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CONFIDENCE INTERVALS FOR THE SCALE PARAMETER OF A TWO-PARAMETER WEIBULL DISTRIBUTION: ONE SAMPLE PROBLEM

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Abstract: The problem of interval estimating for the scale parameter θ in a two parameter Weibull distribution is addressed. The pivotal quantities whose percentiles can be used to construct confidence limits for the scale parameter θ are derived. Therefore in this paper, an exact, asymptotic and approximate $(1-\alpha)100\%$ confidence intervals for the scale parameter θ of the two parameter Weibull distribution for the case of the one sample problem are derived. The three confidence intervals are simple and easy to compute. A Monte Carlo simulation study is performed to compare the efficiencies of the three confidence interval methods in terms of two criteria, coverage probabilities and average widths. The simulation results showed that the proposed confidence intervals perform well in terms of coverage probability and average width. Additionally, when the three methods are compared, it is found that the performance of the method depends on the value of the shape parameter β, scale parameters θ and sample size n used. The three methods are illustrated using a real-life data set which also supported the findings of the simulation study to some extent.

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1. Introduction

The Weibull distribution was introduced several decades ago by Waalobi Weibull [1]. It is a flexible distribution that can encompass characteristics of several other distributions. For example, it approximates the normal distribution as the shape parameter is about 3.6 in which case the skewness become zero, [2]. Also, it becomes the exponential and Rayleigh distributions when the shape parameter is equal to one and two, respectively [3]. This property has given rise to widespread applications. The Weibull distribution has many applications in statistics and other areas. For further details on applications of the Weibull distribution, we refer the readers to, for example; [4], [5], [6] and [7].

The general theory of confidence interval estimation was developed by [8] and widely used technique of constructing a confidence interval (CI) of the parameter for a probability distribution is based on the pivotal quantities approach which determines what is known as an exact confidence interval as mentioned by [9] and [10]. The pivotal quantity method is valid for any sample size n as mentioned by [11], [12] and [13]. Therefore, a confidence interval (CI) can be defined as a range of values that gives the user for a sense of precise statistic estimates of the parameter, [14]. When a large sample size n is applied, an asymptotic confidence interval is mostly used to construct a sequence of the estimator $\hat{\theta}_n$ of θ with a probability density function $f(.;\theta)$ that is asymptotically normally distributed with mean θ and variance $\sigma_n^2(\theta)$ ([15], [16], [17]).

Because of its importance, many estimation methods have been proposed for the Weibull distribution for both complete and censored samples data. Recently, many main estimation methods have been proposed by many authors. The most common estimation method is the maximum likelihood estimation (MLE) which has attractive efficiency properties and is asymptotically unbiased. The use of the proposed estimation methods depends on the area of application. For further details on the main estimation methods, we refer the readers to [18], [7], [19], [20] and [21] among others.

In this paper, we derive an exact, asymptotic and approximate $(1-\alpha)100\%$ confidence intervals for the scale parameter (θ) of the two parameter Weibull distribution for the case of the one sample problem using pivotal-based approach. The evaluation of the efficiency for these proposed confidence intervals will be proved via conducting an extensive Monte-Carlo simulation study to compare the coverage probability (CP) and the average width (AW). Furthermore, the three methods will be illustrated using a real-life data in order to demonstrate how the proposed confidence intervals can be applied in practice and support the findings of the simulation study.

The structure for the rest of this paper is organized as follows: In Section 2, materials and methods are discussed. In Section 3, the three proposed confidence interval methods for the scale parameter (θ) of the two parameter Weibull distribution are derived. A Monte-Carlo simulation study has been conducted in Section 4. In Section 5, a real-life data are analyzed to illustrate the implementation of the methods. Finally, some concluding remarks are presented in Section 6.

2. Materials and methods

In this section, we will discuss the criteria for the efficiency comparison among the considered confidence intervals and the essential conditions for the work in this study. In addition, we will derive the pivotal quantity that will be used in this study to construct the proposed $(1-\alpha)100\%$ confidence interval (CI) for the population mean of the one parameter exponential distribution.

2.1. Criteria for the efficiency comparison

The efficiency comparison criteria among the three estimation methods of the $(1-\alpha)100\%$ confidence intervals are the coverage probability (CP) and the average width (AW) of the resulting confidence intervals. It is acknowledged that the CP and AW are useful criteria for evaluating the confidence intervals. Let CI=(L($\mathbf{X}$),U($\mathbf{X}$)) be a confidence interval of a parameter θ based on the data $\mathbf{X}$ having the nominal $(1-\alpha)100\%$ confidence level, where L($\mathbf{X}$) and U($\mathbf{X}$), respectively, are the lower and upper endpoints of this confidence interval. The following definitions provide the efficiency comparison criteria in this study:

Definition 1. The coverage probability (CP) associated with a confidence interval CI=(L($\mathbf{X}$),U($\mathbf{X}$)) for the unknown parameter θ of a probability density function $f(x;\theta)$ is measured by $P_\theta\{\theta \in (L(\mathbf{X}), U(\mathbf{X}))\}$ (see [16]).

Definition 2. The length of a confidence interval, W=U($\mathbf{X}$) - L($\mathbf{X}$), is simply the difference between the upper U(**X**) and lower L(**X**) endpoints of a confidence interval CI=(L(**X**),U(**X**)). The expected length of a confidence interval CI=(L(**X**),U(**X**)) is given by $E_\theta(W)$ (see [22], [23], [24]).

2.2. Essential conditions for the study

Throughout the following discussion, the essential conditions for the work in this study are denoted by **(C1)–(C3)** and will be given as follows:

(C1) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population of two parameter Weibull distribution with shape parameter β (known) and scale parameter θ such that β and $\theta \in \Omega$ where $\Omega = \{(\beta, \theta) : \theta < \beta < \infty; 0 < \theta < \infty\}$. The probability density function (pdf) of the two parameter Weibull random variable X is given by equation (1) below:

$$f(x;\beta,\theta) = \begin{cases} \dfrac{\beta}{\theta}\; x^{\beta-1}\; e^{-\dfrac{x^\beta}{\theta}} & ; \quad x > 0,\;\; \beta > 0,\;\; \theta > 0, \\ 0 & ; \quad \text{Otherwise.} \end{cases} \tag{1}$$

The cumulative distribution function (CDF) of the two parameter Weibull distribution with shape parameter β (known) and scale parameter θ is given by equation (2) below:

$$F(x;\beta,\theta) = P(X \le x) = \begin{cases} 1 - e^{-\dfrac{x^\beta}{\theta}} & ; \;\; x \ge 0,\;\; \beta > 0,\;\; \theta > 0, \\ 0 & ; \quad \text{Otherwise.} \end{cases} \tag{2}$$

For X has two parameter Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$, we have:

(i)

$$\mu = E(X) = \theta^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) \tag{3}$$

