SUMS OF THE POWERS OF RECIPROCAL OF POLYGONAL NUMBERS

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Abstract: We address the question of expressing the sums of the powers of polygonal numbers in closed forms using some basic functions. We obtain explicit expressions for the closed form expressions for the sums of the squares of reciprocals of polygonal numbers, the sums of the cubes of reciprocals of polygonal numbers the sums of the fourth-powers of reciprocals of polygonal numbers. These closed form expressions are composed of digamma function, Riemann zeta function and the Hurwitz zeta function. It has been possible to obtain the general result for the sums of an arbitrary power of reciprocals of square numbers. An outline is given to extend the result to the general case of the sums of the powers of reciprocals of polygonal numbers.

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1. Introduction

A polygonal number is a number represented as dots arranged in the shape of a regular polygon. For example, four dots can be arranged in the shape of a square of side 2. Nine dots can be arranged in the shape of a square of side 3. Sixteen dots can be arranged in a square of side 4. In general, s^2

dots can be arranged in the shape of a square of side s . We have chosen this example as squares are easy to visualize and straightforward to compute. The *square numbers* form the sequence $1, 4, 9, \dots, n^2$. The *triangular numbers* are represented in Figure 1.

Figure 1: Triangular Numbers

The triangular numbers are generated by the formula

$$T_n = \frac{n^2 + n}{2} = {}^{n+1}C_2 = \binom{n+1}{2}, \quad (1)$$

where ${}^{n+1}C_2 = \frac{n(n+1)}{2}$ is a binomial coefficient. For the triangular numbers, the sequence is $1, 3, 6, 10, \dots, \frac{1}{2}n(n+1)$. We shall cite the various integer sequences occurring in this study, by the unique identity assigned to each of them in *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [1]. So, the triangular numbers are represented by the Sequence A000217.

The square numbers are represented by Figure 2 and the corresponding sequence in the OEIS is the Sequence A000290.

Figure 2: Square Numbers

The pentagonal numbers (Figure 3) are $1, 5, 12, \dots, \frac{1}{2}n(3n-1)$ and are denoted by the Sequence A000326 in the OEIS.

The hexagonal numbers (Figure 4) are $1, 6, 15, 28, 45, 66, \dots, n(2n-1)$ and are denoted by the Sequence A000384 in the OEIS.

The first element of any polygonal sequence is the number 1. In OEIS, the number 0 is included in the sequence. We shall not follow this convention. The procedure of enlarging the polygons is illustrated by the red layers in the four figures.

In general, the n -th polygonal number of a s -sided polygon or s -gonal is

Figure 3: Pentagonal Numbers

Figure 4: Hexagonal Numbers

$$P(s, n) = \frac{(s-2)n^2 - (s-4)n}{2}. \quad (2)$$

By definition, $s \geq 3$. The polygonal numbers are also known as figurate numbers [2].

The n -th s -gonal number is related to the triangular number T_n as

$$P(s, n) = (s-2)T_{n-1} + n = (s-3)T_{n-1} + T_n. \quad (3)$$

The polygonal numbers satisfy the following recurrence relations

$$\begin{aligned} P(s, n+1) &= P(s, n) + (s-2)n + 1, \\ P(s+1, n) &= P(s, n) + T_{n-1} = P(s, n) + \frac{1}{2}n(n-1), \\ 2P(s, n) &= P(s+k, n) + P(s-k, n). \end{aligned} \quad (4)$$

The generating function for the triangular numbers is

$$G_s(x) = \frac{x}{(1-x)^3} = \sum_{n \geq 1} T_n x^n. \quad (5)$$

The generating function for the s -gonal numbers is

$$G_s(x) = \frac{(s-3)x^2 + x}{(1-x)^3} = \sum_{n \geq 1} P(s, n) x^n. \quad (6)$$

s	Name	Formula	$n = 1$	$n = 2$	$n = 3$	$n = 4$
3	Triangular	$\frac{n(n+1)}{2}$	1	3	6	10
4	Square	n^2	1	4	9	16
5	Pentagonal	$\frac{3n^2-n}{2}$	1	5	12	22
6	Hexagonal	$\frac{4n^2-2n}{2}$	1	6	15	28
7	Heptagonal	$\frac{5n^2-3n}{2}$	1	7	18	34
8	Octagonal	$\frac{6n^2-4n}{2}$	1	8	21	40
9	Nonagonal	$\frac{7n^2-5n}{2}$	1	9	24	46
10	Decagonal	$\frac{8n^2-6n}{2}$	1	10	27	52

Table 1: Table of Values of Polygonal Numbers

In Table 1, we summarize the numerical data for the first four terms of the ten polygonal numbers.

