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SYMMETRY REDUCTIONS AND INVARIANT SOLUTIONS OF A NONLINEAR FOKKER-PLANCK EQUATION BASED ON THE SHARMA-TANEJA-MITTAL ENTROPY

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Abstract: A nonlinear Fokker-Planck equation that arises in the framework of statistical mechanics based on the Sharma-Taneja-Mittal entropy is studied via Lie symmetry analysis. The equation describes kinetic processes in anomalous mediums where both super-diffusive and subdiffusive mechanisms arise contemporarily and competitively. We perform complete group classification of the equation based on two parameters that characterise the underlying Sharma-Taneja-Mittal entropy. For arbitrary values of the parameters, the equation is found to admit a two-dimensional symmetry Lie algebra. We identify and catalogue all the cases in which the equation admits additional Lie point symmetries. We also perform symmetry reductions of the equation and obtain second-order ordinary differential equations that describe all essentially different invariant solutions of the equation.

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1. Introduction

The classical operational calculus is intended mainly to solving initial value problems both for ordinary differential equations (ODEs) and for partial differential equations (PDEs) with constant coefficients, [15].

Adaptations of the Fokker-Planck equation serve as mathematical models of various problems that arise in physical and biological sciences (see Caldas et al [1] and the references therein). In statistical mechanics, for example, the Fokker-Planck equation is used to describe kinetic processes in anomalous mediums. In this application, the linear version of the equation is considered appropriate for the description of a wide variety of physical phenomena characterized by short-range interactions and/or short-time memories, typically associated with normal diffusion. The nonlinear Fokker-Planck equation, on the other hand, is associated with anomalous diffusion, generally associated with non-Gaussian distributions Scarfone and Wada, [2].

In this paper we study a nonlinear Fokker-Planck equation (derived in Scarfone and Wada [2]) in the framework of statistical mechanics based on a two-parameter entropy known as the Sharma-Taneja-Mittal (STM) entropy. The equation is a $(1+1)$-partial differential equation, namely

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(xu) - \Omega\,\frac{\partial^2}{\partial x^2}\left[u^{1+r}\left(\frac{r+\kappa}{2\,\kappa}\,u^{\kappa} - \frac{r-\kappa}{2\,\kappa}\,u^{-\kappa}\right)\right] = 0, \tag{1}$$

where Ω is a constant diffusion coefficient, r and κ ($\neq 0$) are *deformation* parameters that characterise the underlying entropy and essentially define the family of equations (1). The dependent variable $u \equiv u(x,t)$ is the normalized density distribution describing a conservative particle system in the velocity-time space (x,t). We shall refer to this equation as the Sharma-Taneja-Mittal nonlinear Fokker-Planck (STM-NFP) equation. For a detailed account of the context and derivation of the equation the interested reader may consult Scarfone and Wada [2] wherein Equation (1) is studied, along with two well-known special cases, via Lie symmetry analysis. Admitted Lie point symmetries are found and used to construct invariant solutions.

The invariant solutions of the STM-NFP equation reported in Scarfone and Wada [2] are based on two infinitesimal symmetry generators

$$X_1 = \partial_t, \quad X_2 = e^{-t}\partial_x, \tag{2}$$

the only ones admitted by the whole family of equations represented by (1) for arbitrary r and κ. In this paper, we determine all instances, depending on specifications of r and κ, under which equation (1) admits additional symmetries besides those in (2). This is called complete group classification of (1). Related work is carried out by Ivanova et al. [4, 5, 6] who report results of group classification of various versions of nonlinear diffusion-convection equations. Numerous other papers have been devoted to group classification of diverse differential equations. In particular, many classes of nonlinear evolution equations depending on arbitrary functions of one, or at most two, variables have been studied via this method [3, 9, 7, 8, 10, 11, 12, 13, 14, 15].

In the case of the equation under consideration in this paper, (1), we have determined via the method of group classification that there are basically four cases corresponding to $r = \pm k$ and $r = -1 \pm k$ in which equation (1) admits additional symmetries. We have also constructed a number of invariant solutions of the equation.