(ii)

$$\sigma^2 = Var(X) = \theta^{\frac{2}{\beta}} \left(\Gamma\left(\frac{2}{\beta} + 1\right) - \left(\Gamma\left(\frac{1}{\beta} + 1\right)\right)^2\right) \tag{4}$$

(iii)

$$\sigma = SD(X) = \sqrt{Var(X)} = \sqrt{\theta^{\frac{2}{\beta}} \left(\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2 \right)}$$
$$= \theta^{\frac{1}{\beta}} \sqrt{\left(\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2 \right)} \quad (5)$$

(iv) The theoretical coefficient of variation (γ), which is a useful indicator, is obtained as:

$$\gamma = \frac{\sigma}{\mu} = \frac{\theta^{\frac{1}{\beta}} \sqrt{\left(\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2 \right)}}{\theta^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta}+1\right)}$$
$$= \frac{\sqrt{\left(\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2 \right)}}{\Gamma\left(\frac{1}{\beta}+1\right)} \quad (6)$$

(C2) Let $\chi^2_{(\frac{\alpha}{2},2n)}$ and $\chi^2_{(1-\frac{\alpha}{2},2n)}$, respectively, be the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles points (quantiles) of the chi-square distribution with $2n$ degrees of freedom where $n > 0$.

(C3) Let $Z_{\frac{\alpha}{2}}$ and $Z_{1-\frac{\alpha}{2}}$, respectively, be the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles points (quantiles) of the standard normal distribution, $Z \sim N(0,1)$, which satisfy the following relation: $P(|Z| < Z_{1-\frac{\alpha}{2}}) = P(-Z_{1-\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}}) = P(Z_{\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}}) = 1-\alpha$.

2.3. The pivotal quantity derivation for the exact and approximate methods

In this section, we will derive the pivotal quantities for the exact and approximate confidence interval methods considered in this paper.

Definition 3. If $Q = q(X_1, X_2, \ldots, X_n; \theta)$ is a random variable that is a function only of $X_1, X_2, \ldots, X_n$ and θ, then Q is called a pivotal quantity if its probability distribution does not depend on θ or any other unknown parameter (see [25], page 363).

2.3.1. The pivotal quantity derivation for the exact method

In this section, we will derive the pivotal quantity that will be used later to construct the exact $(1-\alpha)100\%$ confidence interval (CI) for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ, that is $Weibull(\beta, \theta^{\frac{1}{\beta}})$.

Definition 4. If $X_i \sim f(x_i;\theta)$ and if $F(x;\theta)$ is the cumulative distribution function (CDF) of X_i , then $1-F(x_i;\theta) \sim Uniform(\theta, 1)$, and consequently for a random sample of size n; $X_1, X_2, \ldots, X_n$; it follows that the pivotal quantity:

$$Q = q(X_1, X_2, \ldots, X_n; \theta) = -2\sum_{i=1}^{n} \ln[1 - F(x_i;\theta)] \sim \chi^2_{(2n)} \tag{7}$$

(see [25], page 366).

Lemma 2.1. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables from a Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X_i \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$, and if $F(x;\beta,\theta) = 1 - e^{-x^\beta/\theta}$ is the cumulative distribution function (CDF) of X_i, then the pivotal quantity is given by $Q = q(X_1, X_2, \ldots, X_n; \beta, \theta) = \frac{2}{\theta}\sum_{i=1}^{n} X_i^\beta \sim \chi^2_{(2n)}$.*

Proof. To prove this, we use Definition 4 as follows:

$$\begin{aligned} Q = q(X_1, X_2, \ldots, X_n; \beta, \theta) &= -2\sum_{i=1}^{n} \ln[1 - F(X_i;\beta,\theta)] \\ = -2\sum_{i=1}^{n} \ln\left[1 - \left(1 - e^{-X_i^\beta/\theta}\right)\right] &= -2\sum_{i=1}^{n} \ln e^{-X_i^\beta/\theta} \\ = -2\sum_{i=1}^{n} -\frac{X_i^\beta}{\theta} &= \frac{2}{\theta}\sum_{i=1}^{n} X_i^\beta \sim \chi^2_{(2n)}. \end{aligned} \tag{8}$$

2.3.2. The pivotal quantity derivation for the approximate method

In this section, we will derive the pivotal quantity based on the suggestions given by [2] and [26]. This pivotal quantity will be used in this study to construct the proposed $(1-\alpha)100\%$ approximate confidence interval (CI) for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ, that is $Weibull(\beta, \theta^{\frac{1}{\beta}})$. Let X be a random variable from a gamma distribution with the shape and scale parameters are β (known) and θ, respectively, that is $X \sim Gamma(\beta, \theta)$. The probability density function (pdf) of the random variable X is given by equation (9) below:

$$f(x;\beta,\theta) = \begin{cases} \dfrac{1}{\theta^{\beta}\Gamma(\beta)} \; x^{\beta-1} \; e^{-\dfrac{x}{\theta}} & ; \quad x > 0, \;\; \beta > 0, \;\; \theta > 0, \\ 0 & ; \quad \text{Otherwise,} \end{cases} \tag{9}$$

where $\Gamma(x) =$ The gamma function $= \int_0^\infty t^{x-1}e^{-t}dt$. When the shape parameter $\beta = 1$, the gamma distribution reduces to the one parameter exponential distribution with a scale parameter θ, that is $X \sim Exp(\theta)$. According to [2] when X follows an exponential distribution with mean θ, that is $X \sim Exp(\theta)$, the power transformation $X^{\frac{1}{\beta}}$ has a Weibull distribution with shape parameter β (known) and scale parameter θ. That is,

$$X^* = X^{\frac{1}{\beta}} \sim Weibull(\beta, \theta^{\frac{1}{\beta}}). \tag{10}$$

According to [26], the use of $\beta = 3.6$ makes a good approximation to a normal curve, then $X^* = X^{1/3.6}$ is approximately normally distributed with mean, variance and standard deviation that can be given as follows:

$$\mu_{X^*} = E(X^*) = E(X^{\frac{1}{\beta}}) = \theta^{\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta})$$

$$= \theta^{\frac{1}{3.6}}\Gamma(1+\frac{1}{3.6}) = \theta^{\frac{1}{3.6}}\Gamma\left(\frac{4.6}{3.6}\right) = 0.90111\;\theta^{\frac{1}{3.6}}, \tag{11}$$

$$\sigma^2_{X^*} = Var(X^*) = Var(X^{\frac{1}{\beta}}) = \theta^{\frac{2}{\beta}}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right]$$

$$= \theta^{\frac{2}{3.6}}\left[\Gamma\left(1+\frac{2}{3.6}\right) - \left(\Gamma\left(1+\frac{1}{3.6}\right)\right)^2\right]$$

