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ON A SUBCLASS OF STARLIKE FUNCTIONS ASSOCIATED WITH GENERALIZED CARDIOID

Yuzaimi Yunus[1 §], Ajab Bai Akbarally[2], Suzeini Abdul Halim[3]

[1] Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA, Cawangan Melaka
Kampus Jasin, 77300 Melaka, MALAYSIA

[2] Universiti Teknologi MARA, 40450 Shah Alam
Selangor, MALAYSIA

[3] Institute of Mathematical Sciences
Universiti Malaya, 50603 Kuala Lumpur, MALAYSIA

Abstract: The purpose of this paper is to investigate a subclass of analytic functions associated with generalized cardioid in the open unit disk. The geometric properties of functions in the subclass are investigated. Subsequently, the bound for initial coefficients, the Fekete-Szegö inequality and second Hankel determinant inequality for functions belonging to this class are obtained. Furthermore, we find the sharp estimate for Toeplitz determinant, $T_2(2)$ for this class.

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1. Introduction

Let us denote by $\mathcal{A}$ the class of analytic functions f in the open unit disk

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[§]Correspondence author

$\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Denote $\mathcal{S}$ as a subclass of $\mathcal{A}$ consisting of univalent functions. Some subclasses of $\mathcal{S}$ play an important role in geometric function theory such as the class of starlike functions, $\mathcal{S}^*$ and class of convex functions, $\mathcal{C}$. These classes are characterized by quantities $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ respectively. Many authors have generalized the aforementioned classes namely by applications of operators to obtain new subclasses, for example [28].

A function f is said subordinate to g, denoted $f \prec g$, if there is an analytic function $\omega(z)$ defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. If g is univalent in $\mathbb{U}$ then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Ma and Minda [13] gave unified representation of various subclasses of f by using subordination. They introduced the following classes that include some well-known classes:

$$S^*(h) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec h(z) \right\},$$

and

$$K(h) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec h(z) \right\}.$$

The function $h(z)$ maps $\mathbb{U}$ onto a set bounded in the right half-plane. Recently, several authors defined many interesting subclasses of S^* by restricting the value of $zf'(z)/f(z)$ lying in specific domain in the right half plane. Sokol and Stankiewicz [26] introduced the class S^*_L consisting of functions $f \in \mathcal{A}$ satisfying $|(zf'(z)/f(z))^2 - 1| < 1 \quad z \in \mathbb{U}$, where $zf'(z)/f(z)$ lies in the domain bounded by the right-half of the lemniscate of Bernoulli. This class can be written in the form of subordination as $S^*_L := \mathcal{S}^*(\sqrt{1+z}) = \{f \in \mathcal{A} : zf'(z)/f(z) \prec \sqrt{1+z}\}$. Some results associated with this class can be found in [1, 8, 17, 23]. Aouf, Dziok and Sokol [3] studied the class $S^*(q_c)$, where $q_c = \sqrt{1+cz}$ which can be reduced to class S^*_L. Also, Mendiratta, Nagpal and Ravichandran [15] investigated and introduced the class S^*_{RL} where $zf'(z)/f(z)$ lies in the domain bounded by the left-half shifted lemniscate of Bernoulli. With the same concept, Sharma, Jain and Ravichandran [22] studied the class S^*_C which lies in the domain bounded by a cardioid. For list of some classes that are defined by subordination, one can see [5, 29].

Motivated by the work of Sharma et al. [22], a class defined by subordination such that $zf'(z)/f(z)$ lies in a specific curve in the right half plane is introduced. We denote the class as $S^*_{GC}(k)$, the class of analytic functions in the unit disk with $zf'(z)/f(z)$ lying in the interior of the general cardioid in the right half plane.

Let $h_k(z) : \mathbb{U} \to \mathbb{C}$ be a function defined by

$$h_k(z) = 1 + 2kz + kz^2, \quad h(0) = 1 \quad \text{where} \quad 0 < k \leq 2/3. \tag{2}$$

We define $S^*_{GC}(k)$ as a subclass of $\mathcal{A}$ where

$$S^*_{GC}(k) := S^*(h_k) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + 2kz + kz^2 \right\}. \tag{3}$$

A function f belongs to $S^*_{GC}(k)$ if and only if $zf'(z)/f(z)$ lies in the region bounded by a general cardioid Ω_k, on the right half plane given by:

$$\Omega_k := \{x + iy : (x^2 + y^2 - 2x + 1 - 5k^2)^2 + 4k^2(x^2 + y^2 - 2x + 1 - 5k^2) - 8k^3x - 8k^4 + 8k^3 = 0\},$$

where $0 < k \leq 2/3$.