The sums of the reciprocals of polygonal numbers has been obtained by other authors, [3]-[5]. The results in Ref. [5] require detailed comments as follows. The approach in Ref. [5] is entirely different from the one presented here. Firstly, the authors in Ref. [5] focus on the finite sums using Harmonic numbers and related functions (generalized Harmonic numbers, generalized Harmonic function and error function). Secondly, the authors extend their results on ‘finite sums’ to ‘infinite sums’ using the ‘integral representation’ of the latter. The results in Ref. [5] are essentially confined to the *sums of the squares of reciprocals of polygonal numbers*. As for the sums of the ‘higher powers’ of reciprocals of polygonal numbers, they present a recipe for the *finite* case using partial fractions with the aid of the symbolic package *Mathematica*. However, this recipe is *not* used to derive any expressions for specific cases. As for the *infinite* sums of the ‘higher powers’ of reciprocals of polygonal numbers, there is only a conjecture, stating that “the said sum has a closed form structurally similar to the sums of the squares of reciprocals of polygonal numbers”. Whereas in the present study, results are presented in detail for the sums of ‘higher powers’ of reciprocals of polygonal numbers. The results presented in this article do validate the conjecture in Ref. [5].

In Section 2, we shall give the formulae for the sums of the reciprocals of triangular, square and other low-order polygonal numbers. Then, we shall derive the general result for the *sums of the reciprocals of polygonal numbers*. In Section 3, we shall derive the general result for the *sums of the squares of reciprocals of polygonal numbers*. Section 4 has the general result for the

sums of the cubes of reciprocals of polygonal numbers. Section 5 has the general result for the *sums of the fourth-powers of reciprocals of polygonal numbers.* In Section 6, we shall address the question of extending our results to the *sums of the powers of reciprocals of polygonal numbers.* A general result is obtained for the *sums of the powers of reciprocals of square numbers.* In the same section, we outline the procedure to obtain the result for an arbitrary power. Section 7, our final section has a discussion for the possible generalizations along with our concluding remarks.

2. Sums of the reciprocals of polygonal numbers

Let us examine the sums of the reciprocals of the first few sequences of polygonal numbers. We start with the triangular numbers

$$\begin{aligned}
 S_3 &= \sum_{n=1}^{\infty} \frac{1}{T_n} \\
 &= \sum_{n=1}^{\infty} \frac{2}{n^2 + n} \\
 &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \\
 &= 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
 &= 2.
 \end{aligned} \tag{7}$$

The series for S_3 is a telescoping series and is summed straightaway [6]. Next is the sum of the reciprocals of square numbers

$$\begin{aligned}
 S_4 &= \sum_{n=1}^{\infty} \frac{1}{P(4, n)} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
 &= \zeta(2) \\
 &= \frac{\pi^2}{6}.
 \end{aligned} \tag{8}$$

The sum for S_4 is the famous result due to Euler. It can be obtained using the Fourier series (essentially, the Fourier expansion of x^2 , see [7]). It can

also be obtained from the more general result for the even-order *Riemann-Zeta function* [8]. The Riemann-Zeta function is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \quad (9)$$

Next, we consider the case of the pentagonal numbers and have

$$\begin{aligned} S_5 &= \sum_{n=1}^{\infty} \frac{1}{P(5, n)} \\ &= \sum_{n=1}^{\infty} \frac{2}{3n^2 - n} \\ &= 3 \ln 3 - \frac{\sqrt{3}}{3} \pi. \end{aligned} \quad (10)$$

Lastly, we have the case of the hexagonal numbers

$$\begin{aligned} S_6 &= \sum_{n=1}^{\infty} \frac{1}{P(6, n)} \\ &= \sum_{n=1}^{\infty} \frac{2}{4n^2 - 2n} \\ &= 2 \ln 2. \end{aligned} \quad (11)$$

