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# THE MULTI-LAYER HELE-SHAW MODEL WITH CONSTANT VISCOSITY FLUIDS CAN NOT MINIMIZE THE SAFFMAN-TAYLOR INSTABILITY

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**Abstract:** The Saffman-Taylor instability occurs when a less viscous Stokes fluid is displacing a more viscous one, in a rectangular Hele-Shaw cell. This could be an useful model for the study of secondary oil recovery from a porous medium with low-pressure reserves. In some previous papers was considered a large number of liquid layers with constant viscosities inserted between the initial fluids, in order to minimize this instability. We highlight some strong contradictions related to the linear stability of this flow pattern.

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**Key Words:** Hele-Shaw displacements; multi-layer model; hydrodynamic stability

## 1. Introduction

We consider a Stokes flow in a 2D Hele-Shaw cell (first studied in [11]) parallel with the plane $xOy$. The flow is in the positive direction of the $Ox$ axis. The cell gap is $b$, the cell length is denoted by $l$. The gravity effects are neglected. The viscosity, velocity and pressure are denoted by $\nu_S, \mathbf{u} = (\underline{u}, \underline{v}, \underline{w}), p$. We consider $\epsilon = b/l << 1$ and we neglect $w$. The flow equations are

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$$p_x = -\frac{12\nu_S}{b^2}u, \quad p_y = -\frac{12\nu_S}{b^2}v, \quad p_z = 0, \quad u_x + v_y = 0, \tag{1}$$

$$u = (1/b)\int_0^b \underline{u}(x,y,z)dz, \quad v = (1/b)\int_0^b \underline{v}(x,y,z)dz,$$

where the lower indices $x, y, z$ are denoting the partial derivatives. The above equations are quite similar to the Darcy's law for the flow in a porous medium with the permeability $(b^2/12)$. A rigorous justification is given in [1], [13].

A sharp interface exists between two immiscible displacing fluids in a Hele-Shaw cell. Saffman and Taylor [18] have proven the well know result: the interface is unstable when the displacing fluid is less viscous. Moreover, the fingering phenomenon appears in this case, studied in [12], [17]. A surface tension on the interface is limiting the range of unstable disturbances. Without surface tension, the growth rates of perturbations are always positive and become infinite with increasing wave numbers. These properties were obtained from the formula (11) of [18] and have been confirmed by a large number of experiments.

The linear stability of the displacement by air of a Newtonian fluid in a 3D Hele-Shaw cell was studied in [16]. Unlike in the 2D case, the displacement could be almost stable even if the displacing fluid is less viscous but the surface tension on the air-fluid interface is large enough. Without surface tension, the growth rate is bounded for increasing wavenumbers of the perturbations.

An important problem is to minimize the Saffman-Taylor instability, by using some "intermediate" liquids (possible with surfactant properties) inserted between the initial displacing fluids.

An intermediate fluid with variable *increasing* viscosity in a middle layer between the displacing fluids can minimize the Saffman-Taylor instability - see the theoretical, experimental and numerical results given in [2], [8], [9], [10], [15], [19], [20], [23]. The displacements with variable viscosity in Hele-Shaw cells and porous media are studied in [14], [21], [22].

The three-layer model with an a priori unknown viscosity in a middle layer between the two initial displacing fluids was first considered in [9]. An optimal intermediate viscosity (which minimizes the instability) was obtained by a numerical procedure.

The Hele-Shaw displacement with $N$ intermediate layers (the $N$-layer Hele-Shaw model) was studied in [3], [4], [5], [6]. The corresponding (positive) growth rates become very small for $N$ large enough, in the case of the intermediate constant viscosities with positive jumps in the flow direction.

In this paper we show that some properties of the $N$-layer Hele-Shaw model with constant intermediate viscosities could contradict the experimental and theoretical results. We prove that a large surface tension on the interface between the displacing fluid and the intermediate region leads us to very large growth rates, independent of $N$. Moreover, the growth constants corresponding to neighboring interfaces can have very different magnitudes. A very unstable interface can exists (with large deformations over small time intervals), followed by an almost stable one. The distance between two neighboring interfaces could decrease during the displacement process. This is in contradiction with a basic assumption used in the four cited above papers: the length of the intermediate layers must be *constant*.