The function (2) is univalent in $\mathbb{U}$ and it is easy to see that $h_k(\mathbb{U}) = \Omega_k$, therefore $f(z)$ belongs to S^*_{GC} if and only if $zf'(z)/f(z) \prec h_k$. This gives the structural formula for functions in $S^*_{GC}(k)$. A function $f \in S^*_{GC}(k)$ if and only if there exists an analytic function q, $q \prec h_k$ such that

$$f(z) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} dt \right). \tag{4}$$

Choosing $q = h_k$ in (4), we see that the function

$$f_0(z) = z\exp\left(2kz + \frac{kz^2}{2} \right) = z + 2kz^2 + \left(2k^2 + \frac{k}{2} \right) z^3 + ...$$

belongs to the class $S^*_{GC}(k)$.

Interestingly, the main focus of univalent function theory is to investigate on the coefficients of functions. The extensive focus is to estimate the bounds of coefficients and this includes Hankel determinant. Hankel determinant of f was defined by Pommerenke [18] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $q \geq 1$ and $n \geq 1$. The uniqueness of Hankel determinant is that it has constant entries along reverse diagonal. The subject of investigation for this determinant varied from rate of growth of $H_q(n)$ as $n \to \infty$ ([16]) to determination of precise bounds on $H_q(n)$ for some classes for specific q and n in the unit disk, $\mathbb{U}$. For $n = 1$, $q = 2$ and $a_1 = 1$, $H_2(1) = |a_3 - a_2^2|$ and the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds for functions analytic and univalent in $\mathbb{U}$, [4]. The more general functional $|a_3 - \mu a_2^2|$ is known as Fekete-Szegö, studied by [7, 10]. For $n = q = 2$, $H_2(2) = |a_2 a_4 - a_3^2|$ is known as second Hankel determinant. Janteng, Halim and Darus [9] obtained sharp bound for $H_2(2)$ for the class R. For more details on the class R, one can refer to [14]. For other studies about Hankel determinant of various classes, one can refer to [11, 20, 21, 27].

Closely related to Hankel determinant is Toeplitz determinant. The Toeplitz determinant of f for $q \geq 1$ and $n \geq 1$ is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix},$$

particularly for $q = n = 2$, $T_2(2) = a_2^2 - a_3^2$; $q = 2, n = 3, T_2(3) = a_3^2 - a_4^2$; $q = 3, n = 1, T_3(1) = 1 + 2a_2^2 + 2a_2^2 a_3 - a_3^2$ and $q = 3, n = 2, T_3(2) = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)$. For studies on Toeplitz determinant, see [19, 24].

We shall need the following lemmas. Let P be the class of functions p satisfying Re $p(z) > 0$, $z \in \mathbb{U}$, and in the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{5}$$

Lemma 1. ([4]) *Let the function $p \in P$ be given by (5), then $|c_n| \leq 2$ for each n.*

Lemma 2. ([12]) *Let the function $p \in P$ be given by (5). Then for some complex valued x with $|x| \leq 1$ and some complex valued ξ with $|\xi| \leq 1$ we have*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\xi.$$

Lemma 3. ([13]) *Let $p \in P$. If ν is a real parameter then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4v - 2, & \text{if } \nu \geq 1. \end{cases}$$

Lemma 4. ([6]) *Let $p \in P$ be of the form* (5) *and $\lambda \in \mathbb{C}$. Then*

$$|c_n - \lambda c_k c_{n-k}| \leq 2 \max\{1, |2\lambda - 1|\},$$

for $1 \leq k \leq n - 1$.

Lemma 5. ([2]) *For $0 < a < \sqrt{2}$, let r_a be given by*

$$r_a = \begin{cases} (\sqrt{1-a^2} - (1-a^2))^{1/2}, & \text{if } 0 < a \leq 2\sqrt{2}/3, \\ \sqrt{2} - a, & \text{if } 2\sqrt{2}/3 \leq a < \sqrt{2}, \end{cases}$$

then $\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 1\}$.