$$= \theta^{\frac{2}{3.6}}\left[\Gamma\left(\frac{5.6}{3.6}\right) - \left(\Gamma\left(\frac{4.6}{3.6}\right)\right)^2\right]$$

$$= \theta^{\frac{2}{3.6}}\left[0.88929 - (0.90111)^2\right] = 0.07729\;\theta^{\frac{2}{3.6}}, \tag{12}$$

$$\sigma_{X^*} = SD(X^*) = \sqrt{\sigma^2_{X^*}} = \sqrt{0.07729\;\theta^{\frac{2}{3.6}}} = 0.27801\;\theta^{\frac{1}{3.6}}, \tag{13}$$

that is,

$$X^{*} = X^{\frac{1}{3.6}} \sim N\left(0.90111\,\theta^{\frac{1}{3.6}}, 0.07729\,\theta^{\frac{2}{3.6}}\right), \tag{14}$$

approximately as suggested by [26], and therefore the sampling distribution of the sample mean $(\overline{X}^{*})$ for the power transformed data which given as follows:

$$\overline{X}^{*} = \frac{\sum_{i=1}^{n} X_i^{*}}{n}$$

$$= \frac{\left(X_1^{*} = X_1^{1/3.6}\right) + \left(X_2^{*} = X_2^{1/3.6}\right) + \cdots + \left(X_n^{*} = X_n^{1/3.6}\right)}{n}, \tag{15}$$

will be approximately normally distributed with mean, variance and standard deviation that can be given as follows:

$$\mu_{\overline{X}^{*}} = E(\overline{X}^{*}) = \mu_{X^{*}} = E(X^{\frac{1}{\beta}}) = \theta^{\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta}) = 0.90111\,\theta^{\frac{1}{3.6}}, \tag{16}$$

$$\sigma^2_{\overline{X}^{*}} = Var(\overline{X}^{*}) = \frac{\sigma^2_{X^{*}}}{n} = \frac{0.07729\,\theta^{\frac{2}{3.6}}}{n}, \tag{17}$$

$$\sigma_{\overline{X}^{*}} = SD(\overline{X}^{*}) = \sqrt{\sigma^2_{\overline{X}^{*}}} = \frac{\sigma_{X^{*}}}{\sqrt{n}} = \frac{0.27801\,\theta^{\frac{1}{3.6}}}{\sqrt{n}}, \tag{18}$$

that is,

$$\overline{X}^{*} = \frac{\sum_{i=1}^{n} X_i^{*}}{n} \sim N\left(0.90111\,\theta^{\frac{1}{3.6}}, \frac{0.07729\,\theta^{\frac{2}{3.6}}}{n}\right). \tag{19}$$

Based on the above results, we can modify the result of the central limit theorem regarding the sampling distribution of the sample mean $\overline{X}^{*}$ for the quantity $Z = \dfrac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} \sim N(0,1)$ using the suggested power transformation. The modified Z^{*} using the power transformation $X^{*} = X^{1/3.6}$ is given as follows:

$$Z^{*} = \frac{\overline{X}^{*} - \mu_{\overline{X}^{*}}}{\sigma_{\overline{X}^{*}}} = \frac{\overline{X}^{*} - 0.90111\,\theta^{\frac{1}{3.6}}}{\frac{0.27801\,\theta^{\frac{1}{3.6}}}{\sqrt{n}}}$$

$$= \frac{\sqrt{n}\left(\overline{X}^{*} - 0.90111\,\theta^{\frac{1}{3.6}}\right)}{0.27801\,\theta^{\frac{1}{3.6}}} \sim N(0,1). \tag{20}$$

In this study, the Z^{*} will be the pivotal quantity that will be used in our proposed method to construct the proposed $(1-\alpha)100\%$ approximate confidence interval (CI) for the for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ.

3. The confidence intervals for the scale parameter of the Weibull distribution

In this section, for $0 < \alpha < 1$, the following three methods of $(1-\alpha)100\%$ confidence interval are studied for the efficiency comparisons. They are the three confidence interval methods for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ, namely, the exact method, the asymptotic method and the normal approximation confidence interval method.

3.1. The exact confidence interval for the scale parameter of the Weibull distribution

In this section, we will obtain the $(1-\alpha)100\%$ exact confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ.

Lemma 3.1. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables from a Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X_i \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$, then by using the pivotal quantity $Q = q(X_1, X_2, \ldots, X_n; \beta, \theta) = \frac{2}{\theta}\sum_{i=1}^{n} X_i^{\beta} \sim \chi^2_{(2n)}$, the $(1-\alpha)100\%$ exact confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ will be given by $CI_{Exact} = \left(\frac{2\sum_{i=1}^{n} X_i^{\beta}}{\chi^2_{(1-\alpha/2,2n)}}, \frac{2\sum_{i=1}^{n} X_i^{\beta}}{\chi^2_{(\alpha/2,2n)}}\right)$.*

Proof. To prove this, we need to consider the significance level α based on the relation given in condition (C2) where $\chi^2_{(\alpha/2,2n)}$ and $\chi^2_{(1-\alpha/2,2n)}$ are hold by this condition, then the $(1-\alpha)100\%$ exact confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ can be derived as follows:

$$P\left(\chi^2_{(\alpha/2,2n)} < Q < \chi^2_{(1-\alpha/2,2n)}\right) = 1-\alpha,$$

$$P\left(\chi^2_{(\alpha/2,2n)} < \frac{2}{\theta}\sum_{i=1}^{n} X_i^{\beta} < \chi^2_{(1-\alpha/2,2n)}\right) = 1-\alpha,$$

$$P\left(\frac{\chi^2_{(\alpha/2,2n)}}{2\sum_{i=1}^{n} X_i^{\beta}} < \frac{1}{\theta} < \frac{\chi^2_{(1-\alpha/2,2n)}}{2\sum_{i=1}^{n} X_i^{\beta}}\right) = 1-\alpha,$$

$$P\left(\frac{2\sum_{i=1}^{n}X_i^{\beta}}{\chi^2_{(1-\alpha/2,2n)}} < \theta < \frac{2\sum_{i=1}^{n}X_i^{\beta}}{\chi^2_{(\alpha/2,2n)}}\right) = 1-\alpha. \tag{21}$$

Hence, the $(1-\alpha)100\%$ exact confidence interval for the scale parameter (θ) is given by $CI_{Exact} = \left(\frac{2\sum_{i=1}^{n}X_i^{\beta}}{\chi^2_{(1-\alpha/2,2n)}}, \frac{2\sum_{i=1}^{n}X_i^{\beta}}{\chi^2_{(\alpha/2,2n)}}\right)$.