The paper is organised as follows. In Section 2, we introduce elements of Lie symmetry analysis of differential equations. Group classification of (1) is done in Section 3. In this section we determine all the instances depending on r and κ when the principal Lie algebra of (1) is extended. In Section 4, we construct adjoint representations of the symmetry Lie algebras of (1) corresponding to all the instances when the equation admits additional symmetries. Furthermore, we compute corresponding optimal systems of admitted one-dimensional subalgebras and perform symmetry reductions of (1). Finally, we give concluding remarks in Section 5.

2. Preliminaries

Lie symmetry analysis is one of the most powerful methods for finding analytical solutions of differential equations. It has its origins in studies by the Norwegian mathematician Sophus Lie who began to investigate continuous groups of transformations that leave differential equations invariant. Accounts of the subject and its application to differential equations are covered in many books (see, for example, [21, 22, 23] for an introduction and [18, 19, 20, 17] for a more detailed exposition). Central to methods of Lie symmetry analysis is invariance of a differential equation under a continuous group of transformations. Consider a one-parameter Lie group of point transformations

$$\begin{aligned} \tilde{x} &= x + \varepsilon\, \xi(x,t,u) + O(\varepsilon^2) \\ \tilde{t} &= t + \varepsilon\, \tau(x,t,u) + O(\varepsilon^2) \\ \tilde{u} &= u + \varepsilon\, \eta(x,t,u) + O(\varepsilon^2) \end{aligned} \tag{3}$$

depending on a continuous parameter ε. This transformation is characterised by its its infinitesimal generator,

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{4}$$

A general $(1+1)$ partial differential equation with a dependent variable u and independent variables (x,t),

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \tag{5}$$

is invariant under (3) if and only if

$$X^{(2)}\Delta = 0 \quad \text{when} \quad \Delta = 0, \tag{6}$$

where $X^{(2)}$ is the second prolongation of X given by

$$X^{(2)} = X + \eta_i^{(1)}\partial_{u_i} + \eta_{i_1 i_2}^{(2)}\partial_{u_{i_1 i_2}}, \quad i_1, i_2 = 1, 2, \tag{7}$$

with

$$\eta_i^{(1)} = D_i\eta - \left(D_i\xi^j\right)u_j, \quad \eta_{i_1 i_2}^{(2)} = D_{i_2}\eta_{i_1}^{(1)} - \left(D_{i_2}\xi^j\right)u_{i_1 j}, \quad i, i_k, j = 1, 2, \tag{8}$$

where $u_i = \frac{\partial u}{\partial x^i}$, $u_{i_1 i_2} = \frac{\partial^2 u}{\partial x^{i_1}\partial x^{i_2}}$, $i, i_j = 1, 2$, $(x^1, x^2) = (x, t)$, $(\xi^1, \xi^2) = (\xi, \tau)$ and D_i denotes the total differential operator with respect to x^i:

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + u_{ijk}\frac{\partial}{\partial u_{jk}} + \cdots \tag{9}$$

The Einstein summation convention is adopted in (7), (8) and (9). The invariance condition (6) yields an over-determined system of linear partial differential equations (determining equations) for the symmetry group of equation (5). The infinitesimals ξ, τ and η are then determined as a general solution to the determining equations. If the infinitesimals contain more than one arbitrary constant, the resulting multi-parameter infinitesimal generator is split into single parameter generators, which constitute a basis for the symmetry Lie algebra of (5).

When equation (5) has arbitrary elements (parameters and/or functions) the nature and dimension of the admitted symmetry Lie algebra typically depends on these arbitrary elements. For many equations modelling natural phenomena it is desirable that they possess non-trivial symmetry. This is because in such cases it is possible to obtain a lot of information about solutions of the equation, including reduction of multidimensional equations to ordinary differential equations, constructing classes of exact and approximate solutions. Furthermore, mathematical models must be of such a form that they are consistent with important natural principles of physics such as conservation laws of energy, momentum, etc. It turns out that this beauty of equations of mathematical physics is often encoded in the symmetries admitted by the equation Fushchych [24]. From this point of view, therefore, we seek to specify the arbitrary elements in the equation in such a way as to increase the dimension of the admitted symmetry Lie algebra. This is the essence of the group classification method. Seminal work on group classification was done by Sophus Lie [25] (see also Ovsiannikov [19]) who investigated linear second-order partial differential equations (PDEs) with two independent variables. For a more detailed account of the method of group classification of differential equations the reader is refereed to Gazizov and Ibragimov [16] and Olver [17] (and the references contained therein).