The aim of this paper is to investigate the geometric properties of functions in the class $S^*_{GC}(k)$. We also seek the upper bound for initial coefficients, Fekete-Szegö functional, second Hankel determinant and also several Toeplitz determinants.

2. Main results

Firstly, we shall prove the following lemma.

Lemma 6. *For $1 - k < a < 1 + 3k$, let r_a be given by*

$$r_a = \begin{cases} a - 1 + k, & \text{if } 1 - k < a < 1 + k, \\ 1 + 3k - a, & \text{if } 1 + k \leq a < 1 + 3k, \end{cases}$$

and R_a be given by

$$R_a = \begin{cases} 1 + 3k - a, \quad \text{if } 1 - k < a \leq 1/3(k+3), \\ \sqrt{(1-a)(1-a-2k) + 5k^2 + k(k+1-a)^2/(a-1)}, \\ \qquad\qquad \text{if } 1/3(k+3) \leq a < 1 + 3k, \end{cases}$$

then $\{w : |w - a| < r_a\} \subseteq \Omega \subseteq \{w : |w - a| < R_a\}$.

Proof. Let $\psi(z) = 1 + 2kz + kz^2$. Then on the boundary of $\psi(\mathbb{U})$ we have $\psi(e^{it}) = w$. The parametric equations of $w = u + iv$ are given by

$$u(t) = 1 + 2k\,\cos\,t + k\,\cos\,2t, \quad v(t) = 2k\,\sin\,t + k\,\sin\,2t.$$

Since the curve $w = \psi(e^{it})$ is symmetric with respect to the real axis, so it is sufficient to consider the interval $0 \leq t \leq \pi$. The square of the distance from the point $(a, 0)$ to the points on the curve is given by

$$\begin{aligned}
z(t) &= (u(t) - a)^2 + (v(t))^2 \\
&= (1 - a + 2k\,\cos\,t + k\,\cos\,2t)^2 + (2k\,\sin\,t + k\,\sin\,2t)^2 \\
&= (1 - a)^2 + 5k^2 + (4k(1 - a) + 4k^2)\,\cos\,t + 2k(1 - a)\,\cos\,2t, \\
z'(t) &= -4k\,\sin\,t(1 - a + k + 2(1 - a)\,\cos\,t), \\
z''(t) &= -(4k(1 - a) + 4k^2)\cos t - 8k(1 - a)\cos 2t.
\end{aligned}$$

Solving $z'(t) = 0$ yields stationary points at $t = 0, \pi$ and $\cos t = (1 - a + k)/2(a - 1)$. First we obtain the radius for a disk, r_a, so that the disk is inside Ω.

Let $1 - k < a < 1 + k$, $z''(\pi) > 0$ shows that $z(t)$ is minimum at $t = \pi$. Then

$$\begin{aligned}
r_a &= \sqrt{z(\pi)} \\
&= \sqrt{k^2 - 2k(1 - a) + (1 - a)^2} \\
&= k - 1 + a.
\end{aligned}$$

For $1 + k \leq a < 1 + 3k$, $z''(0) > 0$ then $z(t)$ is minimum at $t = 0$. Therefore

$$\begin{aligned}
r_a &= \sqrt{z(0)} \\
&= \sqrt{9k^2 + 6k(1 - a) + (1 - a)^2} \\
&= 3k - 1 + a.
\end{aligned}$$

Next we determine the radius for a disk, R_a, so that Ω lie inside the disk.

Let $1 - k < a \leq 1/3(3 + k)$, $z''(0) < 0$ shows that $z(t)$ is maximum at $t = 0$,

$$\begin{aligned}
R_a &= \sqrt{z(0)} \\
&= \sqrt{9k^2 + 6k(1 - a) + (1 - a)^2} \\
&= 3k - 1 + a.
\end{aligned}$$

If $1/3(3+k) \leq a < 1+3k$, it shown that $z(t)$ is maximum at $t = t_0$ where $\cos t_0 = (1-a+k)/2(a-1)$. Then

$$\begin{aligned} R_a &= \sqrt{z(t_0)} \\ &= \sqrt{(1-a)(1-a-2k)+5k^2+k(1+k-a)^2/(a-1)}. \end{aligned}$$

□

Our first theorem is to determine the bounds for $|a_n|$ in the class S^*_{GC}. The similar method in [25] is used to determine it.