3.2. The asymptotic confidence interval for the scale parameter of the Weibull distribution

An asymptotic confidence interval is valid only for a sufficiently large sample size (n). This confidence interval is based on a pivotal quantity given by reduced normal random variable $Z = \frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}} \sim N(0,1)$ as $n \to \infty$ where $\hat{\theta}$ is the maximum likelihood estimator (MLE) for the scale parameter (θ) and $\sigma_{\hat{\theta}}$ is the standard error of $\hat{\theta}$. Therefore we need to derive both $\hat{\theta}$ and $\sigma_{\hat{\theta}}$.

Lemma 3.2. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables from a Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X_i \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$, then the maximum likelihood estimator (MLE) of the scale parameter (θ) is $\hat{\theta} = \frac{\sum_{i=1}^{n}X_i^{\beta}}{n}$.*

Proof.

$$f(x;\beta,\theta) = \begin{cases} \frac{\beta}{\theta}\ x^{\beta-1}\ e^{-\frac{x^{\beta}}{\theta}} & ;\quad x>0,\ \ \beta>0,\ \ \theta>0, \\ 0 & ;\quad \text{Otherwise,} \end{cases}$$

$$L(\beta,\theta) = \prod_{i=1}^{n} f(x_i;\beta,\theta)$$

$$= \prod_{i=1}^{n}\frac{\beta}{\theta}\ x_i^{\beta-1}\ e^{-\frac{x_i^{\beta}}{\theta}} = \left(\frac{\beta}{\theta}\right)^n \prod_{i=1}^{n} x_i^{\beta-1}\ e^{-\frac{\sum_i^n x_i^{\beta}}{\theta}},$$

$$\ln L(\beta,\theta) = \ln\left(\frac{\beta}{\theta}\right)^n \prod_{i=1}^{n} x_i^{\beta-1}\ e^{-\frac{\sum_i^n x_i^{\beta}}{\theta}}$$

$$= n\ln\beta - n\ln\theta + (\beta-1)\sum_{i=1}^{n}\ln x_i \; - \frac{\sum_i^n x_i^\beta}{\theta},$$

$$\frac{d\ln L(\beta,\theta)}{d\theta} = 0 \to \frac{-n}{\theta} + \frac{\sum_i^n x_i^\beta}{\theta^2} = 0,$$

$$\frac{-n\theta + \sum_i^n x_i^\beta}{\theta^2} = 0 \to -n\theta + \sum_i^n x_i^\beta = 0,$$

$$\hat{\theta} = \frac{\sum_{i=1}^n x_i^\beta}{n}. \tag{22}$$

Lemma 3.3. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables from a Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X_i \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$, then the maximum likelihood estimator (MLE) of the scale parameter (θ) is $\hat{\theta} = \frac{\sum_{i=1}^n X_i^\beta}{n}$, then the standard error of $\hat{\theta}$ is $\sigma_{\hat{\theta}} = \frac{\theta}{\sqrt{n}}$.*

Proof. To prove that, we need first to find the distribution for the maximum likelihood estimator (MLE) $\hat{\theta}$ by using the transformation method as follows:

$$f(x;\beta,\theta) = \begin{cases} \dfrac{\beta}{\theta}\; x^{\beta-1}\; e^{-\dfrac{x^\beta}{\theta}} & ; \quad x>0, \;\; \beta>0, \;\; \theta>0, \\ 0 & ; \quad \text{Otherwise.} \end{cases}$$

Let $Y = X^\beta$ defines a one-to-one transformation implies that the inverse transformation is $w(y) = x = y^{1/\beta}$ and therefore the derivative (usually called the Jacobian) is $J = w'(y) = \frac{d}{dy}w(y) = \frac{dx}{dy} = \frac{1}{\beta}y^{\frac{1}{\beta}-1}$ is continuous and nonzero on $B = \{y : y > 0\}$ then the probability density function (pdf) of the random variable $Y = X^\beta$ by using the transformation method will be derived as follows:

$$f(y) = f\left(w(y)\right)\left|\frac{d}{dy}w(y)\right|, \quad y \in B,$$

$$f(y) = f\left(y^{\frac{1}{\beta}}\right)\left|\frac{1}{\beta}\; y^{\frac{1}{\beta}-1}\right|, \quad y>0,$$

$$f(y) = \frac{\beta}{\theta}\; y^{\frac{\beta-1}{\beta}}\; e^{-\dfrac{y}{\theta}}\; \frac{1}{\beta}\; y^{\frac{1}{\beta}-1}, \quad y>0,$$

$$f(y) = \frac{1}{\theta}\ e^{-\dfrac{y}{\theta}}, \quad y > 0, \tag{23}$$

that is, $Y = X^\beta \sim Exp(\theta)$, then we can use the moment generating function (mgf) properties for $Y = X^\beta$ to find the standard error of $\hat{\theta}$ as follows:

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f(y)dy = \frac{1}{1-\theta t} = (1-\theta t)^{-1}, \tag{24}$$

but $\hat{\theta} = \frac{\sum_{i=1}^n X_i^\beta}{n} = \frac{\sum_{i=1}^n y_i}{n} = \overline{y}$ and therefore the moment generating function (mgf) for the maximum likelihood estimator (MLE) $\hat{\theta}$ can be derived as follows:

$$\begin{aligned} M_{\hat{\theta}}(t) &= M_{\overline{Y}}(t) = M_{\frac{\sum_{i=1}^n Y_i}{n}}(t) \\ &= \prod_{i=1}^n M_{Y_i}\left(\frac{t}{n}\right) = \left(M_Y\left(\frac{t}{n}\right)\right)^n \\ &= \left(\left(1-\frac{\theta t}{n}\right)^{-1}\right)^n = \left(\left(1-\frac{\theta t}{n}\right)\right)^{-n}, \end{aligned} \tag{25}$$

then

$$E(\hat{\theta}) = M'_{\hat{\theta}}(0) = \theta\left(1-\frac{\theta t}{n}\right)^{-n-1}|_{t=0} = \theta, \tag{26}$$

$$\begin{aligned} E(\hat{\theta}^2) = M''_{\hat{\theta}}(0) &= \frac{n+1}{n}\,\theta^2\left(1-\frac{\theta t}{n}\right)^{-n-2}|_{t=0} \\ &= \frac{n+1}{n}\theta^2 = \left(1+\frac{1}{n}\right)\theta^2, \end{aligned} \tag{27}$$

$$Var(\hat{\theta}) = \sigma^2_{\hat{\theta}} = E(\hat{\theta}^2) - \left(E(\hat{\theta})\right)^2 = \left(1+\frac{1}{n}\right)\theta^2 - \theta^2 = \frac{\theta^2}{n}, \tag{28}$$

and therefore the standard error of $\hat{\theta}$ is given as follows:

$$\sigma_{\hat{\theta}} = \sqrt{\sigma^2_{\hat{\theta}}} = \sqrt{\frac{\theta^2}{n}} = \frac{\theta}{\sqrt{n}}. \tag{29}$$

