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# ON $\alpha$-ABSORBING SUBMODULES

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**Abstract:** Let $R$ be a commutative ring with identity. For an $R$-module $M$, the notion of $\alpha$-absorbing submodules and weakly $\alpha$-absorbing submodules are defined. We study some basic properties of $\alpha$-absorbing submodules and weakly $\alpha$-absorbing submodules. Also, we give some characterizations of them.

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**Key Words:** $\alpha$-absorbing submodule; $\alpha$-prime submodule; weakly $\alpha$-absorbing submodule

## 1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and all modules are unital left $R$-modules. We recall that a proper submodule $P$ of $M$ is called a prime submodule of $M$ if for every $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in (P : M)$.

In [2], the notion of $\alpha$-prime submodules is introduced, a proper submodule $P$ of $M$ is said to be an $\alpha$-prime submodule of $M$ provided that whenever $r \in R$ and $m \in M$ such that $r(m + m) \in P$, we have $r + r \in (P : M)$ or $m + m \in P$. Clearly, every prime submodule is $\alpha$-prime.

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On the $\mathbb{Z}$-module $\mathbb{Z}$, $p\mathbb{Z}$ is an $\alpha$-prime submodule of $\mathbb{Z}$ if and only if $p = 0$ or $p$ is a prime number or $p = 2q$ where $q$ is a prime number. This result obtains that for all prime numbers $q$, $2q\mathbb{Z}$ is an $\alpha$-prime submodule of $\mathbb{Z}$ but $2q\mathbb{Z}$ is not a prime submodule of $\mathbb{Z}$.

In [1], A. Darani and F. Soheilina defined a proper submodule $P$ of $M$ to be a 2-absorbing submodule if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P$, we have $rs \in (P : M)$ or $rm \in P$ or $sm \in P$.

On the $\mathbb{Z}$-module $\mathbb{Z}$, $p\mathbb{Z}$ is a 2-absorbing submodule of $\mathbb{Z}$ if and only if $p = 0$ or $p$ is a prime number or $p = qr$ where $q$ and $r$ are prime numbers.

In this paper we extend the concept of 2-absorbing submodules to $\alpha$-absorbing submodules.

**Definition 1.** A proper submodule $P$ of $M$ is an $\alpha$-absorbing submodule of $M$ if for each $r, s \in R$ and every $m \in M$ such that $rs(m + m) \in P$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

**Example 2.** Every $\alpha$-prime submodule of a module is an $\alpha$-absorbing submodule. However, $8\mathbb{Z}$ is $\alpha$-absorbing but is not $\alpha$-prime.

**Example 3.** Every 2-absorbing submodule of a module is an $\alpha$-absorbing submodule. However, $8\mathbb{Z}$ is $\alpha$-absorbing but is not 2-absorbing submodule.

Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and let $n \in \mathbb{Z}$. Note that $n\mathbb{Z}$ is a $\alpha$-absorbing submodule of $\mathbb{Z}$ if and only if for all $r, s, m \in \mathbb{Z}$, if $n \mid 2rsm$, then $n \mid 2rs$ or $n \mid 2rm$ or $n \mid 2sm$.

**Lemma 4.** *In the $\mathbb{Z}$-module $\mathbb{Z}$, if $n = 0, p, pq$ or $2pq$ where $p$ and $q$ are prime integers, then $n\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$.*

*Proof.* Since every $\alpha$-prime submodule of a module is an $\alpha$-absorbing submodule, we have $\{0\}$ and $p\mathbb{Z}$ are $\alpha$-absorbing submodules where $p$ is a prime integer. Next, let $p$ and $q$ be prime integers. Since every 2-absorbing submodule of a module is an $\alpha$-absorbing submodule, $pq\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$. To show that $2pq\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$, let $r, s, m \in \mathbb{Z}$ be such that $2pq \mid 2rsm$. Then $pq \mid rsm$. Hence $\Big(p \mid rs \text{ or } p \mid m\Big)$ and $\Big(q \mid rs \text{ or } q \mid m\Big)$. All cases imply that $2pq \mid 2rs$ or $2pq \mid 2rm$ or $2pq \mid 2sm$. Therefore $2pq\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$. □

**Proposition 5.** *Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and let $n \in \mathbb{Z}$. Then $n\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0$ or $n$ is a prime number or $n = pq$ where $p$ and $q$ are prime numbers or $n = 2pq$ where $p$ and $q$ are prime numbers.*

*Proof.* ($\rightarrow$) Let $n \in \mathbb{Z}$. Assume that $n\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$. Suppose that $n \neq 0$ and $n$ is not a prime number and $n \neq pq$ where $p$ and $q$ are prime numbers. Then $n = ab$ where $a$ and $b$ are integers with $1 < a, b < n$. Hence $a$ is not prime or $b$ is not prime. Without loss of generality, we assume that $a$ is not prime. Then $a = cd$ where $c$ and $d$ are integers with $1 < c, d < a$. Hence $n = ab = cdb$. This means $2cdb \in n\mathbb{Z}$. That is $n \mid 2cdb$. Therefore $n \mid 2cd$ or $n \mid 2cb$ or $n \mid 2db$. since $n = ab = cdb$, we have three cases to consider.

*Case 1. $ab \mid 2cd$.*

Then $ab \mid 2a$. Hence $b = 2$. This implies that $n = 2cd$. Next, we suppose for a contradiction that $c$ is not a prime number or $d$ is not a prime number.

Subcase 1.1. $c$ is not a prime number.

Then $c = c_1c_2$ where $c_1$ and $c_2$ are integers with $1 < c_1, c_2 < c$. Hence $n = 2c_1c_2d$. This means $n \mid 2c_1c_2d$. Since $n\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$, $n \mid 2c_1c_2$ or $n \mid 2c_1d$ or $n \mid 2c_2d$. since $n = 2c_1c_2d$, we have $d \mid 1$ or $c_2 \mid 1$ or $c_1 \mid 1$ which are contradictions.