Theorem 7. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{GC}$ then*

$$\sum_{n=2}^{\infty} \left(\left(\frac{n}{3k+1} \right)^2 - 1 \right) |a_n|^2 \leqslant 1 - \frac{1}{(3k+1)^2}.$$

Proof. Suppose $f \in S^*_{GC}$. Then $zf'(z)/f(z) = 1 + 2k\omega(z) + k\omega^2(z)$, where ω is an analytic function in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Since $f(z)(1 + 2k\omega(z) + k\omega^2(z)) = zf'(z)$ and $z = re^{i\theta}$, thus

$$\begin{aligned} 2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} &= \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\ &= \int_0^{2\pi} \left| \frac{re^{i\theta} f'(re^{i\theta})}{1 + 2k\omega(re^{i\theta}) + k\omega^2(re^{i\theta})} \right|^2 d\theta \\ &\geq \int_0^{2\pi} \frac{|re^{i\theta} f'(re^{i\theta})|^2}{(1+3k)^2} d\theta \\ &= \frac{1}{(3k+1)^2} \int_0^{2\pi} |re^{i\theta} f'(re^{i\theta})|^2 d\theta \\ &= \frac{2\pi}{(3k+1)^2} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}, \end{aligned}$$

where $0 < r < 1$ and $a_1 = 1$. Then, we get

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \left(\left(\frac{n}{3k+1} \right)^2 - 1 \right) \leq 0. \tag{6}$$

By letting $r \to 1^-$, yields the required result. □

We have the following corollary.

Corollary 8. *If $f(z) = z + \sum_{n=m}^{\infty} a_n z^n \in S^*_{GC}$, then*

$$|a_n| \leq \sqrt{\frac{3k(2+3k)}{n^2 - 1 - 3k(2+3k)}} \text{ for } \quad n = m > \sqrt{1+3k} \quad \text{and} \quad m \in \mathbb{N}.$$

The next theorem gives the necessary and sufficient condition for special functions $z + A_n z^n$ to be in the class S^*_{GC}.

Theorem 9. *A function $f(z) = z + A_n z^n \quad (n = 2, 3, ...)$ belongs to the class S^*_{GC} if and only if*

$$|A_n| \leq \frac{1 - n + \sqrt{(n-1)^2 + 4k(k+n-1)}}{2(n+k-1)}.$$

Proof. The function in the form $f(z) = z + A_n z^n \in S^*$ if $|A_n| \leq 1/n$. Note that $S^*_{GC} \subset S^*$. Suppose that $w = zf'(z)/f(z)$. Function w maps $\mathbb{U}$ onto the disk

$$\left| w - \frac{1 - n|A_n|^2}{1 - |A_n|^2} \right| < \frac{(n-1)|A_n|}{1 - |A_n|^2}.$$

Since $a := (1 - n|A_n|^2)/(1 - |A_n|^2) < 1$ then by Lemma 6 for the disk to lie inside Ω if and only if

$$\begin{aligned} \frac{(n - 1|A_n|)}{1 - |A_n|^2} &\leq \frac{1 - n|A_n|^2}{1 - |A_n|^2} - 1 + k \\ &= \frac{(1 - k - n)|A_n|^2 + k}{1 - |A_n|^2} \end{aligned}$$

which gives $(n+k-1)|A_n|^2 + (n-1)|A_n| - k \leq 0$. Solving the inequality yields the result. □

We state the following initial coefficients and Fekete-Szegö estimate for the class.