3. Lie point symmetries of the STM-NFP equation

Clearly, when $r = -1/2$ and $\kappa = \pm 1/2$ equation (1) is reduced to a first-order linear equation and therefore admits an infinite dimensional Lie algebra. These cases will not be considered. For other values of the parameters, suppose

$$X = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u, \tag{10}$$

where ξ, τ and η are arbitrary functions, is admitted by Equation (1). The invariance condition

$$X^{(2)}\left\{\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(xu) - \Omega\,\frac{\partial^2}{\partial x^2}\left[u^{1+r}\left(\frac{r+\kappa}{2\,\kappa}\,u^{\kappa} - \frac{r-\kappa}{2\,\kappa}\,u^{-\kappa}\right)\right]\right\}\Bigg|_{(1)} = 0, \tag{11}$$

where $X^{(2)}$ is the second-prolongation of X defined in (7), translates in the the usual way into the following system of determining equations, thanks to the handiness of Mathematica [26]:

$$\xi_u = 0, \tag{12}$$

$$\tau_u = 0, \tag{13}$$

$$\tau_x = 0, \tag{14}$$

$$\xi + \xi_t - x\,\xi_x + x\,\tau_t - 2\,(\kappa - r)\,u^{r-\kappa-1}\,\varphi(-\kappa,r)\,\eta_x$$
$$+\,2\,(\kappa + r)\,u^{r+\kappa-1}\,\varphi(\kappa,r)\,\eta_x - \,u^{r-\kappa}\,\varphi(-\kappa,r)\,(\xi_{xx} - 2\,\eta_{xu})$$

$$- u^{\kappa+r}\,\varphi(\kappa,r)\;(\xi_{xx} - 2\,\eta_{xu}) = 0, \tag{15}$$

$$\begin{aligned}&\eta + u\,\tau_t - \eta_t - u\,\eta_u + x\,\eta_x + u^{r-\kappa}\,\varphi(-\kappa,r)\,\eta_{xx}\\ &+ u^{\kappa+r}\,\varphi(\kappa,r)\,\eta_{xx} = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\eta\;(r+\kappa-1)\;(\kappa+r)\;u^{r+\kappa-1}\,\varphi(\kappa,r) + \eta\,u^{r-\kappa-1}\,\vartheta(\kappa,r)\\ &+ \left[(r-\kappa)\;u^{r-\kappa}\,\varphi(-\kappa,r) + (\kappa+r)\;u^{\kappa+r}\,\varphi(\kappa,r)\right]\;(\tau_t + \eta_u - 2\,\xi_x)\\ &+ \left(u^{1+\kappa+r}\,\varphi(\kappa,r) + \frac{\Omega\,u^{1-\kappa+r}\,\vartheta(\kappa,r)}{2\,\kappa\,\varphi(\kappa,-r)}\right)\;\eta_{uu} = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}&\eta\;(r-\kappa)\;u^{r-\kappa-1}\,\varphi(-\kappa,r) + \eta\;(\kappa+r)\;u^{r+\kappa-1}\,\varphi(\kappa,r)\\ &- u^{r-\kappa}\,\varphi(-\kappa,r)\;(2\,\xi_x - \tau_t) - u^{\kappa+r}\,\varphi(\kappa,r)\;(2\,\xi_x - \tau_t) = 0,\end{aligned} \tag{18}$$

where

$$\begin{aligned}\varphi(\kappa,r) &= \frac{\Omega\;(\kappa+r)\;(1+\kappa+r)}{2\,\kappa},\\ \vartheta(\kappa,r) &= \frac{\Omega\;\left[1-(\kappa-r)^2\right]\;(\kappa-r)^2}{2\,\kappa}.\end{aligned}$$