Subcase 1.2. $d$ is not a prime number.

Then $c = d_1d_2$ where $d_1$ and $d_2$ are integers with $1 < d_1, d_2 < c$. Hence $n = 2cd_1d_2$. This implies that $n \mid 2cd_1d_2$. Since $n\mathbb{Z}$ is an $\alpha$-absorbing submodule of $\mathbb{Z}$, $n \mid 2cd_1$ or $n \mid 2cd_2$ or $n \mid 2d_1d_2$. Since $n = 2cd_1d_2$, we have $d_2 \mid 1$ or $d_1 \mid 1$ or $c \mid 1$ which lead to contradictions.

In Case 1, we conclude that $n = 2cd$ where $c$ and $d$ are prime numbers.

*Case 2. $ab \mid 2cb$.*

Then $abd \mid 2cdb$. Since $a = cd$, we have $abd \mid 2ab$. Hence $d \mid 2$. Therefore $d = 2$. This obtains that $n = 2cb$. Similarly to Case 1, we suppose for a contradiction that $c$ is not a prime number or $b$ is not a prime number.

Subcase 2.1. $c$ is not a prime number.

Then $c = c_1c_2$ where $c_1$ and $c_2$ are integers with $1 < c_1, c_2 < c$. Hence $n = 2c_1c_2b$. Thus $n \mid 2c_1c_2b$. This implies that $n \mid 2c_1c_2$ or $n \mid 2c_1b$ or $n \mid 2c_2b$. We have that $b \mid 1$ or $c_2 \mid 1$ or $c_1 \mid 1$ which lead to contradictions.

Subcase 2.2. $b$ is not a prime number.

Then $b = b_1b_2$ where $b_1$ and $b_2$ are integers with $1 < b_1, b_2 < b$. Hence $n = 2cb_1b_2$. Then $n \mid 2cb_1b_2$. This implies that $n \mid 2cb_1$ or $n \mid 2cb_2$ or $n \mid 2b_1b_2$. So $b_2 \mid 1$ or $b_1 \mid 1$ or $c \mid 1$ which lead to contradictions.

In Case 2, we conclude that $n = 2cb$ where $c$ and $b$ are prime numbers.

*Case 3.* $ab \mid 2db$.

Then $abc \mid 2cdb$. Since $a = cd$, we have $abc \mid 2ab$. Hence $c \mid 2$. Therefore $c = 2$ which leads to $n = 2db$. We suppose for a contradiction that $d$ is not a prime number or $b$ is not a prime number.

Subcase 3.1. $d$ is not a prime number.

Then $d = d_1d_2$ where $d_1$ and $d_2$ are integers with $1 < d_1, d_2 < d$. Hence $n = 2d_1d_2b$. Then $n \mid 2d_1d_2b$. So $n \mid 2d_1d_2$ or $n \mid 2d_1b$ or $n \mid 2d_2b$. This means $b \mid 1$ or $d_2 \mid 1$ or $d_1 \mid 1$ which lead to contradictions.

Subcase 3.2. $b$ is not a prime number. Then $b = b_1b_2$ where $b_1$ and $b_2$ are integers with $1 < b_1, b_2 < b$. Hence $n = 2db_1b_2$. Thus $n \mid 2db_1b_2$. This implies that $n \mid 2db_1$ or $n \mid 2db_1$ or $n \mid 2b_1b_2$. Therefore $b_2 \mid 1$ or $b_1 \mid 1$ or $d \mid 1$ which lead to contradictions.

In Case 3, we conclude that $n = 2db$ where $d$ and $b$ are prime numbers. Therefore $n = 2pq$ where $p$ and $q$ are prime numbers.

($\leftarrow$) This part follows from Lemma 4. □

**Example 6.** The intersection of two $\alpha$-absorbing submodules need not to be $\alpha$-absorbing submodule. For example, $5\mathbb{Z}$ and $12\mathbb{Z}$ are $\alpha$-absorbing submodules of $\mathbb{Z}$ but $5\mathbb{Z} \cap 12\mathbb{Z} = 60\mathbb{Z}$ is not an $\alpha$-absorbing submodules of $\mathbb{Z}$.

**Example 7.** The intersection of two 2-absorbing submodules need not to be $\alpha$-absorbing submodule. For example, $15\mathbb{Z}$ and $21\mathbb{Z}$ are 2-absorbing submodules of $\mathbb{Z}$ but $15\mathbb{Z} \cap 21\mathbb{Z} = 105\mathbb{Z}$ is not an $\alpha$-absorbing submodules of $\mathbb{Z}$.

## 2. Properties of $\alpha$-absorbing submodules

Let $(G, +)$ be a group and $H \subseteq G$. We denote the symbol $\beta(H)$ by $\{h+h \mid h \in H\}$ and $\alpha(H)$ by $\{h \mid h + h \in H\}$. It is clear that $\beta(H) \subseteq H \subseteq \alpha(H)$. If $I$ is an ideal of $R$, then $\alpha(I)$ and $\beta(I)$ are ideals of $R$. Furthermore, if $N$ is a submodule of $M$, then $\alpha(N)$ and $\beta(N)$ are submodules of $M$.

Let $M$ be a left $R$-module. If $N$ is a submodule of an $R$-module $M$, by $(N : M)$ we mean $\{r \in R \mid rM \subseteq N\}$. For an element $x \in M$ and a submodule $N$ of $M$, we will denote $\{r \in R \mid rx \in N\}$ with the short form $(N : x)$.