Theorem 10. *Let $f \in S^*_{GC}$ be of form (1), then*

$$|a_2| \leq 2k,$$

$$|a_3| \leq \frac{k}{2} \begin{cases} 2, & \text{if} \quad 0 < k \leq \frac{1}{4}, \\ 4k+1, & \text{if} \quad \frac{1}{4} \leq k \leq \frac{2}{3}, \end{cases}$$

$$|a_4| \leq \frac{k}{3}\begin{cases} 4k^2 - 3k + 2, & \textit{if} \quad 0 < k \leq \frac{1}{3}, \\ k(4k+3), & \textit{if} \quad \frac{1}{3} \leq k \leq \frac{2}{3}, \end{cases}$$

and for any non-zero complex number μ

$$|a_3 - \mu a_2^2| \leq k \ max\left\{1, \left|\frac{1}{2} - 2(2\mu - 1)k\right|\right\}.$$

Proof. If $f \in S^*_{GC}$, then from the condition (3)

$$\frac{zf'(z)}{f(z)} = 1 + 2k\omega(z) + k\omega^2(z), \tag{7}$$

where $\omega(z)$ is analytic function in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Let

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + ... \quad (z \in \mathbb{U}) \tag{8}$$

then p is analytic and $p \in P$. Rearrange (8)

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\big(c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 - c_1 c_2 + c_1^3)z^3 + ...\big).$$

Now, by expanding the right and left sides of (7), we obtain

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + ... \tag{9}$$

and

$$1 + 2k\omega(z) + k\omega^2(z) = 1 + kc_1 z + k(c_2 - \frac{c_1^2}{4})z^2 + k(c_3 - \frac{c_1 c_2}{2})z^3 + ... \quad . \tag{10}$$

Then, by comparing (9) and (10) we get

$$a_2 = kc_1, \tag{11}$$

$$a_3 = \frac{k}{2}(c_2 - (\frac{1}{4} - k)c_1^2), \tag{12}$$

$$a_4 = \frac{k}{3}\big(c_3 - \frac{1}{2}(1-3k)c_1 c_2 + \frac{k}{8}(4k-3)c_1^3\big), \tag{13}$$

From well known Lemma 1 and (11), we get

$$|a_2| \leq 2k. \tag{14}$$

From (12), we have

$$|a_3| \leq \frac{k}{2}\left|c_2 - \left(\frac{1}{4} - k\right)c_1^2\right|. \tag{15}$$

Using Lemma 3, we obtain

$$\left|c_2 - \left(\frac{1}{4} - k\right)c_1^2\right| \leq \begin{cases} 2, & \text{if } \ 0 < k \leq \frac{1}{4}, \\ 4k+1 & \text{if } \ \frac{1}{4} \leq k \leq \frac{2}{3}, \end{cases} \tag{16}$$

with the value ν from the lemma is $\nu = \left(\frac{1}{4} - k\right)$. Therefore, using (15) and (16) we get

$$|a_3| \leq \frac{k}{2}\begin{cases} 2, & \text{if } \ 0 < k \leq \frac{1}{4}, \\ 4k+1, & \text{if } \ \frac{1}{4} \leq k \leq \frac{2}{3}. \end{cases} \tag{17}$$

Now, we find the estimate bound for a_4. From (13) and apply triangle inequality, we have

$$|a_4| \leq \frac{k}{3}\left(\left|c_3 - \frac{1}{2}(1-3k)c_1c_2\right| + \frac{k}{8}(4k-3)|c_1|^3\right).$$

Using Lemma 4 yields,

$$\left|c_3 - \frac{1}{2}(1-3k)c_1c_2\right| \leq \begin{cases} 2, & \text{if } \ 0 < k < \frac{1}{3}, \\ 6k, & \text{if } \ \frac{1}{3} \leq k \leq \frac{2}{3}. \end{cases} \tag{18}$$

Then, together from (18) and Lemma 1, we get

$$|a_4| \leq \frac{k}{3}\begin{cases} 4k^2 - 3k + 2 & \text{if } \ 0 < k \leq \frac{1}{3}, \\ 4k^2 + 3k & \text{if } \ \frac{1}{3} \leq k \leq \frac{2}{3}. \end{cases} \tag{19}$$

Next, for functional $a_3 - \mu a_2^2$ where μ is non-zero complex number, using (11) and (12), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{k}{2}\left(c_2 - \left(\frac{1}{4} - k\right)c_1^2\right) - \mu k^2 c_1^2 \\ &= \frac{k}{2}\left(c_2 - \left(\frac{1}{4} + (2\mu - 1)k\right)c_1^2\right). \end{aligned}$$

Applying Lemma 4, with $\lambda = \frac{1}{4} + (2\mu - 1)k$, we obtain

$$|a_3 - \mu a_2^2| \leq k \max\left\{1, \left|\frac{1}{2} - 2(2\mu - 1)k\right|\right\}.$$

□

In the next theorem we look at the second Hankel determinant bound for functions in S^*_{GC}.