Lemma 3.4. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables from a Weibull distribution with shape parameter β (known) and scale parameter θ, that is $X_i \sim Weibull(\beta, \theta^{\frac{1}{\beta}})$. If the maximum likelihood estimator (MLE) of the scale parameter (θ) is $\hat{\theta} = \frac{\sum_{i=1}^n X_i^\beta}{n}$ and the standard error of $\hat{\theta}$ is $\sigma_{\hat{\theta}} = \frac{\theta}{\sqrt{n}}$, then by using the pivotal quantity (or* *z-transform)* $Z = \frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}} = \frac{\frac{\sum_{i=1}^{n} X_i^{\beta}}{n} - \theta}{\frac{\theta}{\sqrt{n}}} \sim N(0,1)$ *as* $n \to \infty$*, the* $(1-\alpha)100\%$ *asymptotic (approximate or large sample) confidence interval for the scale parameter* (θ) *of the two parameters Weibull distribution with shape parameter* β *(known) and scale parameter* θ *will be* $CI_{Asymptotic} = \left(\frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{1-\frac{\alpha}{2}}}, \frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{\frac{\alpha}{2}}} \right)$.

Proof. To prove this, we need to consider the significance level α based on the relation given in condition (C3) where $Z_{\frac{\alpha}{2}}$ and $Z_{1-\frac{\alpha}{2}}$ are hold by this condition, then the $(1-\alpha)100\%$ asymptotic confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ can be derived as follows:

$$P(Z_{\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}}) = 1-\alpha,$$

$$P\left(Z_{\frac{\alpha}{2}} < \frac{\frac{\sum_{i=1}^{n} X_i^{\beta}}{n} - \theta}{\frac{\theta}{\sqrt{n}}} < Z_{1-\frac{\alpha}{2}} \right) = 1-\alpha,$$

$$P\left(Z_{\frac{\alpha}{2}} < \frac{\frac{\sum_{i=1}^{n} X_i^{\beta}}{\sqrt{n}} - \sqrt{n}\theta}{\theta} < Z_{1-\frac{\alpha}{2}} \right) = 1-\alpha,$$

$$P\left(Z_{\frac{\alpha}{2}} < \frac{\sum_{i=1}^{n} X_i^{\beta}}{\theta\sqrt{n}} - \sqrt{n} < Z_{1-\frac{\alpha}{2}} \right) = 1-\alpha,$$

$$P\left(\sqrt{n} + Z_{\frac{\alpha}{2}} < \frac{\sum_{i=1}^{n} X_i^{\beta}}{\theta\sqrt{n}} < \sqrt{n} + Z_{1-\frac{\alpha}{2}} \right) = 1-\alpha,$$

$$P\left(\frac{n+\sqrt{n}Z_{\frac{\alpha}{2}}}{\sum_{i=1}^{n} X_i^{\beta}} < \frac{1}{\theta} < \frac{n+\sqrt{n}Z_{1-\frac{\alpha}{2}}}{\sum_{i=1}^{n} X_i^{\beta}} \right) = 1-\alpha,$$

$$P\left(\frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{1-\frac{\alpha}{2}}} < \theta < \frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{\frac{\alpha}{2}}} \right) = 1-\alpha. \tag{30}$$

Hence, the $(1-\alpha)100\%$ asymptotic confidence interval for the scale parameter (θ) is $CI_{Asymptotic} = \left(\frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{1-\frac{\alpha}{2}}}, \frac{\sum_{i=1}^{n} X_i^{\beta}}{n+\sqrt{n}Z_{\frac{\alpha}{2}}} \right)$.

3.3. The approximate confidence interval for the scale parameter of Weibull distribution

In this section, the approximate $(1-\alpha)100\%$ confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ based on the pivotal quantity (Z^*) given in equation (20) is constructed. We will refer to our proposed confidence interval by $CI_{Proposed}$. The proposed $(1-\alpha)100\%$ approximate confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ is stated as follows:

Step 1: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n hold in condition (C1).

Step 2: : Calculate $X^* = X^{1/3.6}$ for the random sample $X_1, X_2, \ldots, X_n$ to get the new random sample $X_1^*, X_2^*, \ldots, X_n^*$, where $X_1^* = X_1^{1/3.6}, X_2^* = X_2^{1/3.6}$, $\ldots, X_n^* = X_n^{1/3.6}$.

Step 3: Calculate the sample mean (X^*) for the transformed data in Step 2 as follows:

$$\overline{X}^* = \frac{\sum_{i=1}^{n} X_i^*}{n}$$

$$= \frac{\left(X_1^* = X_1^{1/3.6}\right) + \left(X_2^* = X_2^{1/3.6}\right) + \cdots + \left(X_n^* = X_n^{1/3.6}\right)}{n}.$$

Step 4: Let $Z_{\frac{\alpha}{2}}$ and $Z_{1-\frac{\alpha}{2}}$ hold in condition (C3).

Step 5: Consider the pivotal quantity $Z^* = \dfrac{\sqrt{n}\left(\overline{X}^* - 0.90111\,\theta^{\frac{1}{3.6}}\right)}{0.27801\,\theta^{\frac{1}{3.6}}}$ which was derived in equation (20) and the significance level α, then based on the relation given in condition (C3), the proposed $(1-\alpha)100\%$ confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ $(CI_{Proposed})$ will be derived as follows:

$$P(Z_{\frac{\alpha}{2}} < Z^* < Z_{1-\frac{\alpha}{2}}) = 1-\alpha,$$

$$P\left(Z_{\frac{\alpha}{2}} < \frac{\sqrt{n}\left(\overline{X}^* - 0.90111\,\theta^{\frac{1}{3.6}}\right)}{0.27801\,\theta^{\frac{1}{3.6}}} < Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

$$P\left(Z_{\frac{\alpha}{2}} < \sqrt{n}\left[\frac{\overline{X}^*}{0.27801\ \theta^{\frac{1}{3.6}}} - 3.24129\right] < Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

$$P\left(\frac{Z_{\frac{\alpha}{2}}}{\sqrt{n}} < \left[\frac{\overline{X}^*}{0.27801\ \theta^{\frac{1}{3.6}}} - 3.24129\right] < \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1-\alpha,$$

$$P\left(\frac{Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 3.24129 < \left[\frac{\overline{X}^*}{0.27801\ \theta^{\frac{1}{3.6}}}\right] < \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 3.24129\right) = 1-\alpha,$$

$$P\left(\frac{\overline{X}^*}{\left(\frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 3.24129\right)(0.27801)} < \theta^{\frac{1}{3.6}} < \frac{\overline{X}^*}{\left(\frac{Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 3.24129\right)(0.27801)}\right) = 1-\alpha,$$

$$P\left(\frac{\overline{X}^*}{\left(\frac{(0.27801)Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111\right)} < \theta^{\frac{1}{3.6}} < \frac{\overline{X}^*}{\left(\frac{(0.27801)Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111\right)}\right) = 1-\alpha,$$

$$P\left(\left[\frac{\overline{X}^*}{\left(\frac{(0.27801)Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111\right)}\right]^{3.6} < \theta < \left[\frac{\overline{X}^*}{\left(\frac{(0.27801)Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111\right)}\right]^{3.6}\right) = 1-\alpha.$$