**Proposition 8.** *A proper submodule $P$ of an $R$-module $M$ is $\alpha$-absorbing if and only if for all submodules $N$ of $M$ and for all $r, s \in R$,*

$$\text{if } rs\beta(N) \subseteq P, \text{ then } rs + rs \in (P : M) \text{ or } r\beta(N) \subseteq P \text{ or } s\beta(N) \subseteq P.$$

*Proof.* $(\rightarrow)$ Assume that $P$ is an $\alpha$-absorbing submodule of $M$. Let $N$ be a submodule of $M$ and $r, s \in R$ such that $rs\beta(N) \subseteq P$. Suppose that $rs + rs \notin (P : M)$ and $r\beta(N) \nsubseteq P$ and $s\beta(N) \nsubseteq P$. Then $rx \notin P$ for some $x \in \beta(N)$ and $sy \notin P$ for some $y \in \beta(N)$. There exist $n, m \in N$ such that $x = n+n$ and $y = m+m$. This implies $rn+rn = rx \notin P$ and $sm+sm = sy \notin P$. We see that $rs(n + n) = rsx \in rs\beta(N) \subseteq P$. Since $P$ is $\alpha$-absorbing and $rs + rs \notin (P : M)$ and $r(n + n) \notin P$, we have $s(n + n) \in P$. We see again that $rs(m+m) = rsy \in rs\beta(N) \subseteq P$. Since $P$ is $\alpha$-absorbing and $rs+rs \notin (P : M)$ and $s(m + m) \notin P$, we have $r(m + m) \in P$. We note here that

$$s(n + n) \in P \text{ and } r(m + m) \in P. \tag{1}$$

It follows that $rs(m + m + n + n) \in P$. Since $P$ is $\alpha$-absorbing and $rs + rs \notin (P : M)$, we have $r(m + m + n + n) \in P$ or $s(m + m + n + n) \in P$. By the result (1), $r(n + n) \in P$ or $s(m + m) \in P$ which lead to a contradiction.

$(\leftarrow)$ Let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in P$. Then $rs\beta(Rm) \subseteq P$. This implies that $rs+rs \in (P : M)$ or $r\beta(Rm) \subseteq P$ or $s\beta(Rm) \subseteq P$. Hence $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Therefore $P$ is an $\alpha$-absorbing submodule of $M$. □

**Proposition 9.** *If $N_1$ and $N_2$ are $\alpha$-prime submodules of an $R$-module $M$, then $N_1 \cap N_2$ is an $\alpha$-absorbing submodule of $M$.*

*Proof.* Assume that $N_1$ and $N_2$ are $\alpha$-prime submodules of an $R$-module $M$. Let $r, s \in R$ and $m \in M$ be such that $rs(m + m) \in N_1 \cap N_2$. Since $N_1$ and $N_2$ are $\alpha$-prime submodules of $M$, $\Big(rs + rs \in (N_1 : M) \text{ or } m + m \in N_1\Big)$ and $\Big(rs + rs \in (N_2 : M) \text{ or } m + m \in N_2\Big)$. There are 4 cases to be considered :

(i) $rs + rs \in (N_1 : M)$ and $rs + rs \in (N_2 : M)$

(ii) $rs + rs \in (N_1 : M)$ and $m + m \in N_2$

(iii) $m + m \in N_1$ and $rs + rs \in (N_2 : M)$

(iv) $m + m \in N_1$ and $m + m \in N_2$.

In Case ((i)), this case implies that $rs + rs \in (N_1 \cap N_2 : M)$.

Next, Case ((ii)) is considered. Since $(rs + rs)M \subseteq N_1$ and $m \in M$, $r(sm + sm) = (rs + rs)m \in N_1$. Since $N_1$ is $\alpha$-prime, $r + r \in (N_1 : M)$ or $s(m + m) \in N_1$. These results imply that $r(m + m) \in N_1 \cap N_2$ or $s(m + m) \in N_1 \cap N_2$.

The proof of Case ((iii)) is similar to Case ((ii)).

Finally, we consider Case ((iv)). If $m+m \in N_1$ and $m+m \in N_2$, then $r(m+m) \in N_1 \cap N_2$ or $s(m+m) \in N_1 \cap N_2$. Therefore the intersection of each $\alpha$-prime submodules of $M$ is an $\alpha$-absorbing submodule of $M$. $\square$

Note that for a submodule $N$ of $M$ and an ideal $I$ of $R$, we denote $(N :_M I)$ by $\{m \in M \mid Im \subseteq N\}$.

**Proposition 10.** *Let $P$ be a submodule of $M$. Then the following statement are equivalent:*

*(i) $P$ is an $\alpha$-absorbing submodule of $M$.*

*(ii) If $I$ is an ideal of $R$ such that $IM \nsubseteq P$, then $(P :_M I)$ is an $\alpha$-absorbing submodule of $M$.*

*(iii) If $r \in R$ such that $rM \nsubseteq P$, then $(P :_M Rr)$ is an $\alpha$-absorbing submodule of $M$.*

*Proof.* $(i) \to (ii)$ Assume that $P$ is an $\alpha$-absorbing submodule of $M$. Let $I$ be an ideal of $R$ such that $IM \nsubseteq P$. Then $(P :_M I) \neq M$. Let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in (P :_M I)$. Then $Irs(m+m) \subseteq P$. Since $Irs(m+m) = rsI(m+m) = rs\beta(Im)$, we have $rs\beta(Im) \subseteq P$. By Proposition 8, $rs+rs \in (P : M)$ or $r\beta(Im) \subseteq P$ or $s\beta(Im) \subseteq P$. Since $(P : M) \subseteq ((P :_M I) : M)$ and $\beta(Im) = I(m+m)$, $rs+rs \in ((P :_M I) : M)$ or $r(m+m) \in (P :_M I)$ or $s(m+m) \in (P :_M I)$. Therefore $(P :_M I)$ is an $\alpha$-absorbing submodule of $M$.

$(ii) \to (iii)$ This part is obvious.