Theorem 11. *Let f of the form* (1) *be in S^*_{GC}. Then*

$$|a_2a_4 - a_3^2| \leq k^2. \tag{20}$$

Proof. Substitute (11), (12) and (13) into the functional $a_2a_4 - a_3^2$, we have

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{k^2}{3}c_1\left(c_3 - \frac{1}{2}(1-3k)c_1c_2 + \frac{k}{8}(4k-3)c_1^3\right) \\ &\quad - \frac{k^2}{4}\left(c_2^2 - 2c_1^2c_2\left(\frac{1}{4}-k\right) + \left(\frac{1}{4}-k\right)^2 c_1^4\right) \\ &= \frac{k^2}{24}\left(8c_1c_3 - 6c_2^2 - c_1^2c_2 - \frac{1}{8}(16k^2+3)c_1^4\right). \end{aligned}$$

Applying Lemma 2 and simplifying it,

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{k^2}{24}\left(8c_1c_3 - 6c_2^2 - c_1^2c_2 - \frac{1}{8}(16k^2+3)c_1^4\right) \\ &= \frac{k^2}{24}\Bigg(8c_1\frac{1}{4}(c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\ &\quad + 2(4-c_1^2)(1-|x|^2)z) - \frac{6}{4}(c_1^4 + 2c_1^2(4-c_1^2)x + x^2(4-c_1^2)^2) \\ &\quad - \frac{c_1^2}{2}(c_1^2 + x(4-c_1^2)) - \frac{1}{8}(16k^2+3)c_1^4\Bigg) \\ &= \frac{k^2}{24}\Bigg(2c_1^4 - \frac{3}{2}c_1^4 - \frac{1}{2}c_1^4 + 4c_1^2(4-c_1^2)x - 3c_1^2(4-c_1^2)x \\ &\quad - \frac{1}{2}c_1^2(4-c_1^2)x - 2c_1^2(4-c_1^2)x^2 - \frac{3}{2}(4-c_1^2)^2x^2 \\ &\quad + 4c_1(4-c_1^2)(1-|x|^2)z\Bigg) \\ &= \frac{k^2}{24}\Bigg(-\frac{1}{8}(16k^2+3)c_1^4 + \frac{1}{2}c_1^2(4-c_1^2)x \\ &\quad - \frac{1}{2}(c_1^2+12)(4-c_1^2)x^2 + 4c_1(4-c_1^2)(1-|x|^2)z\Bigg). \end{aligned}$$

Using Lemma 1 and assuming without loss generality $c = c_1 \in [0, 2]$. Applying triangle inequality, we obtain

$$|a_2a_4 - a_3^2| \le \frac{k^2}{24}\bigg(\frac{1}{8}(16k^2+3)c^4 + \frac{1}{2}c^2(4-c^2)|x| \\ + \frac{1}{2}(c^2+12)(4-c^2)|x|^2 + 4c(4-c^2)(1-|x|^2)\bigg) := G(c,|x|).$$

Differentiate $G(c,|x|)$ with respect to $|x|$, we get

$$\frac{\partial G(c,|x|)}{\partial |x|} = \frac{1}{2}c^2(4-c^2) + (4-c^2)(c^2-8c+12)|x|.$$

It is clear that $\frac{\partial G(c,|x|)}{\partial |x|} > 0$, which shows that $G(c,|x|)$ is an increasing function with $|x|$ in closed interval $|x| \in [0,1]$. Therefore, $G(c,|x|) \le G(c,1)$, and

$$\begin{aligned}\max G(c,|x|) &= G(c,1) \\ &= \frac{k^2}{24}\bigg(\frac{1}{8}(16k^2+3)c^4 + 16c - 4c^3 + \frac{1}{2}c^2(4-c^2) \\ &\qquad + \frac{1}{2}(4-c^2)(c^2-8c+12)\bigg) \\ &= \frac{k^2}{24}\left(\frac{1}{8}(16k^2-5)c^4 - 2c^2 + 24\right) := F(c), \quad \text{(say)}\end{aligned}$$

where $k \in (0, 2/3]$ is a constant. Then

$$\begin{aligned}F'(c) &= \frac{k^2}{24}\left(\frac{1}{2}(16k^2-5)c^3 - 4c\right) \\ &= \frac{k^2c}{48}\left((16k^2-5)c^2 - 8\right), \\ F''(c) &= \frac{k^2}{48}\left(3(16k^2-5)c^2 - 8\right).\end{aligned}$$