Our results are mainly obtained by using the boundary conditions on the interfaces, which are based on the Laplace-Young law and are also related with the amplitudes of the velocity perturbations.

The paper is laid out as follows. In Section 2 we describe the three-layers Hele-Shaw model with variable intermediate viscosity, introduced in [9]. In Section 3 we study the multi-layer model with constant intermediate viscosities and we get the properties mentioned above. We conclude in the last section.

## 2. The flow model

The three-layer Hele-Shaw model with a variable intermediate viscosity was first studied [9]. Even if this flow model is described in many papers (see [3] and the references therein), we give here some details, for the clarity of our exposure.

An amount of polymer solute with a variable concentration $c$ and variable viscosity $\nu$ is injected with the positive velocity $U$ in a rectangular Hele-Shaw cell saturated with a fluid with viscosity $\nu_O$, during a time interval $TI$. The adsorption, dispersion and diffusion of the solute in the equivalent porous medium are neglected. The viscosity $\nu$ can be considered as a series of powers of $c$, with constant coefficients - see [7], [8]. We consider a dilute solute and we have a first order development of $\nu$ as a function of $c$. Then $\nu$ is invertible in terms of $c$ and from the continuity equation for the solute (which is $Dc/Dt = 0$) we get $D\nu/Dt = 0$. It follows

$$\nu_t + u\nu_x + v\nu_y = 0. \tag{2}$$

After a time interval $TI$, a displacing fluid with viscosity $\nu_W$ is injected in the porous medium, with the same velocity $U$. On this way we obtain an intermediate fluid layer with the variable viscosity $\nu$ which is moving with the velocity $U$ far upstream.

Without loss of generality, the intermediate region is taken to be the interval

$$Ut - L < x < Ut.$$

We have three incompressible fluids with viscosities $\nu_W$ (displacing fluid), $\nu$ (intermediate layer) and $\nu_O$ (displaced fluid). The flow is governed by the Darcy's equations (1) with viscosity $\mu_S$ below:

$$\mu_S = \mu_W, \quad x < Ut - L; \quad \mu_S = \mu, \quad x \in (Ut - L, Ut);$$

$$\mu_S = \mu_O, \quad x > Ut; \tag{3}$$

$$\mu_W = 12\nu_W/b^2; \quad \mu = 12\nu/b^2; \quad \mu_O = 12\nu_O/b^2. \tag{4}$$

In [9] was studied the linear stability of the following basic state:

$$u = U, \ v = 0; \quad x = Ut - L, \ x = Ut;$$

$$P_x = -\mu_d U, \ P_y = 0; \quad \mu = \mu(x - Ut). \tag{5}$$

The basic viscosity $\mu$ in the middle layer is obtained from the equation (2) which gives us

$$\mu_t + U\mu_x = 0. \tag{6}$$

The boundary conditions on the interfaces are based on the Laplace-Young law: the pressure jump is given by the surface tension multiplied with the interfaces curvature. Moreover, the component $u$ of the velocity is continuous and the interfaces are material. The basic interfaces are straight lines, thus the basic pressure $P$ is continuous (but his gradient is not).

We introduce the moving reference frame $\overline{x} = x - Ut, \quad \tau = t$. However, we still use the notation $x$, $t$ instead of $\overline{x}, \tau$. Therefore the equation (6) leads us to $\mu_\tau = 0$, $\mu = \mu(x)$.

The perturbations of the basic velocity, pressure, viscosity are denoted by $u', v', p', \mu'$. The perturbation of the interface near a point $x = a$ is denoted by $\eta(a, y, t)$ - see the equation (14) below. We insert $u', v', p', \mu'$ in the equations (3), (6). As in [9] pag.82, formulas (2.14)-(2.16), we get the stability system

$$p'_x = -\mu u' - \mu' U, \quad p'_y = -\mu v', \quad u'_x + v'_y = 0, \tag{7}$$

$$\mu'_t + u'\mu_x = 0. \tag{8}$$

A Fourier decomposition is used for the perturbation $u'$:

$$u'(x, y, t) = f(x)[\cos(ky) + \sin(ky)]e^{\sigma t}, \ k \geq 0, \tag{9}$$

where $f(x)$ is the amplitude, $\sigma$ is the growth constant and $k$ are the wave numbers. From (7)-(9) we get the Fourier decompositions for the perturbations $v', p', \mu'$:

$$v' = (1/k) f_x [-\sin(ky) + \cos(ky)] e^{\sigma t},$$
$$p' = (\mu/k^2) f_x [-\cos(ky) - \sin(ky)] e^{\sigma t},$$
$$\mu' = (-1/\sigma) \mu_x f [\cos(ky) + \sin(ky)] e^{\sigma t}. \tag{10}$$

The cross derivation of the relations $(7)_1, (7)_2$ leads us to

$$\mu u'_y + \mu'_y U = \mu_x v' + \mu v'_x.$$

From (9) - (10) we get the equation of the amplitude $f$:

$$-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{-L, 0\}. \tag{11}$$

Outside the intermediate region we have

$$-f_{xx} + k^2 f = 0, \quad x \notin (-L, 0). \tag{12}$$

The perturbations decay to zero in the far field, thus

$$f(x) = f(-L) e^{k(x+L)}, \ \forall x \leq -L;$$

$$f(x) = f(0) e^{-kx}, \ \forall x \geq 0. \tag{13}$$

We suppose that at $x = a$ exist a viscosity jump $[\mu^+(a) - \mu^-(a)]$ and a surface tension $T(a)$, where $^+,^-$ are the right and left limit values. The amplitude $f$ is continuous at $x = a$ but a jump of $f_x$ is possible. The perturbed interface near $a$ is denoted by $\eta(a, y, t)$. In the first approximation we have $\eta_t = u$, therefore we can consider

$$\eta(a, y, t) = (1/\sigma) f(a) [\cos(ky) + \sin(ky)] e^{\sigma t}. \tag{14}$$

The limit values $p^+(a), \quad p^-(a)$, are obtained by using $P(a)$, the Taylor first order expansion of $P$ near $a$ and the expression $(10)_2$ of $p'$ in $a$. We have $P_x^{+,-}(a) = -\mu^{+,-}(a) U$, then we get

$$p^+(a) = P^+(a) + P_x^+(a)\eta + p'^+(a) = P^+(a)$$

$$-\mu^+(a)\{\frac{Uf(a)}{\sigma} + \frac{f_x^+(a)}{k^2}\}[\cos(ky) + \sin(ky)] e^{\sigma t}, \tag{15}$$

$$p^-(a) = P^-(a) + P_x^-(a)\eta + p'^-(a) = P^-(a)$$
$$-\mu^-(a)\{\frac{Uf(a)}{\sigma} + \frac{f_x^-(a)}{k^2}\}[\cos(ky) + \sin(ky)]e^{\sigma t}. \tag{16}$$

The Laplace-Young law is

$$p^+(a) - p^-(a) = T(a)\eta_{yy}, \tag{17}$$

where $T(a)$ is the surface tension acting in the point $a$ and $\eta_{yy}$ is the approximate value of the curvature of the perturbed interface.

We use the "ad hoc" notation $F_a = F(a)$. $P$ is continuous, thus from (15) - (17) it follows

$$-\mu_a^+[\frac{Uf_a}{\sigma} + \frac{f_x^+(a)}{k^2}] + \mu_a^-[\frac{Uf_a}{\sigma} + \frac{f_x^-(a)}{k^2}] = -\frac{T_a}{\sigma}f_a k^2,$$
$$\mu_a^- f_x^-(a) - \mu_a^+ f_x^+(a) = \frac{k^2 Uf[\mu_a^+ - \mu_a^-] - k^4 T_a}{\sigma} f_a. \tag{18}$$

The linear stability system of the basic state (5) is given by (11) - (13) with the boundary conditions below

$$\mu_L^- f_x^-(-L) - \mu_L^+ f_x^+(-L) = \frac{k^2 Uf[\mu_L^+ - \mu_L^-] - k^4 T_L}{\sigma} f_L. \tag{19}$$

$$\mu_0^- f_x^-(0) - \mu_0^+ f_x^+(0) = \frac{k^2 Uf[\mu_0^+ - \mu_0^-] - k^4 T_0}{\sigma} f_0. \tag{20}$$