$(iii) \to (i)$ Assume that $(iii)$ holds. Since the ring $R$ contains the identity $1_R$ and $(P :_M R) = P$, we have $P$ is an $\alpha$-absorbing submodule of $M$. $\square$

For an element $x \in M$ and a submodule $N$ of $M$, we will denote $\{r \in R \mid rx \in N\}$ with the short form $(N : x)$.

**Proposition 11.** *Let $P$ be an $\alpha$-absorbing submodule of $M$. Let $m \in M$ and $r, s \in R$ such that $rs+rs \notin (P : M)$. Then*

$$(P : rs(m+m)) = (P : r(m+m)) \cup (P : s(m+m)).$$

*Proof.* Let $a \in (P : rs(m+m))$. Then $rs(am+am) \in P$. Since $P$ is an $\alpha$-absorbing submodule of $M$ and $rs+rs \notin (P : M)$, $ar(m+m) \in P$ or

$as(m+m) \in P$. Therefore $a \in (P : r(m+m)) \cup (P : s(m+m))$. Next, the reverse inclusion part is obvious. $\square$

For an element $r \in R$ and a submodule $N$ of $M$, we will denote a submodule $\{m \in M \mid rm \in N\}$ of $M$ by $N_r$.

**Proposition 12.** *Let $P$ be a submodule of $M$. Then the following statement are equivalent :*

*(i) $P$ is an $\alpha$-absorbing submodule of $M$.*

*(ii) For each $r, s \in R$, if $rs + rs \notin (P : M)$, then $\alpha(P_{rs}) = \alpha(P_r) \cup \alpha(P_s)$.*

*(iii) For each $r, s \in R$, if $rs + rs \notin (P : M)$, then $\alpha(P_{rs}) = \alpha(P_r)$ or $\alpha(P_{rs}) = \alpha(P_s)$.*

*Proof.* $(i) \to (ii)$ Assume that $P$ is an $\alpha$-absorbing submodule of $M$. Let $r, s \in R$ be such that $rs + rs \notin (P : M)$. First, we will show that $\alpha(P_{rs}) \subseteq \alpha(P_r) \cup \alpha(P_s)$. Let $m \in \alpha(P_{rs})$. Then $rs(m+m) \in P$. Since $P$ is an $\alpha$-absorbing submodule of $M$, $r(m+m) \in P$ or $s(m+m) \in P$. Hence $m+m \in P_r$ or $m+m \in P_s$. Therefore $m \in \alpha(P_r) \cup \alpha(P_s)$. To show that $\alpha(P_r) \cup \alpha(P_s) \subseteq \alpha(P_{rs})$, let $m \in \alpha(P_r) \cup \alpha(P_s)$. Then $r(m+m) \in P$ or $s(m+m) \in P$. Therefore $rs(m+m) \in P$. This obtains that $m \in \alpha(P_{rs})$.

$(ii) \to (iii)$ This part is obvious.

$(iii) \to (i)$ Assume that for each $r, s \in R$, if $rs+rs \notin (P : M)$, then $\alpha(P_{rs}) = \alpha(P_r)$ or $\alpha(P_{rs}) = \alpha(P_s)$. To show that $P$ is an $\alpha$-absorbing submodule of $M$, let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in P$ and $rs + rs \notin (P : M)$. Then $m \in \alpha(P_{rs})$. By assumption, $\alpha(P_{rs}) = \alpha(P_r)$ or $\alpha(P_{rs}) = \alpha(P_s)$. Then $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. Hence $r(m+m) \in P$ or $s(m+m) \in P$. Therefore $P$ is an $\alpha$-absorbing submodule of $M$. $\square$

The next results are inspired by [5].

**Lemma 13.** *Let $I$ be an ideal of $R$ and $P$ be an $\alpha$-absorbing submodule of $M$. If $a \in R$ and $m \in M$ and $aI(m+m) \subseteq P$, then $a(m+m) \in P$ or $I(m+m) \subseteq P$ or $Ia \subseteq \alpha((P : M))$.*

*Proof.* Let $a \in R$ and $m \in M$ be such that $aI(m+m) \subseteq P$. We suppose that $a(m+m) \notin P$ and $Ia \nsubseteq \alpha((P : M))$. There exists an element $b \in I$ such that $ab+ab \notin (P : M)$. This implies that $ab(m+m) \in P$. Since $P$ is an $\alpha$-absorbing submodule of $M$, $b(m+m) \in P$. To show that $I(m+m) \subseteq P$, let $c \in I$. Then

$a(b+c)(m+m) \in P$. Since $P$ is $\alpha$-absorbing, $a(b+c)+a(b+c) \in (P:M)$ or $(b+c)(m+m) \in P$. If $(b+c)(m+m) \in P$, then by $b(m+m) \in P$ it follows that $c(m+m) \in P$. Next, assume that $a(b+c)+a(b+c) \in (P:M)$. Since $ab+ab \notin (P:M)$, $ac+ac \notin (P:M)$. This implies $ac(m+m) \in P$. Since $P$ is $\alpha$-absorbing, $c(m+m) \in P$. This shows that $I(m+m) \subseteq P$. □

**Lemma 14.** *Let $I$ and $J$ be ideals of $R$ and $P$ be an $\alpha$-absorbing submodule of $M$. If $m \in M$ and $IJ(m+m) \subseteq P$, then $I(m+m) \subseteq P$ or $J(m+m) \subseteq P$ or $IJ \subseteq \alpha((P:M))$.*

*Proof.* Let $m \in M$ and $IJ(m+m) \subseteq P$. We assume that $I(m+m) \nsubseteq P$ and $J(m+m) \nsubseteq P$. To show that $IJ \subseteq \alpha((P:M))$, let $x \in I$ and $y \in J$. Since $I(m+m) \nsubseteq P$, there exists $a \in I$ such that $a(m+m) \notin P$ and $aJ(m+m) \subseteq P$. By Lemma 13, $aJ \subseteq \alpha((P:M))$. Moreover,

$$aJ \subseteq \alpha((P:M)) \text{ for all } a \in I \backslash (P : m+m). \tag{2}$$