Hence, the proposed $(1-\alpha)100\%$ approximate confidence interval for the scale parameter (θ) of the two parameters Weibull distribution with shape parameter β (known) and scale parameter θ ($CI_{Proposed}$) is obtained in equation (31),

$$CI_{Proposed} = \left(\left[\frac{\overline{X}^*}{\left(\frac{(0.27801) Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111 \right)} \right]^{3.6}, \left[\frac{\overline{X}^*}{\left(\frac{(0.27801) Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111 \right)} \right]^{3.6} \right), \tag{31}$$

where $Z_{\frac{\alpha}{2}}$ and $Z_{1-\frac{\alpha}{2}}$ hold in condition (C3). Let

$$k_1 = \left(\frac{(0.27801) Z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111 \right)$$

and

$$k_2 = \left(\frac{(0.27801) Z_{\frac{\alpha}{2}}}{\sqrt{n}} + 0.90111 \right)$$

be the two constants, then equation (31) can be simplified in the form of the following equation:

$$CI_{Proposed} = \left(\left[\frac{\overline{X}^*}{k_1} \right]^{3.6}, \left[\frac{\overline{X}^*}{k_2} \right]^{3.6} \right). \tag{32}$$

Now, the constants k_1 and k_2 are required for the most common confidence interval used in real applications, i.e., the confidence level of 95% ($\alpha = 0.05$). Hence, the constants k_1 and k_2 for sample sizes not greater than 100 are provided in Table 1.

4. The simulation study and results

In this section, in order to compare the efficiencies of the three methods for 95% confidence intervals of the scale parameter θ for Weibull distribution, an extensive Monte-Carlo simulation study was conducted by using SAS version 9.4 programming to examine the coverage probabilities (CP) and average widths

Table 1: The values of k_1 and k_2 for confidence level $(1-\alpha)100\% = 95\%$

n	k_1	k_2	n	k_1	k_2	n	k_1	k_2
2	1.28641	0.51581	35	0.99322	0.80901	68	0.96719	0.83503
3	1.21571	0.58651	36	0.99193	0.81029	69	0.96671	0.83551
4	1.17356	0.62866	37	0.99069	0.81153	70	0.96624	0.83598
5	1.14480	0.65742	38	0.98950	0.81272	71	0.96578	0.83644
6	1.12356	0.67866	39	0.98836	0.81386	72	0.96533	0.83689
7	1.10706	0.69516	40	0.98727	0.81495	73	0.96489	0.83733
8	1.09376	0.70846	41	0.98621	0.81601	74	0.96445	0.83777
9	1.08274	0.71948	42	0.98519	0.81703	75	0.96403	0.83819
10	1.07342	0.72880	43	0.98421	0.81801	76	0.96361	0.83861
11	1.06540	0.73682	44	0.98326	0.81896	77	0.96321	0.83901
12	1.05841	0.74381	45	0.98234	0.81988	78	0.96281	0.83941
13	1.05224	0.74998	46	0.98145	0.82077	79	0.96242	0.83980
14	1.04674	0.75548	47	0.98059	0.82163	80	0.96203	0.84019
15	1.04180	0.76042	48	0.97976	0.82246	81	0.96165	0.84057
16	1.03733	0.76489	49	0.97895	0.82327	82	0.96128	0.84094
17	1.03327	0.76895	50	0.97817	0.82405	83	0.96092	0.84130
18	1.02954	0.77268	51	0.97741	0.82481	84	0.96056	0.84166
19	1.02612	0.77610	52	0.97667	0.82555	85	0.96021	0.84201
20	1.02295	0.77927	53	0.97596	0.82626	86	0.95987	0.84235
21	1.02002	0.78220	54	0.97526	0.82696	87	0.95953	0.84269
22	1.01728	0.78494	55	0.97458	0.82764	88	0.95920	0.84302
23	1.01473	0.78749	56	0.97393	0.82830	89	0.95887	0.84335
24	1.01234	0.78988	57	0.97328	0.82897	90	0.95855	0.84367
25	1.01009	0.79213	58	0.97266	0.82956	91	0.95823	0.84399
26	1.00797	0.79425	59	0.97205	0.83017	92	0.95792	0.84430
27	1.00598	0.79624	60	0.97146	0.83076	93	0.95761	0.84461
28	1.00409	0.79813	61	0.97088	0.83134	94	0.95731	0.84491
29	1.00230	0.79993	62	0.97031	0.83191	95	0.95702	0.84520
30	1.00059	0.80163	63	0.96976	0.83246	96	0.95672	0.84550
31	0.99898	0.80324	64	0.96922	0.83300	97	0.95644	0.84578
32	0.99744	0.80478	65	0.96870	0.83352	98	0.95615	0.84607
33	0.99596	0.80626	66	0.96818	0.83404	99	0.95587	0.84635
34	0.99456	0.80766	67	0.96768	0.83454	100	0.95560	0.84662

(AW) of the three confidence intervals. Twenty-four populations of Weibull distribution with shape parameter ($\beta = 1.0$, 1.5, 3.5, 10.0) and scale parameter ($\theta = 1.5$, 2.0, 2.5, 3.0, 3.5, 4.0) were each generated of the size $N = 100{,}000$. For each population, the sample sizes of $n = 5$, 10, 20, 40, 50 were randomly generated 50,000 times. For each set of samples, the common 95% confidence intervals of parameter θ were constructed for the three methods. The coverage probability (CP) and the average width (AW) are obtained by using the following two formulas:

$$CP = \frac{\#(L \leq \theta \leq U)}{50,000},$$

$$AW = \frac{\sum_{i=1}^{50,000}(U_i - L_i)}{50,000}. \tag{33}$$

The simulation results are shown in Table 2 to Table 7. For situations of a scale parameter θ equals 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, and a shape parameter β equals 1, the results show that the coverage probabilities of the three methods close to the nominal level (0.95) and the average widths of exact and Proposed methods tend to be no difference for almost all sample sizes. In addition, the average width of Asymptotic method is wider than those of exact and Proposed methods for a small sample size ($n = 5$, 10), but the average widths of the three methods tend to be no difference for the larger sample sizes ($n > 10$) for these situations. It also shows that the average widths of the three methods tend to decrease when the sample size increases for all the scale and shape parameters.

