1}) - p(\zeta_{2n+1}, \zeta_{2n+1}) - p(\zeta_{2n+2}, \zeta_{2n+2}) - \cdots \\
& \quad - p(\zeta_{2m-1}, \zeta_{2m-1}) - p(\zeta_{2m}, \zeta_{2m}) \\
& \leq p(\zeta_{2n}, \zeta_{2n+1}) + p(\zeta_{2n+1}, \zeta_{2n+2}) + \cdots \\
& \quad + p(\zeta_{2m-1}, \zeta_{2m}) + p(\zeta_{2m}, \zeta_{2m+1}) \\
& \leq \psi(\zeta_{2n-1}, \zeta_{2n}) - \psi(\zeta_{2n}, \zeta_{2n+1}) \\
& \quad + \psi(\zeta_{2n}, \zeta_{2n+1}) - \psi(\zeta_{2n+1}, \zeta_{2n+2}) \\
& \quad + \cdots \\
& \quad + \psi(\zeta_{2m-2}, \zeta_{2m-1}) - \psi(\zeta_{2m-1}, \zeta_{2m}) \\
& \quad + \psi(\zeta_{2m-1}, \zeta_{2m}) - \psi(\zeta_{2m}, \zeta_{2m+1}) \\
& = \psi(\zeta_{2n-1}, \zeta_{2n}) - \psi(\zeta_{2m}, \zeta_{2m+1}) \\
& \to 0 \qquad \text{as } n, m \to \infty.
\end{aligned}$$

Again from (p_4), we have

$$\begin{aligned}
p(\zeta_{2n}, \zeta_{2m}) & \leq p(\zeta_{2n}, \zeta_{2m+1}) + p(\zeta_{2m+1}, \zeta_{2m}) - p(\zeta_{2m+1}, \zeta_{2m+1}) \\
& \to 0 \qquad \text{as } n, m \to \infty.
\end{aligned}$$

Clearly, $\{\zeta_{2n}\}$ is a Cauchy sequence in (Ω, p). In a similar way, one proves that $\{\omega_{2n}\}$ is Cauchy in (Ω, p).

By proceeding the similar track as mentioned in Theorem 8, we get the CFP for $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$. □

Corollary 11. *Let (Ω, p) be a PMS and let $\mathcal{A}, \mathcal{F}, \mathcal{G} : \Omega \to \Omega$ be so that*

$$(a) \quad p(\mathcal{A}\nu, \mathcal{A}\mu) \leq \alpha(\psi(\mathcal{F}\nu, \mathcal{G}\mu))\psi(\mathcal{F}\nu, \mathcal{G}\mu) - \psi(\mathcal{A}\nu, \mathcal{A}\mu)$$

where $\psi, \phi : \Omega \to [0, \infty)$ are lower semi-continuous and $\alpha : [0, \infty) \to (0, 1)$ is continuous. Suppose that
(b) $\mathcal{A}(\Omega) \subseteq \mathcal{G}(\Omega)$ and $\mathcal{A}(\Omega) \subseteq \mathcal{F}(\Omega)$;
(c) either $(\mathcal{A}, \mathcal{F})$, or $(\mathcal{A}, \mathcal{G})$ are ω - compatible;
(d) either $\mathcal{F}(\Omega)$, or $\mathcal{G}(\Omega)$ is complete.
Then $\mathcal{A}, \mathcal{F}$ and $\mathcal{G}$ have a unique CFP of the form μ.

Theorem 12. *Let* (X, p) *be a PMS and let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$, $\mathcal{G} : \Omega \to \Omega$ *be so that*
(i) $p(\mathcal{A}\nu, \mathcal{B}\mu) \leq \beta(\alpha(\mathcal{F}\nu, \mathcal{G}\mu))\alpha(\mathcal{F}\nu, \mathcal{G}\mu) - \alpha(\mathcal{A}\nu, \mathcal{B}\mu)$
where $\alpha : \Omega \times \Omega \to [0, \infty)$ *is lower semi-continuous and*
$\beta : [0, +\infty) \to (0, 1)$ *is continuous.*
(ii) $\mathcal{A}(\Omega) \subseteq \mathcal{G}(\Omega)$ *and* $\mathcal{B}(\Omega) \subseteq \mathcal{F}(\Omega)$*;*
(iii) $(\mathcal{A}, \mathcal{F})$ *and* $(\mathcal{B}, \mathcal{G})$ *are* ω *- Compatible;*
(iv) *either* $\mathcal{F}(\Omega)$*, or* $\mathcal{G}(\Omega)$ *is complete.*
Then $\mathcal{A}, \mathcal{B}, \mathcal{F}$ *and* $\mathcal{G}$ *have a unique CFP of the form* μ*.*

3. Conclusion

In this paper, we provided some common fixed point results by using Caristi type contractions in the class of partial metric spaces.

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