Since $J(m+m) \nsubseteq P$, there exists $b \in J$ such that $b(m+m) \notin P$ and $bI(m+m) \subseteq P$. By Lemma 13, $bI \subseteq \alpha((P:M))$. Moreover,

$$bI \subseteq \alpha((P:M)) \text{ for all } b \in J \backslash (P : m+m). \tag{3}$$

Hence

$$ab, ay \text{ and } xb \text{ are elements of } \alpha((P:M)). \tag{4}$$

Since $a+x \in I$ and $b+y \in J$, $(a+x)(b+y)(m+m) \in P$. Since $P$ is $\alpha$-absorbing, $(a+x)(b+y)+(a+x)(b+y) \in (P:M)$ or $(a+x)(m+m) \in P$ or $(b+y)(m+m) \in P$.

If $(a+x)(m+m) \in P$, then $x(m+m) \notin P$, hence $x \in I \backslash (P : m+m)$, so $xy \in \alpha((P:M))$ by equation (2). Similarly, by equation (3), if $(b+y)(m+m) \in P$, then $xy \in \alpha((P:M))$. Finally, assume that $(a+x)(b+y)+(a+x)(b+y) \in (P:M)$. The equation (4) implies that $xy \in \alpha((P:M))$. □

**Proposition 15.** *A proper submodule $P$ of an $R$-module $M$ is $\alpha$-absorbing if and only if for all ideals $I$ and $J$ of $R$ and for all submodule $N$ of $M$,*

$$\text{if } IJ\beta(N) \subseteq P, \text{ then } I\beta(N) \subseteq P \text{ or } J\beta(N) \subseteq P \text{ or } IJ \subseteq \alpha((P:M)).$$

*Proof.* Let $P$ be a proper submodule of $M$.

$(\rightarrow)$ Assume that $P$ is $\alpha$-absorbing. Let $I$ and $J$ be ideals of $R$ and $N$ be submodule of $M$ such that $IJ\beta(N) \subseteq P$ and $IJ \nsubseteq \alpha((P:M))$. Note that

$n + n \in \beta(N)$ for all $n \in N$. By Lemma 14, $J(x + x) \subseteq P$ or $I(x + x) \subseteq P$ for all $x \in N$. If $J(x + x) \subseteq P$ for all $x \in N$, then $J\beta(N) \subseteq P$. Similarly, if $I(x+x) \subseteq P$ for all $x \in N$, then $I\beta(N) \subseteq P$. Next, suppose for a contradiction that there exist elements $x$ and $x'$ of $N$ such that $I(x + x) \nsubseteq P$ and $J(x' + x') \nsubseteq P$. By Lemma 14, we have $J(x + x) \subseteq P$ and $I(x' + x') \subseteq P$. Then $IJ(x + x' + x + x') \subseteq P$. The Lemma 14 implies that $I(x + x' + x + x') \subseteq P$ or $J(x + x' + x + x') \subseteq P$. Since $J(x + x) \subseteq P$ and $I(x' + x') \subseteq P$, we have $I(x + x) \subseteq P$ or $J(x' + x') \subseteq P$ which leads to a contradiction. Hence $I\beta(N) \subseteq P$ or $J\beta(N) \subseteq P$.

($\leftarrow$) This part follows from Proposition 8. □

## 3. Properties of $\alpha$-prime ideals and $\alpha$-absorbing ideals

We recall from [2] that a proper ideal $I$ of a ring $R$ is called an $\alpha$-prime ideal of $R$ if for all $a, b \in R$, if $a(b+b) \in R$, then $a+a \in I$ or $b+b \in I$. We have the following propositions.

**Proposition 16.** *Let $P$ be an $\alpha$-absorbing submodule of $M$. Then $(P : M)$ is an $\alpha$-prime ideal of $R$ if and only if $(P : m)$ is an $\alpha$-prime ideal of $R$ for all $m \in M\backslash\alpha(P)$.*

*Proof.* ($\rightarrow$) Assume that $(P : M)$ is an $\alpha$-prime ideal of $R$. Let $m \in M$ be such that $m + m \notin P$. Let $a, b \in R$ be such that $a(b + b) \in (P : m)$. Then $a(b + b)m \in P$. Since $P$ is an $\alpha$-absorbing submodule of $M$, $ab + ab \in (P : M)$ or $a(m + m) \in P$ or $b(m + m) \in P$. If $a(m + m) \in P$ or $b(m + m) \in P$, then $a + a \in (P : m)$ or $b + b \in (P : m)$. Next, assume that $ab + ab \in (P : M)$. Since $(P : M)$ is an $\alpha$-prime ideal of $R$, $a + a \in (P : M)$ or $b + b \in (P : M)$. Since $(P : M) \subseteq (P : m)$, $a + a \in (P : m)$ or $b + b \in (P : m)$. This shows that $(P : m)$ is an $\alpha$-prime ideal of $R$.