We suppose $L = a = 0$. Then from (12), (13), (18) we can recover the Saffman - Taylor growth constant formula. Indeed, we have

$$-f_{xx} + k^2 f = 0,$$
$$f(x) = f(a)e^{k(x-a)},\ x \le a; \quad f(x) = f(a)e^{-k(x-a)},\ x \ge a,$$

and the relation (18) is giving the following results

$$\sigma_{ST} = \frac{kU(\mu_O - \mu_W) - T(a)k^3}{\mu_O + \mu_W}; \tag{21}$$

$$\mu_O > \mu_W \quad \text{and} \quad k^2 < U(\mu_O - \mu_W)/T(a) \Rightarrow \sigma_{ST} > 0; \tag{22}$$

$$T(a) = 0 \quad \Rightarrow \lim_{k\to\infty} \sigma_{ST} = \infty. \tag{23}$$

**Remark 1.** Consider a constant intermediate viscosity $\mu_1$, such that

$$\mu_W < \mu_1 < \mu_O.$$

Thus the stability system is given by the equation (12) and

$$-f_{xx} + k^2 f = 0, \quad x \in (-L, 0), \tag{24}$$

$$f(x) = f(-L)e^{k(x+L)}, \ \forall x \leq -L; \quad f(x) = f(0)e^{-kx}, \ \forall x \geq 0 \tag{25}$$

with boundary conditions (19)-(20).

**Remark 2.** We can inject several polymer-solutes with constant concentrations $c_i$ during the time intervals $T_i$, $1 \leq i \leq N$. We divide the middle region in $N$ small intervals (layers) separated by the interfaces $x_i$ such that

$$x_0 = 0 > x_1 > x_2 > ... > x_N = -L. \tag{26}$$

On each small interval we have the constant viscosities $\mu_i$ such that

$$\mu_O = \mu_0 > \mu_1 > \mu_2 > ... > \mu_{N-1} > \mu_N = \mu_W \tag{27}$$

and the amplitude equations

$$-\mu_i (f_i)_{xx} + \mu_i k^2 f_i = 0. \tag{28}$$

For $a = x_i$, $0 \leq i \leq N$ we have the boundary conditions (18). This is the multi-layer Hele-Shaw model, studied in [3], [4], [5] [6]. The amount of fluid between two neighboring interfaces and the length of each layer are *constant*, due to the mass conservation principle. □

## 3. The linear stability analysis

The main point is to study the possible solutions of the equation $-f_{xx}+k^2 f = 0$.

i) Like in [3], [6], the solution of (24) is $f(x) = Ae^{kx} + Be^{-kx}$, where $A, B$ are constant. The perturbations $u'$ must be "small", otherwise we exceed the frame of the linear stability. We must impose the conditions

$$\max_k |f(x)| < \infty; \quad \lim_{k \to \infty} |f(x)| < \infty. \tag{29}$$

Therefore, in fact we need $B = 0$ and the amplitude is

$$f(x) = Ae^{kx}. \tag{30}$$

We use (25) and (30), thus we get

$$f_x^-(-L) = kf(-L), \quad f_x^+(0) = -kf(-L),$$

$$f_x^+(-L) = kf(-L) = f_x^-(0).$$

Therefore the possible eigenvalues of the problem (19)-(20) and (24)-(25) are, say,

$$\sigma_L = \frac{kU(\mu_1 - \mu_W) - k^3 T(-L)}{\mu_W - \mu_1},$$

$$\sigma_O = \frac{kU(\mu_O - \mu_1) - k^3 T(0)}{\mu_O + \mu_1}. \tag{31}$$

We recall $\mu_W < \mu_1$, then from the above relation we obtain:

$$T(-L) > 0,\ T(0) > 0,\ k \to \infty \Rightarrow \sigma_L \to \infty, \quad \sigma_O \to -\infty. \tag{32}$$

$$T(-L) = 0,\ T(0) > 0,\ k \to \infty \Rightarrow \sigma_L \leq 0, \quad \sigma_O \to -\infty. \tag{33}$$

The instability appears even if two surface tensions exist. Moreover, if one of the surface tensions is missing, the growth rates are bounded with increasing $k$. These properties are in contradiction with the experimental data and with the Saffman-Taylor stability criterion.

