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MULTIPLICITY OF SOLUTIONS OF DIRICHLET'S PROBLEM FOR SECOND-ORDER p-LAPLACIAN DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract: The existence of infinitely many solutions of Dirichlet's problem for p-Laplacian ordinary differential equation of second order is studied in the paper. The variational method is applied using the symmetric mountain pass theorem.

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1. Introduction

In this paper we study the existence of solutions of the second-order p-Laplacian equation

$$\left(\varphi_p\left(u'\left(x\right)\right)\right)' - a(x)\varphi_q\left(u\left(x\right)\right) + b(x)\varphi_r\left(u\left(x\right)\right) = 0, \quad 0 < x < L \tag{1}$$

coupled with the Dirichlet conditions

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$$u(0) = u(L) = 0. \tag{2}$$

We denote $\varphi_p(t) = |t|^{p-2} t,\ t \in \mathbb{R}$ for $p > 1$ and assume that $1 < p < q < r$, $a(x)$ and $b(x)$ are positive continuous functions on $[0, L]$. Partial cases of Eq. (1) are considered by many authors and appear in biomathematical and phase-transition models. Austin [2] considers the case $p = 2, q = 3, r = 4$ in a model of an aneurysm in the circle of Willis. Grossinho and Sanchez [7] consider the periodic solutions of the equation in this case using variational method. Further periodic and homoclinic solutions are studied in [9]. Systems of p-Laplacian equations are considered in [4] using critical point theory for non-smooth functionals. Higher-order equations are studied in [8] using the generalized Clark's theorem. It is applied to fourth-order p-Laplacian equations in [10]. The case $p = 2$ is considered in the thesis of Kalcheva [5], Chapter 2.

Further, by (P) will denote the problem (1) and (2). It has a variational structure, which means that its solutions can be obtained as critical points of a C^1 functional J on a Sobolev space X, defined as

$$X = W_0^{1,p}(0, L) = \{u \in L^p(0, L) : u' \in L^p(0, L), u(0) = u(L) = 0\}, \tag{3}$$

where $L^p(0, L)$ is the usual Lebesque space. The space X is a separable Banach space with norm

$$\|u\|_X = \left(\int_0^L \left(|u'(x)|^p + |u(x)|^p \right) \mathrm{d}x \right)^{\frac{1}{p}}$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^L |u'(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{4}$$

by the Poincare inequality $\|u\|_X \leq C \|u'\|_{L^p}$, where $\|u\|_{L^p}^p = \int_0^L |u|^p \, \mathrm{d}x$.

Moreover, the embedding $X \in C([0, L])$ is compact, $u \in L^q(0, L)$ for $q \geq p$ and

$$\int_0^L |u(x)|^q \, \mathrm{d}x \leq \|u\|_{L^\infty}^{q-p} \|u\|_{L^p}^p,$$

(see [3], Chapter 8). We suppose that $a(x)$ and $b(x)$ are positive continuous functions and there are constants a_j, b_j for $j = 1, 2$ such that

$$0 < a_1 \leq a(x) \leq a_2, \quad 0 < b_1 \leq b(x) \leq b_2. \tag{5}$$

The functional $J : X \to \mathbb{R}$ is defined as

$$J(u) = \frac{1}{p}\int \left|u'(x)\right|^p \mathrm{d}x + \frac{1}{q}\int_0^L a(x)\left|u(x)\right|^q \mathrm{d}x - \frac{1}{r}\int_0^L b(x)\left|u(x)\right|^r \mathrm{d}x.$$

Under conditions (5), J is C^1 functional, which can be shown in a standard way (see [9],[5]) and

$$\langle J'(x), v\rangle = \int_0^L \left(\varphi_p(u')v' + a(x)\varphi_q(u)v - b(x)\varphi_r(u)v\right)\mathrm{d}x.$$

By a weak solution of problem (1) we mean a function $u \in X$, such that

$$\int_0^L \left(\varphi_p(u')v' + a(x)\varphi_q(u)v - b(x)\varphi_r(u)v\right)\mathrm{d}x = 0, \tag{6}$$

i.e. u is a critical point of J.

By a solution of problem (P) we mean a function $u \in C([0, L])$, such that $\varphi_p(u') \in AC([0, L])$ and $u(x)$ satisfies Eq. (1) for $x \in [0, L]$ and boundary condition (2). Here $AC([0, L])$ denotes the space of absolutely continuous functions on $[0, L]$ (see [1], [3]). Let u be a weak solution of (P), i.e. let (6) hold. Since $w = \varphi_p(u') \in AC([0, L]) = W^{1,1}(0, L)$,

$$\int_0^L wv'\,\mathrm{d}x = -\int_0^L w'v\,\mathrm{d}x,$$

for every $v \in X$. By $u' = \varphi_{p'}(w) = \varphi_p^{-1}(w) \in C([0, L])$, where $\frac{1}{p} + \frac{1}{p'} = 1$. It follows that

$$\int_0^L \left(\left(\left|u'\right|^{p-2}u'\right)' - a(x)\left|u\right|^{q-2}u + b(x)\left|u\right|^{r-2}\left|u\right|\right)v\,\mathrm{d}x = 0$$

for every $v \in \mathbb{C}_0^\infty([0, L])$. Then it follows that u is a solution of (P). Further we look for critical points of functional J in order to find the solution of (P). We will apply variational method and symmetric mountain-pass theorem. Our main result is the following:

Theorem 1. *Let $1 < p < q < r$, $a(x)$ and $b(x)$ be positive continuous functions on $[0, L]$. Then the problem* (P) *has infinitely many pairs of solutions.*

The case $p > r$ is considered in [10] with application of the generalized Clark's theorem. The periodic and homoclinic solutions are studied in [9] in the case $p = 2$ and for higher order equations in [8].