For situations of a scale parameter θ equals 1.5 and a shape parameter β equals 1.5, 3.5, 10.0 the results show that the coverage probabilities of exact and asymptotic methods close to the nominal level (0.95), whereas this of Proposed method closes to one and the average widths of exact and Proposed methods tend to be no difference for all sample sizes.

For the cases of a scale parameter θ equals 2.0, 2.5, 3.0, 3.5, 4.0 and a shape parameter β equals 1.5, 3.5, 10.0 the results show that the coverage probabilities of exact and asymptotic methods close to the nominal level (0.95), whereas this of Proposed method closes to one for a small sample size and it tends to decrease when a sample size increases. However, Proposed method tends to have the shortest average width for all sample sizes in these cases. Especially, for a small sample size ($n = 5$), it is found that the coverage probability of Proposed method close to one and it tends to give the shortest average width when the large scale and shape parameters are considered. For all scale and shape parameters, the results show that the average width of asymptotic method is more wider than those of exact and Proposed methods for a small sample size ($n = 5$).

Table 2: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 1.5$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9504	3.893	0.9567	11.366	0.9523	4.395
10		0.9483	2.256	0.9535	3.027	0.9485	2.531
20		0.9496	1.445	0.9538	1.628	0.9505	1.616
40		0.9487	0.974	0.9500	1.029	0.9498	1.085
50		0.9497	0.864	0.9503	0.902	0.9496	0.961
5	1.5	0.9494	3.892	0.9554	11.363	0.9952	3.994
10		0.9496	2.255	0.9544	3.025	0.9950	2.349
20		0.9500	1.446	0.9527	1.629	0.9940	1.517
40		0.9503	0.975	0.9515	1.029	0.9916	1.025
50		0.9502	0.864	0.9514	0.901	0.9900	0.909
5	3.5	0.9506	3.896	0.9552	11.375	1.0000	3.826
10		0.9508	2.247	0.9562	3.016	1.0000	2.285
20		0.9495	1.445	0.9527	1.628	1.0000	1.488
40		0.9507	0.975	0.9520	1.029	1.0000	1.011
50		0.9518	0.864	0.9528	0.902	1.0000	0.897
5	10	0.9495	3.885	0.9564	11.342	1.0000	3.857
10		0.9501	2.251	0.9556	3.020	1.0000	2.315
20		0.9490	1.446	0.9505	1.629	1.0000	1.510
40		0.9485	0.975	0.9506	1.029	1.0000	1.026
50		0.9517	0.863	0.9519	0.901	1.0000	0.911

Table 3: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 2.0$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9496	5.205	0.9547	15.196	0.9483	5.883
10		0.9499	3.006	0.9544	4.034	0.9487	3.378
20		0.9515	1.927	0.9537	2.171	0.9518	2.152
40		0.9499	1.301	0.9504	1.373	0.9487	1.448
50		0.9504	1.152	0.9517	1.202	0.9503	1.282
5	1.5	0.9532	5.182	0.9567	15.128	0.9927	4.833
10		0.9511	2.999	0.9567	4.024	0.9880	2.842
20		0.9518	1.926	0.9546	2.170	0.9796	1.836
40		0.9517	1.299	0.9540	1.372	0.9563	1.242
50		0.9508	1.152	0.9516	1.202	0.9408	1.101
5	3.5	0.9493	5.175	0.9570	15.106	1.0000	4.150
10		0.9514	2.999	0.9556	4.024	1.0000	2.481
20		0.9500	1.926	0.9531	2.170	0.9986	1.616
40		0.9494	1.299	0.9505	1.372	0.9494	1.097
50		0.9494	1.151	0.9508	1.201	0.8603	0.973
5	10	0.9506	5.186	0.9563	15.139	1.0000	3.971
10		0.9493	3.006	0.9543	4.033	1.0000	2.382
20		0.9503	1.929	0.9537	2.173	1.0000	1.554
40		0.9500	1.300	0.9521	1.373	0.9961	1.056
50		0.9511	1.152	0.9512	1.202	0.8347	0.937

Table 4: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 2.5$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9508	6.486	0.9569	18.936	0.9516	7.321
10		0.9499	3.754	0.9545	5.037	0.9488	4.216
20		0.9511	2.406	0.9533	2.711	0.9484	2.690
40		0.9499	1.623	0.9519	1.714	0.9497	1.807
50		0.9494	1.441	0.9504	1.504	0.9491	1.603
5	1.5	0.9491	6.468	0.9566	18.882	0.9879	5.607
10		0.9511	3.746	0.9554	5.027	0.9777	3.298
20		0.9515	2.406	0.9547	2.710	0.9514	2.130
40		0.9500	1.626	0.9513	1.717	0.8710	1.442
50		0.9516	1.440	0.9525	1.502	0.8162	1.277
5	3.5	0.9491	6.494	0.9562	18.958	0.9996	4.426
10		0.9504	3.753	0.9553	5.036	0.9962	2.646
20		0.9497	2.409	0.9522	2.714	0.8839	1.722
40		0.9490	1.626	0.9500	1.717	0.1166	1.170
50		0.9489	1.440	0.9505	1.502	0.0140	1.038
5	10	0.9493	6.478	0.9560	18.910	1.0000	4.059
10		0.9487	3.761	0.9532	5.047	1.0000	2.436
20		0.9509	2.411	0.9538	2.716	0.7439	1.590
40		0.9522	1.625	0.9523	1.716	0.0000	1.080
50		0.9495	1.438	0.9509	1.501	0.0000	0.958

Table 5: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 3.0$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9486	7.799	0.9559	22.768	0.9498	8.807
10		0.9505	4.500	0.9552	6.038	0.9500	5.053
20		0.9497	2.892	0.9519	3.258	0.9488	3.235
40		0.9507	1.950	0.9522	2.059	0.9500	2.172
50		0.9508	1.729	0.9495	1.804	0.9486	1.925
5	1.5	0.9500	7.759	0.9565	22.652	0.9827	6.322
10		0.9495	4.511	0.9548	6.053	0.9647	3.732
20		0.9503	2.888	0.9524	3.253	0.9100	2.405
40		0.9498	1.949	0.9505	2.058	0.7388	1.627
50		0.9515	1.728	0.9530	1.803	0.6394	1.442
5	3.5	0.9498	7.759	0.9565	22.652	0.9987	4.659
10		0.9505	4.506	0.9554	6.047	0.9595	2.787
20		0.9501	2.889	0.9531	3.254	0.3383	1.815
40		0.9494	1.951	0.9510	2.060	0.0001	1.232
50		0.9489	1.729	0.9500	1.804	0.0000	1.093
5	10	0.9509	7.761	0.9573	22.657	1.0000	4.134
10		0.9491	4.496	0.9558	6.033	0.9840	2.480
20		0.9500	2.892	0.9538	3.258	0.0000	1.619
40		0.9500	1.950	0.9516	2.059	0.0000	1.100
50		0.9499	1.726	0.9507	1.801	0.0000	0.976