If $T(-L) > 0,\ T(0) > 0$ then the growth constant $\sigma_L$ is strongly increasing with increasing $k$. Then the perturbations $u'$ of the first interface, even for small time intervals, become very large. As the perturbations of the second interface (governed by $\sigma_O$) are decreasing for large $k$, it is possible to have a collision between the interfaces. In this case it is not clear if the length of the intermediate region can be constant.

Moreover, any basic solution is subject to perturbations, which can be given (for example) by an unexpected vibration of the porous medium or of the Hele-Shaw cell in the laboratory. We proved that such a perturbation can reduce the distance between the initial interfaces in a very short time interval. From this point of view, even the existence of a basic solution with *constant* intermediate viscosity seems be not so evident.

The above results question the physical validity of the model with an intermediate liquid of constant viscosity. In the above analysis we used only the boundary conditions (18). The amplitude (30) was not considered in [3], [4], [5], [6].

If the intermediate viscosity is not constant, then we have not the solution (30) for the amplitudes. The eigenvalues (31) cannot be obtained. Therefore the model of Gorell and Homsy is useful when the intermediate liquid has a variable viscosity and $f$ is a priori unknown (and dependends on $\sigma$).

ii) For the $N$-layers model, in each inteval $(x_{i-1}, x_i)$ we use the amplitudes

$$f_i(x) = A_i e^{kx}.$$

On each interface $a = x_i$, the possible eigenvalues are obtained from the boundary conditions

$$\sigma_a = \frac{k^2 U f[\mu^+(a) - \mu^-(a)] - k^4 T(a)}{\mu^-(a) f_x^-(a) - \mu^+(a) f_x^+(a)} f(a). \tag{34}$$

In the points $a = -L, a = 0$ we obtain again the relations (31). Therefore the first interface (with the displacing fluid) could be much more unstable, compared to the last interface (with the displaced fluid). All the results obtained in the previous point i) are still valid.

## 4. Conclusions

The interface between two Stokes immiscible fluids in a rectangular Hele-Shaw cell is unstable when the displacing fluid is less viscous. A surface tension on the interface is limiting the range of disturbances which are unstable. If the surface tension is missing, then the growth rates become infinite with increasing wave numbers - see (21)-(23).

An intermediate fluid with a variable viscosity between the displacing fluids can minimize the Saffman-Taylor instability - see [3], [9] and the references therein.

The multi-layer Hele-Shaw model, consisting of $N$ intermediate fluids with constant viscosities was studied in some previous papers, where arbitrary small (positive) growth rates were obtained, if $N$ is large enough. Therefore is seems that this model is giving an important improvement of the flow stability.

We show that the multi-layer Hele-Shaw model can not supress the Saffman-Taylor instability. When the viscosity-jumps are positive in the flow direction, we get growth rates which becomes infinite with increasing wave numbers. The main results of our linear stability analysis are given by the formulas (31)-(33):

1) Large surface tensions give instability.
2) $T(-L) = 0 \Rightarrow \sigma_L \leq 0$.
3) An unstable interface can be followed by a stable one.

The properties 1) - 2) are contradicting the experimental data and the evident consequences of the Saffman-Taylor formula. Moreover, the considered model can not supress the Saffman-Taylor instability. In fact, the displacement described by this model can be even more unstable, compared with the Saffman-Taylor case. The property 3) is contradicting the mass conservation principle. Moreover, it is not clear if the distance $D$ between two neighboring interfaces can remain constant during the displacement process. On the contrary, $D$ could decrease very quickly over time

Our conclusion is: the multi-layer Hele-Shaw model with constant intermediate viscosities does not agree with the known theoretical and experimental results. Moreover, it cannot be used to minimize the Saffman-Taylor instability.

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