The paper is organized as follows. In Section 2 we present the variational formulation of the problem, formulate the symmetric mountain-pass theorem and prove a lemma for the (PS) condition. In Section 3 we prove Theorem 1.

2. Preliminary results

As we mention in Introduction, we denote by X the Sobolev space

$$X = W_0^{1,p}(0,L) = \left\{u \in L^p(0,L) : u' \in L^p(0,L), u(0) = u(L) = 0\right\}$$

equipped with the norm

$$\|u\| = \left(\int_0^L |u'(x)|^p \, dx\right)^{\frac{1}{p}}.$$

We consider the functional $J : X \to \mathbb{R}$

$$J(u) = \frac{1}{p}\int_0^L |u'(x)|^p \, dx + \frac{1}{q}\int_0^L a(x)|u(x)|^q \, dx - \frac{1}{r}\int_0^L b(x)|u(x)|^r \, dx,$$

and look for critical points of J, which are solutions of (P). We will apply the following known result.

Theorem 2 ([11], Theorem 9.12). *Let E be an infinite dimensional Banach space and $I \in \mathbb{C}^1(E,\mathbb{R})$ be an even functional which satisfies Palais-Smale (PS) condition, $I(0) = 0$. If $E = V \bigoplus X$ where V is finite dimensional and I satisfies:*

(I_1) *there are constants ρ, $\alpha > 0$ such that $I(u) \geq \alpha$ if $\|u\|_E = \rho$, $u \in E$;*

(I_2) *for each finite dimensional subspace $E_n \subset E$, there exists R_n, such that $I(u) \leq 0$ if $u \in E_n$, $\|u\|_E \geq R_n$.*

Then, I possesses an unbounded sequence of critical points.

Next, we formulate an inequality due to Lindqvist [6], used in [8] for the proof of (PS) condition.

Lemma 1 ([6], Lemma 4.2). **a)** *If $p \geq 2$ one has*

$$(\varphi_p(x) - \varphi_p(y))(x-y) \geq \frac{2}{p(2^{p-1}-1)}|y-x|^p$$

for $x, y \in \mathbb{R}$;

b) *If* $1 < p < 2$

$$(\varphi_p(x) - \varphi_p(y))(x - y) \geq \frac{2}{p(2^{p-1} - 1)} |y - x|^p$$

where $C(p)$ *depends only in* p.

Now we prove the following auxiliary lemma.

Lemma 2. *Let* $1 < p < q < r$, $a(x)$ *and* $b(x)$ *be continuous positive functions on* $[0, L]$ *and* (5) *be satisfied. Then, the functional* $J : X \to \mathbb{R}$ *satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS)-sequence in X, i.e. $\{J(u_n)\}$ is a bounded sequence and $J'(u_n) \to 0$ in $X^\star$. We have

$$\frac{1}{r}\langle J'(u_n), u_n\rangle = \frac{1}{r}\int_0^L \left(|u_n'(x)|^p + a(x)|u_n(x)|^q - b(x)|u_n(x)|^r\right) \mathrm{d}x$$

and

$$\begin{aligned} J(u_n) - \frac{1}{r}\langle J'(u_n), u_n\rangle \\ = \left(\frac{1}{p} - \frac{1}{r}\right)\int_0^L |u_n'(x)|^p \,\mathrm{d}x + \left(\frac{1}{q} - \frac{1}{r}\right)\int_0^L |u_n(x)|^q \,\mathrm{d}x \\ \geq \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|^p, \end{aligned}$$

which implies that

$$|J(u_n)| + \frac{1}{r}\left\|J'(u_n)\right\|_\star \|u_n\| \geq \left(\frac{1}{p} - \frac{1}{r}\right)|u_n\|^p. \tag{7}$$

Then, the sequence $\{u_n\}$ is bounded in X. Indeed, if we suppose that there is a subsequence $\{u_{n_k}\}$ still denoted by $\{u_n\}$ such that $\|u_n\| \to \infty$, by (7) we obtain

$$\frac{|J(u_n)|}{\|u_n\|^p} + \frac{1}{r}\frac{\|J'(u_n)\|_\star}{\|u_n\|^{p-1}} \geq \frac{1}{p} - \frac{1}{r} > 0,$$

which implies a contradiction as $n \to \infty$, because $|J(u_n)|$ is bounded and $\|J'(u_n)\|_\star \to 0$. Hence $\{u_n\}$ is a bounded sequence in X and let $u_n \rightharpoonup u$ weakly in X. By compact embedding $X \subset C([0, L])$, it follows that:

$$\lim_{n\to\infty}\int_0^L a(x)|u_n(x)|^q \,\mathrm{d}x = \int_0^L a(x)|u(x)|^q \,\mathrm{d}x, \tag{8}$$

$$\lim_{n\to\infty}\int_0^L b(x)\,|u_n(x)|^r\,\mathrm{d}x=\int_0^L b(x)\,|u(x)|^r\,\mathrm{d}x.$$