Additional sums can be evaluated with the help of mathematical handbooks [9]-[11]. Evaluating each of the individual sums S_m is an endless process. The sums are convergent and for arbitrarily large m , they converge to unity. We note the following inequalities

$$\begin{aligned} 1 &< S_s \leq S_3 = 2, \\ S_s &> S_{s'}, \quad \text{if } s < s'. \end{aligned} \quad (12)$$

The general formula for S_s is

$$\begin{aligned} S_s &= \sum_{n=1}^{\infty} \frac{1}{P(s, n)} \\ &= \sum_{n=1}^{\infty} \frac{2}{(s-2)n^2 - (s-4)n} \end{aligned}$$

$$= b \sum_{n=1}^{\infty} \frac{1}{n(n-a)}, \quad (13)$$

where

$$\begin{aligned} a &= \frac{s-4}{s-2}, \quad s \neq 4, \\ b &= \frac{2}{s-4}a = \frac{2}{s-2}, \end{aligned} \quad (14)$$

noting that, $m \geq 3$ by definition of polygonal numbers. The case $s = 4$ leads to the square numbers, whose result is already known from Eq. (8). The sum written compactly in Eq. (13) is related to the Gamma function and its logarithmic derivative known as the Digamma function [12]. The Gamma function is defined by

$$\Gamma(t) = \int_0^{\infty} z^{t-1} e^{-z} dz. \quad (15)$$

The Gamma function obeys the recurrence relation

$$\Gamma(t+1) = t\Gamma(t), \quad (16)$$

with $\Gamma(1) = 1$. For $t = n$, we obtain $\Gamma(n+1) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$, the very familiar factorial function. The Digamma function is defined as the logarithmic derivative of the Gamma function

$$\psi(z) = \frac{d}{dz} \ln(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (17)$$

The recurrence relation for the Gamma function in (16) leads to the recurrence relation for the Digamma function

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \quad (18)$$

The series representation for the Digamma function is

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \quad z \neq -1, -2, -3, \dots, \quad (19)$$

where $-\psi(1) = \gamma = 0.577216\dots$ is the well-known Euler-Mascheroni constant [13] given by

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right). \quad (20)$$

There is also the integral representation given by

$$\gamma = - \int_0^{\infty} e^{-x} \ln x dx. \quad (21)$$

For $s \geq 5$, the quantity $a = (s-4)/(s-2)$ is never an integer. Hence, we can use the value $z = -a = -(s-4)/(s-2)$. This leads to the result

$$\psi(1-a) + \gamma = -a \sum_{n=1}^{\infty} \frac{1}{n(n-a)} = -\frac{a}{b} S_s. \quad (22)$$

Consequently,

$$\begin{aligned} S_s &= \sum_{n=1}^{\infty} \frac{1}{P(s, n)} \\ &= -\frac{2}{s-4} \left[\gamma + \psi \left(\frac{2}{s-2} \right) \right]. \end{aligned} \quad (23)$$

Thus, we have expressed the required sum S_s in a closed form expression, using the Digamma function. This result is also available in [3]. The advantage of using the Digamma function is that it has been studied extensively [12]. The Digamma function can be expressed in a closed form for many values of its arguments. This is precisely the reason, why sums up to S_6 (and higher-orders) can be deduced in simple forms from the more general result presented here. In Eq. (18), we noted the recurrence relation for Digamma function. It would be useful, if additional recurrence relations could be obtained relating $\psi \left(\frac{2}{s-2} \right)$ to $\psi \left(\frac{2}{s'-2} \right)$, where $s' = s+1$. This would enable us to relate $S_{s'} = S_{s+1}$ to S_s . There have been attempts to get the sums of reciprocals of polygonal numbers by using the integral representation of the series in Eq. (13). But such studies [4, 5] have not been able to obtain the closed form expressions, we have in this article.

3. Sums of the squares of reciprocals of polygonal numbers

In the preceding section, we derived the general formula for the *sums of the reciprocals of polygonal numbers* in a neat closed form using digamma function. Can this be done for higher powers? The answer is in the affirmative for the case when the power m has a finite value. This shall be explicitly demonstrated for $m = 2$, $m = 3$ and $m = 4$ in this article along with an outline for higher order powers. In this section, we shall derive the general formula for $m = 2$, which is the *sum of the squares of the reciprocals of polygonal numbers*. As in the preceding section, we shall first do the cases of triangular and square numbers and then extend the case to higher-order polygonal numbers.