Table 6: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 3.5$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9522	9.061	0.9579	26.453	0.9522	10.220
10		0.9521	5.257	0.9552	7.054	0.9498	5.912
20		0.9492	3.373	0.9523	3.800	0.9501	3.769
40		0.9510	2.271	0.9530	2.397	0.9499	2.528
50		0.9489	2.016	0.9500	2.103	0.9481	2.243
5	1.5	0.9505	9.094	0.9564	26.549	0.9793	7.036
10		0.9515	5.260	0.9566	7.058	0.9514	4.136
20		0.9508	3.374	0.9527	3.801	0.8564	2.668
40		0.9495	2.275	0.9519	2.401	0.5896	1.803
50		0.9487	2.016	0.9500	2.104	0.4560	1.599
5	3.5	0.9516	9.079	0.9572	26.506	0.9946	4.872
10		0.9498	5.251	0.9555	7.046	0.8107	2.912
20		0.9492	3.373	0.9512	3.799	0.0252	1.896
40		0.9506	2.276	0.9517	2.403	0.0000	1.288
50		0.9519	2.015	0.9531	2.102	0.0000	1.142
5	10	0.9497	9.110	0.9553	26.597	1.0000	4.202
10		0.9499	5.246	0.9546	7.040	0.1284	2.519
20		0.9508	3.379	0.9537	3.806	0.0000	1.644
40		0.9502	2.275	0.9520	2.402	0.0000	1.117
50		0.9499	2.016	0.9507	2.103	0.0000	0.991

Table 7: The coverage probability (CP) and average width (AW) of 95% CIs for the scale parameter (θ) of a two parameters Weibull distribution when $\theta = 4.0$

n	β	Confidence Interval Methods					
		Exact		Asymptotic		Proposed	
		CP	AW	CP	AW	CP	AW
5	1.0	0.9498	10.363	0.9564	30.254	0.9498	11.708
10		0.9507	5.997	0.9562	8.047	0.9511	6.737
20		0.9502	3.850	0.9536	4.337	0.9493	4.302
40		0.9499	2.599	0.9516	2.744	0.9498	2.897
50		0.9502	2.301	0.9517	2.401	0.9492	2.561
5	1.5	0.9498	10.381	0.9549	30.306	0.9723	7.691
10		0.9503	6.008	0.9548	8.061	0.9314	4.513
20		0.9496	3.856	0.9526	4.344	0.7925	2.916
40		0.9501	2.600	0.9517	2.744	0.4472	1.970
50		0.9497	2.305	0.9513	2.405	0.3007	1.748
5	3.5	0.9492	10.381	0.9554	30.307	0.9812	5.062
10		0.9496	6.016	0.9542	8.073	0.5227	3.026
20		0.9493	3.853	0.9527	4.341	0.0001	1.971
40		0.9510	2.598	0.9523	2.743	0.0000	1.337
50		0.9497	2.303	0.9505	2.403	0.0000	1.187
5	10	0.9511	10.361	0.9573	30.248	0.9984	4.255
10		0.9507	5.990	0.9555	8.038	0.0000	2.553
20		0.9510	3.859	0.9537	4.347	0.0000	1.666
40		0.9497	2.599	0.9508	2.744	0.0000	1.132
50		0.9506	2.303	0.9513	2.403	0.0000	1.005

Table 8: The 95% CIs for the scale parameter (θ) of a Two Parameters Weibull Distribution for Urinary Tract Infection (UTIs) Data

Methods	Confidence Interval Limits		
	Lower Limit	Upper Limit	Width
Exact	0.64756	1.33246	0.68490
Asymptotic	0.66208	1.39998	0.73790
Proposed	0.86515	1.92180	1.05665

5. Real example: *Urinary Tract Infection Data*

In this section, a real-life example is given for the data from a healthcare department to illustrate the application of the three methods of confidence intervals. The data are collected from a large hospital to monitor urinary tract infections (UTIs). The data represent the number of days in between the admission and discharge of male patients. The frequency of patients having discharged from hospital on being acquired the UTIs while in the hospital is mentioned to quickly identify an increased infection rate. The similar data of UTIs were used by [27], [28] and [29]. According to [29] the data follow a Weibull distribution with shape parameter $\beta = 2$. The summary statistics for the data are given as follows: $n = 30, \sum_{i=1}^{n=30} X_i^{\beta} = 26.9702, \overline{X}^* = 0.961129, k_1 = 1.00059, k_2 = 0.80163$.

The resulting 95% confidence intervals for the three confidence interval methods and the corresponding confidence widths are given below in Table 8.

From Table 8, it is found that all the three methods for the 95% confidence intervals of the scale parameter (θ) have the lower and upper limits between 0.64756 to 1.92180, that is, the scale parameter (θ) of a two parameters Weibull distribution for urinary tract infections (UTIs) data seems to be not greater than two with a shape parameter $\beta = 2$. In addition, exact method has the shortest interval width. This conforms to the simulation study that the exact confidence interval performs well efficiency when it compares to the asymptotic and Proposed confidence intervals for the case of a small scale parameter and shape parameter $\beta = 2$.

6. Concluding remarks

An exact, asymptotic and approximate $(1-\alpha)100\%$ confidence intervals for the scale parameter (θ) of a two parameter Weibull distribution for the case of the one sample problem using the pivotal-based approach are derived. A Monte Carlo simulation study is performed to compare the efficiencies in terms of two criteriathe coverage probabilities and average widths of confidence intervalsfor the exact, asymptotic and approximate confidence intervals. It is found that the coverage probabilities of the three confidence intervals are close to the nominal level in cases of the shape parameter β equals 1 and all scale parameters θ for all sample sizes. When a shape parameter β increases, the coverage probabilities of the exact and asymptotic confidence intervals are also close to the nominal level for all sample sizes, whereas the coverage probability of the approximate confidence interval closes to one for a small sample size and it tends to decrease when a sample size increases. When considering the efficiency in term of the average width, it is found that the average widths of the three confidence intervals tend to be no difference in cases of the shape parameter β equals 1 and all scale parameters θ for the large sample sizes. Moreover, in case of the shape parameter β is greater than 1 ($\beta = 1.5,\ 3.5,\ 10.0$) and the scale parameters θ is greater than 1.5 ($\theta = 2.0,2.5,3.0,3.5,4.0$), the approximate confidence interval tends to have the shortest average width for all sample sizes. However, asymptotic confidence interval tends to perform poor efficiency for a small sample size whatever the shape and scale parameters will be. Finally, the approximate confidence interval is easy to compute and it tends to have the coverage probability close to one and have the shortest average width for a small sample size ($n = 5$) and almost all the shape and scale parameters, therefore it can be recommended for the practitioners in these cases.

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