We have $\langle J'(u_n),u_n\rangle\to 0$ as $n\to\infty$ by

$$\left|\langle J'(u_n),u_n\rangle\right|\le\left\|J'(u_n)\right\|_\star\|u_n\|.$$

Then,

$$0=\lim_{n\to\infty}\langle J'(u_n)-J'(u),u_n-u\rangle$$
$$=\lim_{n\to\infty}\int_0^L\left(\varphi_p(u_n')-\varphi_p(u')\right)\left(u_n'-u'\right)\mathrm{d}x. \quad (9)$$

By Lemma 1, for $p\ge 2$ we have

$$\left(\varphi_p(u_n')-\varphi_p(u')\right)\left(u_n'-u'\right)\ge\frac{2}{p\left(2^{p-2}-1\right)}\left|u_n'-u'\right|^p$$

which implies by (9) that $u_n\to u$ strongly in X. If $1<p<2$ by Lemma 1 and Hölder's inequality we have

$$0=\lim_{n\to\infty}\int_0^L\left(\varphi_p(u_n')-\varphi_p(u')\right)\left(u_n'-u'\right)\mathrm{d}x$$
$$\ge\lim_{n\to\infty}\left(\left\|u_n'\right\|_{\mathbb{L}^p}-\left\|u'\right\|_{\mathbb{L}^p}\right)\left(\left\|u_n'\right\|_{\mathbb{L}^p}^{p-1}-\left\|u'\right\|_{\mathbb{L}^p}^{p-1}\right)\ge 0,$$

which shows that $\|u_n\|\to\|u\|$. Since X is an uniformly confex Banach space (see [3]) and $u_n\to u$ weakly in X, then $u_n\to u$ strongly in X which completes the proof of Lemma 2. □

3. Proof of the main result

In this section we prove Theorem 1 and verify the geometric assumptions of Theorem 2.

Since $X=W_0^{1,p}(0,L)\subset C([0,L])\subset L^q(0,L)$ for $q>1$ there are constants k_1 and k_2 such that for $u\in X$

$$\|u\|_{L^q}=\left(\int_0^L|u(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\le k_1\|u\|, \quad (10)$$

$$\|u\|_{L^r}=\left(\int_0^L|u(x)|^r\,\mathrm{d}x\right)^{\frac{1}{r}}\le k_2\|u\|.$$

Proof of Theorem 1

We apply Theorem 2 with $V = \{0\}$. To show (I_1) in X we have by (5) and (10)

$$J(u) = \frac{1}{p}\|u\|^p + \frac{1}{q}\int_0^L a(x)|u|^q \, \mathrm{d}x - \frac{1}{r}\int_0^L b(x)|u|^r \, \mathrm{d}x \geq \frac{1}{p}\|u\|^p - \frac{b_2 k_2}{r}\|u\|^r .$$

Since $r > p$, for $\|u\| = e$ sufficiently small there exists $\alpha > 0$ such that $J(u) \geq \alpha > 0$ if $\|u\| = \rho$.

To prove the condition (I_2) of Theorem 2, let $X_n \subset X$ be an n-dimensional subspace of X and $v_n \in X_n$ is arbitrary point. Since X_n is finite dimensional space there are constants $d_{jn}, j = 1, ..., 4$ such that

$$d_{1n}\|v_n\| \leq \|v_n\|_{\mathbb{L}^q} \leq d_{2n}\|v_n\|,$$
$$d_{3n}\|v_n\| \leq \|v_n\|_{\mathbb{L}^r} \leq d_{4n}\|v_n\|.$$

Then by (5) and last inequalities we have:

$$\begin{aligned} J(v_n) &= \frac{1}{p}\|v_n\|^p + \frac{1}{q}\int_0^L a(x)|v_n(x)|^q \, \mathrm{d}x - \frac{1}{r}\int_0^L b(x)|v_n(x)|^r \, \mathrm{d}x \\ &\leq \frac{1}{p}\|v_n\|^p + \frac{a_2 d_{2n}^q}{q}\|v_n\|^q - \frac{b_1 d_{3n}^r}{r}\|v_n\|^r \\ &= \|v_n\|^p \left(\frac{1}{p} + \frac{a_2 d_{2n}^q}{q}\|v_n\|^{q-p} - \frac{b_1 d_{3n}^r}{r}\|v_n\|^{r-p} \right). \end{aligned}$$

By $1 < p < q < r$, there exists R_n sufficiently large, such that $J(v_n) \leq 0$ for $\|v_n\| \geq R_n$. Then by Theorem 2, the functional J has infinitely many pairs of critical points. □

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