Let us start with the case of the *sum of the squares of the reciprocals of triangular numbers*. The corresponding sequence is 1, 9, 36, 100, ..., $\left(\frac{1}{2}n(n+1)\right)^2$ designated as Sequence A000537 in the OEIS. The required sum denoted by $S_3^{(2)}$ is

$$\begin{aligned}
 S_3^{(2)} &= \sum_{n=1}^{\infty} \frac{1}{T_n^2} \\
 &= \sum_{n=1}^{\infty} \frac{1}{[P(3, n)]^2} \\
 &= \sum_{n=1}^{\infty} \frac{2^2}{[n(n+1)]^2} \\
 &= 2^2 \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{(n+1)} \right\}^2 \\
 &= 2^2 \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)} \right\} \\
 &= 2^2 \left\{ \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \right\} \\
 &= 2^2 \{ \zeta(2) + (-1 + \zeta(2)) - 2 \} \\
 &= 2^2 \left\{ \frac{\pi^2}{6} + \left(-1 + \frac{\pi^2}{6} \right) - 2 \right\} \\
 &= 4 \left(\frac{\pi^2}{3} - 3 \right) \\
 &= \frac{4}{3} \pi^2 - 12.
 \end{aligned} \tag{24}$$

We have used the technique of partial fractions in our derivations. The first and the second sums in the parenthesis are obtained from S_4 in Eq. (8). The third sum is obtained from S_3 in Eq. (7).

Now, we shall consider the case of the *sum of the squares of the reciprocals of square numbers*. The corresponding sequence is 1, 16, 81, ..., n^4 (Sequence A000583 in OEIS). The required sum denoted by $S_4^{(2)}$ is

$$\begin{aligned}
 S_4^{(2)} &= \sum_{n=1}^{\infty} \frac{1}{[P(4, n)]^2} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^4} \\
 &= \zeta(4) \\
 &= \frac{\pi^4}{90}.
 \end{aligned} \tag{25}$$

The series in Eq. (25) is recognized as the $\zeta(4)$, where $\zeta(z)$ is the Riemann zeta function [8].

The general formula for $S_s^{(2)}$ is

$$\begin{aligned}
 S_s^{(2)} &= \sum_{n=1}^{\infty} \frac{1}{[P(s, n)]^2} \\
 &= \sum_{n=1}^{\infty} \frac{2^2}{[(s-2)n^2 - (s-4)n]^2} \\
 &= b^2 \sum_{n=1}^{\infty} \frac{1}{[n(n-a)]^2} \\
 &= b^2 \sum_{n=1}^{\infty} \left\{ \frac{1}{a^2} \left(\frac{1}{n-a} - \frac{1}{n} \right)^2 \right\} \\
 &= \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} + \frac{1}{(n-a)^2} - \frac{2}{n(n-a)} \right\} \\
 &= \frac{b^2}{a^2} \left\{ \zeta(2) + \left(-\frac{1}{a^2} + \zeta(2, -a) \right) - \frac{2}{b} S_s \right\} \\
 &= \frac{b^2}{a^2} \left\{ \frac{\pi^2}{6} + \left(-\frac{1}{a^2} + \zeta(2, -a) \right) + \frac{2}{a} (\gamma + \psi(1-a)) \right\}, \tag{26}
 \end{aligned}$$

where a and b have been defined in Eq. (14). The first sum is familiar to us from S_4 in Eq. (8). The third sum is obtained from S_s in Eq. (22). As for the

second sum, it is expressed using the *Hurwitz Zeta function* [14] defined as

$$\zeta(c, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^c}. \quad (27)$$

The Hurwitz Zeta function is one of the several generalizations of the *Riemann Zeta function* in Eq. (9).