($\leftarrow$) Assume that $(P : m)$ is an $\alpha$-prime ideal of $R$ for all $m \in M\backslash\alpha(P)$. Let $a, b \in R$ be such that $a(b + b) \in (P : M)$. Suppose for a contradiction that $a + a \notin (P : M)$ and $b + b \notin (P : M)$. Let $x, y \in M$ be such that $(a + a)x \notin P$ and $(b + b)y \notin P$. Then $x \notin \alpha(P)$ and $y \notin \alpha(P)$. By assumption, $(P : x)$ and $(P : y)$ are $\alpha$-prime ideals of $R$. Since $a(b+b)x \in P$ and $a(b+b)y \in P$, we have $a(b+b) \in (P : x)$ and $a(b+b) \in (P : y)$. Then $(b+b)x \in P$ and $(a+a)y \in P$. If $x+y+x+y \in P$, then $ax+ay+ax+ay = a(x+y+x+y) \in P$, so $(a+a)x \in P$ which is a contradiction. Hence $x + y \notin \alpha(P)$. Thus $(P : x + y)$ is an $\alpha$-prime ideal of $R$. Since $a(b + b)(x + y) \in P$, $a(b + b) \in (P : x + y)$. This obtains that $a+a \in (P : x+y)$ or $b+b \in (P : x+y)$. Hence $(a+a)(x+y) \in P$ or $(b+b)(x+y) \in P$. Since $(a+a)y \in P$ and $(b+b)x \in P$, we have $a(x+x) \in P$ or $b(y+y) \in P$ which are contadictions. This proves that $a+a \in (P : M)$ or $b+b \in (P : M)$. Therefore $(P : M)$ is an $\alpha$-prime ideal of $R$. □

**Proposition 17.** *Let $P$ be an $\alpha$-absorbing submodule of an $R$-module $M$ and let $m \in M$ and $r \in R \backslash (P : m+m)$. If $(P : M)$ is an $\alpha$-prime ideal of $R$, then $(P : m+m) = (P : r(m+m))$.*

*Proof.* Assume that $(P : M)$ is an $\alpha$-prime ideal of $R$. If $s \in (P : m+m)$, then $s(m+m) \in P$. So $sr(m+m) \in P$. Hence $s \in (P : r(m+m))$. Next, let $s \in (P : r(m+m))$. Then $sr(m+m) \in P$. Since $P$ is an $\alpha$-absorbing submodule of $M$ and $r \notin (P : m+m)$, we have $s(r+r) \in (P : M)$ or $s(m+m) \in P$. If $s(m+m) \in P$, then $s \in (P : m+m)$. Now, assume that $s(r+r) \in (P : M)$. Since $(P : M)$ is an $\alpha$-prime ideal of $R$, $s+s \in (P : M)$ or $r+r \in (P : M)$. If $r+r \in (P : M)$, then $r \in (P : m+m)$ which is a contradiction. Another case, if $s+s \in (P : M)$, then $s(m+m) \in P$. Hence $s \in (P : m+m)$. This proves that $(P : m+m) = (P : r(m+m))$. □

**Definition 18.** An $\alpha$-absorbing ideal of a ring $R$ is an $\alpha$-absorbing submodule of an $R$-module $R$.

Let $I$ be a proper ideal of a ring $R$. Then $I$ is an $\alpha$-absorbing ideal of $R$ if and only if for each $r, s, t \in R$ such that $rs(t+t) \in I$, we have $r(s+s) \in I$ or $r(t+t) \in I$ or $s(t+t) \in I$.

**Proposition 19.** *If $P$ is an $\alpha$-absorbing submodule of an $R$-module $M$, then $(P : M)$ is an $\alpha$-absorbing ideal of $R$.*

*Proof.* Assume that $P$ is an $\alpha$-absorbing submodule of an $R$-module $M$. Let $r, s, t \in R$ such that $rs(t+t) \in (P : M)$. Then $rs(t+t)M \subseteq P$. Suppose that $r(t+t) \notin (P : M)$ and $s(t+t) \notin (P : M)$. Let $x_1, x_2 \in M$ be such that $r(t+t)x_1 \notin P$ and $s(t+t)x_2 \notin P$. Then $rs(t(x_1+x_2)+t(x_1+x_2)) = rs(t+t)(x_1+x_2) \in P$. Since $P$ is an $\alpha$-absorbing submodule of an $R$-module $M$, $rs+rs \in (P : M)$ or $r(t(x_1+x_2)+t(x_1+x_2)) \in P$ or $s(t(x_1+x_2)+t(x_1+x_2)) \in P$.

Next, assume that $r(t(x_1+x_2)+t(x_1+x_2)) \in P$. Since $r(t+t)x_1 + r(t+t)x_2 = r(t(x_1+x_2)+t(x_1+x_2)) \in P$ and $r(t+t)x_1 \notin P$, we have $r(t+t)x_2 \notin P$. Since $rs(tx_2+tx_2) = rs(t+t)x_2 \in P$ and $P$ is $\alpha$-absorbing submodule, we have

$rs + rs \in (P : M)$.

Finally, assume that $s(t(x_1 + x_2) + t(x_1 + x_2)) \in P$. Since $s(t+t)x_1 + s(t+t)x_2 = s(t(x_1+x_2)+t(x_1+x_2)) \in P$ and $s(t+t)x_2 \notin P$, we have $s(t+t)x_1 \notin P$. Since $rs(tx_1+tx_1) = rs(t+t)x_1 \in P$ and $P$ is $\alpha$-absorbing submodule, we have $rs + rs \in (P : M)$.

This complete the proof that $(P : M)$ is an $\alpha$-absorbing ideal of $R$. □

## 4. Weakly $\alpha$-absorbing submodules

A. Darani and F. Soheilina [1] have introduced and studied the concept of a 2-absorbing submodule of an $R$-module over a commutative ring $R$ with identity. Weakly absorbing submodules have been studied in several paper such as [1], [3] and [4]. In this section, we extend the notion of $\alpha$-absorbing submodules and 2-absorbing submodules to weakly $\alpha$-absorbing submodules.