The critical numbers are

$$c = 0 \quad \text{and} \quad c = c^* = \sqrt{\frac{8}{16k^2-5}}.$$

For $c^* = \sqrt{\frac{8}{16k^2-5}}$,

$$F''(c^*) = \frac{k^2}{48}\left(3(16k^2-5)\frac{8}{16k^2-5} - 8\right)$$

$$= \frac{k^2}{3} > 0,$$

then relative minimum occurs at $c = \sqrt{\frac{8}{16k^2-5}}$.

For $c = 0$,

$$\begin{aligned} F''(0) &= \frac{k^2}{48}\big(3(16k^2-5)(0)-8\big) \\ &= \frac{-k^2}{6} < 0, \end{aligned}$$

then relative maximum occurs at $c = 0$ and the relative maximum is

$$\begin{aligned} G(0,1) = F(0) &= \frac{k^2}{24}\left(\frac{1}{8}(16k^2-5)(0) - 2(0) + 24\right) \\ &= k^2. \end{aligned}$$

To find the absolute maximum, we also check at the endpoint $c = 2$.

$$\begin{aligned} G(2,1) = F(2) &= \frac{k^2}{24}\left(\frac{1}{8}(16k^2-5)(16) - 2(4) + 24\right) \\ &= \frac{4}{3}k^4 + \frac{k^2}{4}. \end{aligned}$$

For $k \in (0, 2/3]$, $\quad k^2 > \frac{4}{3}k^4 + \frac{k^2}{4}$. Thus, $|a_2a_4 - a_3^2| \le k^2$. □

The following are Toeplitz determinants bound for functions in S^*_{GC}. Since the functional in Toeplitz determinants are not rotationally invariant, the assumption $c \in [0, 2]$ is invalid.

Theorem 12. *Let f be of the form (1) be in the class S^*_{GC}, then*

$$|T_2(2)| = |a_2^2 - a_3^2| \le \begin{cases} 5k^2, & \text{if } \ 0 < k \le \frac{1}{4}, \\ 4k^2 + \frac{k^2}{4}(4k+1)^2, & \text{if } \ \frac{1}{4} \le k \le \frac{2}{3}, \end{cases}$$

with $0 < k \le 2/3$. The inequality is sharp for the function $f(z) = ze^{k(2z+z^2/2)}$.

Proof. By using the triangle inequality,(14) and (17), we obtain

$$\begin{aligned} |T_2(2)| &= |a_2^2 - a_3^2| \le |a_2|^2 + |a_3|^2 \\ &\le \begin{cases} 4k^2 + k^2, & \text{if } \ 0 < k \le \frac{1}{4}, \\ 4k^2 + \frac{k^2}{4}(4k+1)^2, & \text{if } \ \frac{1}{4} \le k \le \frac{2}{3}. \end{cases} \end{aligned}$$

□

Theorem 13. *Let f be of the form* (1) *be in the class S^*_{GC}, then*

$$|a_3^2 - a_4^2| \leq \begin{cases} k^2 + \frac{k^2}{9}(4k^2 - 3k + 2)^2, & \text{if} \quad 0 < k \leq \frac{1}{4}, \\ \frac{k^2}{4}(4k+1)^2 + \frac{k^2}{9}(4k^2 - 3k + 2)^2, & \text{if} \quad \frac{1}{4} \leq k \leq \frac{1}{3}, \\ \frac{k^2}{4}(4k+1)^2 + \frac{k^2}{9}(4k^2 - 3k)^2, & \text{if} \quad \frac{1}{3} \leq k \leq \frac{2}{3}. \end{cases}$$