4. Sums of the cubes of reciprocals of polygonal numbers

In the preceding sections, we derived the general formulae for the *sums of the reciprocals of polygonal numbers* and the *sums of the squares of reciprocals of polygonal numbers*. In this section, we shall address the question of the *sums of the cubes of reciprocals of polygonal numbers*. We shall follow the procedure very similar to the one used for deriving $S_s^{(2)}$ in Eq. (26). It is as follows

$$\begin{aligned} S_s^{(3)} &= \sum_{n=1}^{\infty} \frac{1}{[P(s, n)]^3} \\ &= \sum_{n=1}^{\infty} \frac{2^3}{[(s-2)n^2 - (s-4)n]^3} \\ &= b^3 \sum_{n=1}^{\infty} \frac{1}{[n(n-a)]^3} \\ &= b^3 \sum_{n=1}^{\infty} \left\{ \frac{1}{a^3} \left(\frac{1}{n-a} - \frac{1}{n} \right)^3 \right\} \\ &= \frac{b^3}{a^3} \sum_{n=1}^{\infty} \left\{ -\frac{1}{n^3} + \frac{1}{(n-a)^3} - 3a \frac{1}{[n(n-a)]^2} \right\} \\ &= \frac{b^3}{a^3} \left\{ -\zeta(3) + \left(-\frac{1}{a^3} + \zeta(3, -a) \right) - \frac{3a}{b^2} S_s^{(2)} \right\}. \end{aligned} \quad (28)$$

Further simplification can be made when required.

5. Sums of the fourth-powers of reciprocals of polygonal numbers

In the preceding sections, we derived the general formulae for the *sums of the reciprocals of polygonal numbers*, the *sums of the squares of reciprocals of*

polygonal numbers and the *sums of the cubes of reciprocals of polygonal numbers*. In this section, we shall address the question of the *sums of the fourth-powers of reciprocals of polygonal numbers*. We shall follow the procedure very similar to the one used for deriving $S_s^{(2)}$ in Eq. (26) and $S_s^{(3)}$ in Eq. (28). It is as follows:

$$\begin{aligned}
 S_s^{(4)} &= \sum_{n=1}^{\infty} \frac{1}{[P(s, n)]^4} \\
 &= \sum_{n=1}^{\infty} \frac{2^4}{[(s-2)n^2 - (s-4)n]^4} \\
 &= b^4 \sum_{n=1}^{\infty} \frac{1}{[n(n-a)]^4} \\
 &= b^4 \sum_{n=1}^{\infty} \left\{ \frac{1}{a^4} \left(\frac{1}{n-a} - \frac{1}{n} \right)^4 \right\} \\
 &= \frac{b^4}{a^4} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^4} + \frac{1}{(n-a)^4} - 4a^2 \frac{1}{[n(n-a)]^3} - 2 \frac{1}{[n(n-a)]^2} \right\} \\
 &= \frac{b^4}{a^4} \left\{ \zeta(4) + \left(-\frac{1}{a^4} + \zeta(4, -a) \right) - \frac{4a^2}{b^3} S_s^{(3)} - \frac{2}{b^2} S_s^{(2)} \right\}. \quad (29)
 \end{aligned}$$

Again, we can do simplifications if needed.

6. Sums of the powers of reciprocals of polygonal numbers

In the preceding sections, we derived the general formulae for the *sums of the reciprocals of polygonal numbers*, the *sums of the squares of reciprocals of polygonal numbers*, the *sums of the cubes of reciprocals of polygonal numbers* and the *sums of the fourth-powers of reciprocals of polygonal numbers*. In this section, we shall address the question of the *sums of the powers of reciprocals of polygonal numbers*. In the preceding derivations, we noticed that square numbers have the simplest formula for the n -th term. So, we shall first analyze the case of the *sums of the powers of reciprocals of square numbers*. The general formula for $S_4^{(m)}$ is

$$\begin{aligned}
 S_4^{(m)} &= \sum_{n=1}^{\infty} \frac{1}{[P(4, n)]^m} \\
 &= \sum_{n=1}^{\infty} \frac{1}{(n^2)^m}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n^{2m}} \\
&= \zeta(2m) \\
&= \frac{(-1)^{m+1} B_{2m} (2\pi)^{2m}}{2(2m)!}, \tag{30}
\end{aligned}$$

where B_{2m} are the *Bernoulli numbers* [15]. The even-order Riemann Zeta function $\zeta(2m)$ can be expressed in a closed form as a rational multiple of π^{2m} using the Bernoulli numbers. Analogous results are *not* known for any of the odd-order Riemann Zeta function $\zeta(2m+1)$.