**Definition 20.** A proper submodule $P$ of $M$ is a weakly $\alpha$-absorbing submodule if for each $r, s \in R$ and every $m \in M$ such that $rs(m+m) \in P\backslash\{0\}$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

By the definition, every $\alpha$-absorbing submodule is weakly $\alpha$-absorbing submodule but the converse does not hold. For example, $\{\bar{0}\}$ is weakly $\alpha$-absorbing but is not $\alpha$-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}_{16}$ because $2 \cdot 2 \cdot (\bar{2} + \bar{2}) = \bar{0}$ and $(2 \cdot 2 + 2 \cdot 2)\mathbb{Z}_{16} \nsubseteq \{\bar{0}\}$ and $2 \cdot (\bar{2} + \bar{2}) \neq \bar{0}$.

For an element $r \in R$ and a submodule $N$ of $M$, we will denote a submodule $\{m \in M \mid rm \in N\}$ of $M$ by $N_r$. Furthermore, for additive identity $0 \in M$, we denote $\{0\}_r$ by $0_r$. That is $0_r = \{m \in M \mid rm = 0\}$.

**Proposition 21.** *Let $P$ be a submodule of $M$. Then the following statements are equivalent:*

*(i) $P$ is a weakly $\alpha$-absorbing submodule of $M$.*

*(ii) For each $r, s \in R$, if $rs + rs \notin (P : M)$, then $\alpha(P_{rs}) = \alpha(P_r) \cup \alpha(P_s) \cup \alpha(0_{rs})$.*

*(iii) For each $r, s \in R$, if $rs + rs \notin (P : M)$, then $\alpha(P_{rs}) = \alpha(P_r)$ or $\alpha(P_{rs}) = \alpha(P_s)$ or $\alpha(P_{rs}) = \alpha(0_{rs})$.*

*Proof.* $(i) \to (ii)$ Assume that $P$ is a weakly $\alpha$-absorbing submodule of $M$. Let $r, s \in R$ be such that $rs+rs \notin (P : M)$. Let $m \in \alpha(P_{rs})$. Then $rs(m+m) \in P$. If $rs(m+m) = 0$, then $m \in \alpha(0_{rs})$. Assume that $rs(m+m) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $r(m+m) \in P$ or $s(m+m) \in P$. Hecne $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. This implies that $m \in \alpha(P_r) \cup \alpha(P_s) \cup \alpha(0_{rs})$. Conversely, let $m \in \alpha(P_r) \cup \alpha(P_s) \cup \alpha(0_{rs})$. Then $r(m+m) \in P$ or $s(m+m) \in P$ or $rs(m+m) = 0$. Hence $rs(m+m) \in P$. Therefore $m \in \alpha(P_{rs})$.

$(ii) \to (iii)$ Clear.

$(iii) \to (i)$ Assume that $(iii)$ holds. Let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in P \backslash \{0\}$ and $rs + rs \notin (P : M)$. Then $m \in \alpha(P_{rs})$ and $rs(m+m) \neq 0$. By assumption, $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. Then $r(m+m) \in P$ or $s(m+m) \in P$. Therefore $P$ is a weakly $\alpha$-absorbing submodule of $M$. □

**Proposition 22.** *Let $P$ be a proper submodule of $M$ and $0_{rs} \subseteq P$ for all $r, s \in R$ with $rs \notin \alpha((P : M))$. Then $P$ is a weakly $\alpha$-absorbing submodule of $M$ if and only if $P$ is an $\alpha$-absorbing submodule of $M$.*

*Proof.* Firstly, it is clear that every $\alpha$-absorbing submodule is weakly $\alpha$-absorbing submodule. Conversely, assume that $P$ is a weakly $\alpha$-absorbing submodule of $M$. Let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in P$ and $rs + rs \notin (P : M)$. Then $m \in \alpha(P_{rs})$. By Proposition 21, we have $m \in \alpha(P_r) \cup \alpha(P_s) \cup \alpha(0_{rs})$. Then $r(m+m) \in P$ or $s(m+m) \in P$ or $rs(m+m) = 0$. If $rs(m+m) = 0$, then $m+m \in 0_{rs} \subseteq P$. So $m+m \in P$. This means $r(m+m) \in P$ or $s(m+m) \in P$. Therefore $P$ is an $\alpha$-absorbing submodule of $M$. □

Let $R_1$ and $R_2$ be commutative rings with identity, $M_i$ be a unital $R_i$-module where $i = 1, 2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module under the operation $(r_1, r_2)(m_1, m_1) = (r_1m_1, r_2m_2)$ for all $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$.

**Proposition 23.** *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let $N_1$ be an $R_1$-submodule of $M_1$. Consider the following statements:*

*(i)* *$N_1$ is an $\alpha$-absorbing submodule of $M_1$.*

*(ii)* *$N_1 \times M_2$ is an $\alpha$-absorbing submodule of $M_1 \times M_2$.*

*(iii)* *$N_1 \times M_2$ is a weakly $\alpha$-absorbing submodule of $M_1 \times M_2$.*

*Then (i) $\to$ (ii) $\to$ (iii). Moreover, if $\beta(M_2) \neq \{0\}$, then (i), (ii) and (iii) are equivalent.*

*Proof.* The proof of $(i) \to (ii)$ and $(ii) \to (iii)$ are straightforward. Next, assume that $\beta(M_2) \neq \{0\}$ and $N_1 \times M_2$ is a weakly $\alpha$-absorbing submodule of $M_1 \times M_2$. Let $w \in M_2$ be such that $w + w \neq 0$. To show that $N_1$ is an $\alpha$-absorbing submodule of $M_1$, let $r, s \in R$ and $m \in M$ be such that $rs(m + m) \in N_1$. Note that $(r, 1)(s, 1)[(m, w) + (m, w)] = (rs, 1)(m + m, w + w) = (rs(m + m), w + w)$. Hence $(r, 1)(s, 1)[(m, w) + (m, w)] \in (N_1 \times M_2)\backslash\{(0, 0)\}$. Since $N_1 \times M_2$ is a weakly $\alpha$-absorbing submodule of $M_1 \times M_2$, $(rs+rs, 1+1) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(r, 1)(m+m, w+w) \in N_1 \times M_2$ or $(s, 1)(m+m, w+w) \in N_1 \times M_2$. These imply that $rs + rs \in (N_1 : M_1)$ or $r(m + m) \in N_1$ or $s(m + m) \in N_1$. Therefore $N_1$ is an $\alpha$-absorbing submodule of $M_1$. □

The following example shows that if $\beta(M_2) = \{0\}$, then the part $(iii) \to (i)$ of Proposition 23 may be not hold.