Proof. Using the same concept of proving as in Theorem 12, by applying triangle inequality,(17) and (19), we have

$$\begin{aligned} |T_2(3)| = |a_3^2 - a_4^2| &\leq |a_3|^2 + |a_4|^2 \\ &\leq \begin{cases} k^2 + \frac{k^2}{9}(4k^2 - 3k + 2)^2, & \text{if} \quad 0 < k \leq \frac{1}{4}, \\ \frac{k^2}{4}(4k+1)^2 + \frac{k^2}{9}(4k^2 - 3k + 2)^2, & \text{if} \quad \frac{1}{4} \leq k \leq \frac{1}{3}, \\ \frac{k^2}{4}(4k+1)^2 + \frac{k^2}{9}(4k^2 + 3k)^2, & \text{if} \quad \frac{1}{3} \leq k \leq \frac{2}{3}. \end{cases} \end{aligned}$$

□

Theorem 14. *Let f be of the form* (1) *be in the class S^*_{GC}, then*

$$\begin{aligned} &|1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \\ &\leq \begin{cases} 1 + 8k^2 + k^2 \; max \left\{1, |1 - \frac{k}{2}(1 + 12k^2)|\right\}, & \text{if } 0 < k < \frac{1}{4}, \\ 1 + 8k^2 + \frac{k^2}{2}(4k+1) \; max \left\{1, |1 - \frac{k}{2}(1 + 12k^2)|\right\}, & \text{if } \frac{1}{4} \leq k \leq \frac{2}{3}. \end{cases} \end{aligned}$$

Proof. Since $f \in S^*_{GC}$ is of the form (1) and applying triangle inequality then

$$|T_3(1)| = |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|.$$

Next, applying Fekete-Szegö functional with $\mu = 2$ we have

$$|a_3 - 2a_2^2| \leq k \text{ max } \left\{1, \left|1 - \frac{k}{2}(1 + 12k^2)\right|\right\}. \tag{21}$$

By considering (14), (17) and (21), we conclude that the proof is complete. □

Theorem 15. *For $f \in S^*_{GC}$ of the form* (1) *then*

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)|$$

$$\leq \begin{cases} 6k^2\left(2k+\frac{k}{3}(4k^2-3k+2)\right), & \textit{if} \quad 0<k\leq\frac{1}{4}, \\ \left(2k+\frac{k}{3}(4k^2-3k+2)\right)\left(5k^2+\frac{k^2}{4}(4k+1)^2\right), & \textit{if} \quad \frac{1}{4}\leq k\leq\frac{1}{3}, \\ \left(2k+\frac{k}{3}(4k^2+3k)\right)\left(5k^2+\frac{k^2}{4}(4k+1)^2\right), & \textit{if} \quad \frac{1}{3}\leq k\leq\frac{2}{3}. \end{cases}$$

Proof. Let $f \in S^*_{GC}$ be of the form (1), then

$$\begin{aligned} |T_3(2)| &= |(a_2-a_4)(a_2^2-2a_3^2+a_2a_4)| \\ &= |a_2-a_4||a_2^2-2a_3^2+a_2a_4| \\ &= |a_2-a_4||a_2^2-a_3^2-a_3^2+a_2a_4| \\ &\leq |a_2-a_4|(|a_2^2-a_3^2|+|a_2a_4-a_3^2|). \end{aligned}$$

Clearly, by triangle inequality $|a_2-a_4| \leq |a_2|+|a_4|$. Therefore

$$|a_2-a_4| \leq |a_2|+|a_4| \\ \begin{cases} 2k+\frac{k}{3}(4k^2-3k+2), & \text{if} \quad 0<k\leq\frac{1}{3}, \\ 2k+\frac{k}{3}(4k^2+3k), & \text{if} \quad \frac{1}{3}\leq k\leq\frac{2}{3}. \end{cases} \tag{22}$$

Also, $|a_2^2-a_3^2-a_3^2+a_2a_4| \leq |a_2^2-a_3^2|+|a_2a_4-a_3^2|$. Then by (20) and Theorem 12, we obtain

$$\begin{aligned} |a_2^2-a_3^2-a_3^2+a_2a_4| &\leq |a_2^2-a_3^2|+|a_2a_4-a_3^2| \\ &\leq \begin{cases} 6k^2, & \text{if} \quad 0<k\leq\frac{1}{4}, \\ 5k^2+\frac{k^2}{4}(4k+1)^2, & \text{if} \quad \frac{1}{4}\leq k\leq\frac{2}{3}. \end{cases} \end{aligned} \tag{23}$$

By applying together (22) and (23), we complete the proof. □

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