Let us now examine the more general summation for the m -th power of a s -gonal numbers denoted by $S_s^{(m)}$, which is

$$\begin{aligned}
S_s^{(m)} &= \sum_{n=1}^{\infty} \frac{1}{[P(s, n)]^m} \\
&= \sum_{n=1}^{\infty} \frac{2^m}{[(s-2)n^2 - (s-4)n]^m} \\
&= b^m \sum_{n=1}^{\infty} \frac{1}{[n(n-a)]^m} \\
&= b^m \sum_{n=1}^{\infty} \left\{ \frac{1}{a^m} \left(\frac{1}{n-a} - \frac{1}{n} \right)^m \right\} \\
&= \frac{b^m}{a^m} \left[(-1)^m \sum_{n=1}^{\infty} \frac{1}{n^m} + \sum_{n=1}^{\infty} \frac{1}{(n-a)^m} + \sum_{n=1}^{\infty} R_{in} \right] \\
&= \frac{b^m}{a^m} \left\{ (-1)^m \zeta(m) + \left(-\frac{1}{a^m} + \zeta(m, -a) \right) + R_{out} \right\}. \tag{31}
\end{aligned}$$

The general formula for $S_s^{(m)}$ is not yet fully determined. The quantity R_{in} is the remainder in the binomial expansion as follows

$$\begin{aligned}
R_{in} &= \left(\frac{1}{n-a} - \frac{1}{n} \right)^m - (-1)^m \frac{1}{n^m} - \frac{1}{(n-a)^m} \\
&= \sum_{i=1}^{m-1} {}^m C_i \frac{1}{[n^i (n-a)^{m-1-i}]}. \tag{32}
\end{aligned}$$

Note, that ${}^m C_i$ are the binomial coefficients. Next, we have to express this remainder in powers of $1/[n(n-a)]$ using partial fractions leading to the expression

$$\begin{aligned} R_{\text{out}} &= \sum_{i=1}^{m-1} c_i \frac{1}{[n(n-a)]^i} \\ &= \sum_{i=1}^{m-1} d_i S_s^{(i)}, \end{aligned} \quad (33)$$

where c_i and d_i are numerical constants. Note, that $S_s^{(1)} \equiv S_s$. This will enable us to express the final remainder R_{out} in lower orders of $S_s^{(m)}$. This has been demonstrated in the derivation of $S_s^{(2)}$, $S_s^{(3)}$ and $S_s^{(4)}$ respectively. The results in Eqs. (31)-(33) validate the conjecture on the form of $S_s^{(m)}$ in Ref. [5].

7. Concluding remarks

We have obtained the general formulae for the the *sums of the reciprocals of polygonal numbers*; the *sums of the squares of reciprocals of polygonal numbers*; the *sums of the cubes of reciprocals of polygonal numbers*; and the *sums of the fourth-powers of reciprocals of polygonal numbers*. The aforementioned sums were expressed using the Digamma function, the Riemann Zeta function and the Hurwitz Zeta function. As for the higher powers, the special case of the square numbers was done in complete generality and we found that $S_4^{(m)} = \zeta(2m)$, where $\zeta(2m)$ is the Riemann Zeta function [8], which can be neatly expressed using the Bernoulli numbers. The procedure used for obtaining the sums up to $m = 4$ can be applied to higher powers. This would require tedious algebraic manipulations using the binomial expansion of $\left(\frac{1}{n-a} - \frac{1}{n}\right)^m$ expressed in powers of $1/[n(n-a)]$, using continued fractions or *otherwise*. We have used this algorithm in our derivations. We have outlined the procedure to work towards the very general expression for the $S_s^{(m)}$.

One can explore obtaining the recurrence relations for $S_s^{(m)}$ in terms of lower order $S_s^{(m)}$. This may be possible using the integral representations of the series we have. This has been tried for the restricted case of the *sums of the reciprocals of polygonal numbers* [4]. Another interesting line of tackling the problem for $S_s^{(m)}$ would be classify the sums into some basic classes. Then use these basic classes to express the higher order sums. This would be analogous to the classification of the quadratic surfaces into seventeen basic forms [16]-[18].

Some of the numerical data can also be obtained using the *Microsoft Excel* [19]-[24]. One can alternately use the powerful symbolic packages, for instance the *Mathematica* [25, 26]. MS Excel is useful in different areas of physics [20]-[23]. It has also found applications in specific problems such as the study of quadratic surfaces in the laboratory [27]-[30]; resistor networks [31]-[34]; chemical physics [35]; and number theory [36].

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