**Example 24.** *Let $M_1 = \mathbb{Z}_{16}$, $M_2 = \mathbb{Z}_2$, $R_1 = R_2 = \mathbb{Z}$. We see that $\beta(M_2) = \{\bar{0}\}$.*

*It is clear that $\{\bar{0}\} \times \mathbb{Z}_2$ is a weakly $\alpha$-absorbing submodule of $\mathbb{Z}_{16} \times \mathbb{Z}_2$. However, $\{\bar{0}\}$ is not an $\alpha$-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}_{16}$.*

**Proposition 25.** *If $P$ is a weakly $\alpha$-absorbing submodule of an $R$-module $M$ and $(P : M)^2\beta(P) \neq 0$, then $P$ is an $\alpha$-absorbing submodule of $M$.*

*Proof.* Assume that $P$ is a weakly $\alpha$-absorbing submodule of an $R$-module $M$ and $(P : M)^2\beta(P) \neq 0$. Let $r, s \in R$ and $m \in M$ be such that $rs(m+m) \in P$. If $rs(m + m) \neq 0$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Assume that $rs(m + m) = 0$. We divide to two cases.

**Case 1.** $rs\beta(P) \neq 0$.

Let $x \in P$ be such that $rs(x+x) \neq 0$. Then $rs(m+m+x+x) = rs(x+x) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $rs + rs \in (P : M)$ or $r(m+m+x+x) \in P$ or $s(m+m+x+x) \in P$. Since $x \in P$, $rs+rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

**Case 2.** $rs\beta(P) = 0$.

**Subcase 2.1.** $s(P : M)(m + m) \neq 0$.

Let $t \in (P : M)$ be such that $st(m + m) \neq 0$. Then $s(r + t)(m + m) = (sr + st)(m+m) = st(m+m) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $s(r+t)+s(r+t) \in (P:M)$ or $s(m+m) \in P$ or $(r+t)(m+m) \in P$. Since $t \in (P:M)$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subcase 2.2.** $s(P:M)(m+m) = 0$.

Since $(P:M)^2\beta(P) \neq 0$, $kf(n+n) \neq 0$ where $k, f \in (P:M)$ and $n \in P$.

Note that we need to show if $sf(n+n) \neq 0$ or $kr(m+m) \neq 0$ or $rf(m+m) \neq 0$, then $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subsubcase 2.2.1.** $sf(n+n) \neq 0$.

Then $s(r+f)(n+n+m+m) = (sr+sf)(n+n+m+m) = sr(n+n)+sr(m+m)+sf(n+n)+sf(m+m) = sf(n+n) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $s(r+f)+s(r+f) \in (P:M)$ or $s(n+n+m+m) \in P$ or $(r+f)(n+n+m+m) \in P$. Since $f \in (P:M)$ and $n \in P$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subsubcase 2.2.2.** $kr(m+m) \neq 0$.

Then $r(k+s)(m+m) = (rk+rs)(m+m) = rk(m+m) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $r(k+s)+r(k+s) \in (P:M)$ or $r(m+m) \in P$ or $(k+s)(m+m) \in P$. Since $k \in (P:M)$, we have $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subsubcase 2.2.3.** $rf(m+m) \neq 0$.

Then $r(f+s)(m+m) = (rf+rs)(m+m) = rf(m+m)+rs(m+m) = rf(m+m) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $r(f+s)+r(f+s) \in (P:M)$ or $r(m+m) \in P$ or $(f+s)(m+m) \in P$. Since $f \in (P:M)$, we have $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Now, we assume that $sf(n+n) = 0$ and $kr(m+m) = 0$ and $rf(m+m) = 0$.

Again, we need to claim that if $kr(n+n) \neq 0$ or $kf(m+m) \neq 0$, then $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subsubcase 2.2.4.** $kr(n+n) \neq 0$.

Then $r(k+s)(n+n+m+m) = rk(n+n)+rk(m+m)+rs(n+n)+rs(m+m) = rk(n+n) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $r(k+s)+r(k+s) \in (P:M)$ or $r(n+n+m+m) \in P$ or $(k+s)(n+n+m+m) \in P$. Since $k \in (P:M)$ and $n \in P$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

**Subsubcase 2.2.5.** $kf(m+m) \neq 0$.

Then $(k+r)(f+s)(m+m) = kf(m+m)+ks(m+m)+rf(m+m)+rs(m+m) = kf(m+m) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $(k+r)(f+s)+(k+r)(f+s) \in (P:M)$ or $(k+r)(m+m) \in P$ or $(f+s)(m+m) \in P$. Since $k, f \in (P:M)$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Here, we assume that $kr(n+n) = 0$ and $kf(m+m) = 0$.

Then $(s+k)(r+f)(m+m+n+n) = sr(m+m)+sr(n+n)+sf(m+m)+sf(n+$

$n)+kr(m+m)+kr(n+n)+kf(m+m)+kf(n+n) = kf(n+n) \neq 0$. Since $P$ is a weakly $\alpha$-absorbing submodule of $M$, $(s+k)(r+f)+(s+k)(r+f) \in (P : M)$ or $(s+k)(m+m+n+n) \in P$ or $(r+f)(m+m+n+n) \in P$. Since $f, k \in (P : M)$ and $n \in P$, $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

Therefore $P$ is an $\alpha$-absorbing submodule of $M$. □


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