From the equations (12)–(14) and (18), we obtain that

$$\xi(x,t,u) = \xi(x,t) \tag{19}$$

$$\tau(x,t,u) = \tau(t) \tag{20}$$

$$\eta(x,t,u) = \frac{u\left[\varphi(-\kappa,r) + \varphi(\kappa,r)\,u^{2\,\kappa}\right]\,(2\,\xi_x - \tau')}{(r-\kappa)\varphi(-\kappa,r) + (\kappa+r)\varphi(\kappa,r)\,u^{2\,\kappa}}. \tag{21}$$

The outstanding equations, (15)–(17), are now expressed in terms of and solved for ξ and τ. Equation (17) in particular becomes

$$\frac{\left[u^{\kappa+r}\,\psi_1(\kappa,r) + u^{3\,\kappa+r}\,\psi_2(\kappa,r) + u^{5\,\kappa+r}\,\psi_3(\kappa,r)\right]\,[\tau' - 2\,\xi_x]}{\psi_0(\kappa,r,u)} = 0, \tag{22}$$

where $'$ denotes the differentiation with respect to t, and

$$\begin{aligned}\psi_0 &= \left[(\kappa-r-1)\;(\kappa-r)^2 + (\kappa+r)^2\;(1+\kappa+r)\;u^{2\,\kappa}\right]^3\\ \psi_1 &= 2\,\kappa\;(1+2\,\kappa)\;\Omega\,(\kappa-r)^3\,(1-\kappa+r)^2\end{aligned}$$

$$
\begin{aligned}
& \quad \times \left[\kappa^4 + r^2\,(1+r)^2 - \kappa^2\,\left(1+2\,r+2\,r^2\right)\right] \\
\psi_2 \quad &= \quad 4\,\kappa\,\Omega\,\left[\kappa^4 + r^2\,(1+r)^2 - \kappa^2\,\left(1+2\,r+2\,r^2\right)\right] \\
& \quad \times \left[r^3\,(1+r)^2 - 2\,\kappa^6 + \kappa^4\,\left(2+5\,r+4\,r^2\right)\right. \\
& \qquad \left. -\,\kappa^2\,r\,\left(1+4\,r+6\,r^2+2\,r^3\right)\right] \\
\psi_3 \quad &= \quad 2\,\kappa\,(2\,\kappa-1)\,\Omega\,(\kappa+r)^3\,(1+\kappa+r)^2 \\
& \quad \times \left[\kappa^4 + r^2\,(1+r)^2 - \kappa^2\,\left(1+2\,r+2\,r^2\right)\right].
\end{aligned}
$$

Clearly equation (22) is solved if and only if $\psi_0(\kappa, r, u) \neq 0$ and

$$\tau' - 2\,\xi_x = 0, \tag{23}$$

or

$$u^{\kappa+r}\,\psi_1(\kappa, r) + u^{3\,\kappa+r}\,\psi_2(\kappa, r) + u^{5\,\kappa+r}\,\psi_3(\kappa, r) = 0. \tag{24}$$

In the light of (19)–(21), the solution of (23) together with (15) and (16) leads to

$$\xi = \varepsilon_1\,e^{-t}, \quad \tau = \varepsilon_2, \quad \eta = 0, \tag{25}$$

from which we obtain the infinitesimal symmetry generators X_1 and X_2 in (2). This means that the principal Lie algebra of (1) is spanned by X_1 and X_2. Instances in which (1) admits additional symmetries are obtained from solutions of (24). Solving this equation for r, by setting $\psi_1(\kappa, r) = \psi_2(\kappa, r) = \psi_3(\kappa, r) = 0$, we obtain

$$r = \pm\kappa \quad \text{or} \quad r = -1 \pm \kappa. \tag{26}$$

We consider each of these cases in turn to determine symmetries admitted by (1) in these instances. This essentially means solving the outstanding determining equations (15) and (16) for ξ and τ. For each of the parameter specifications in (26) the symmetries admitted by (1) include X_1 and X_2 specified in (2). Additional ones are admitted in the various cases as presented below in Table 1.

$$\delta = \begin{cases} \kappa & \text{for (i)} \\ -\kappa & \text{for (ii)} \\ \kappa - \frac{1}{2} & \text{for (iii)} \\ -\kappa - \frac{1}{2} & \text{for (iv).} \end{cases} \tag{27}$$

Table 1: Symmetries of Eqn (1) in the Cases $r = \pm\kappa$ and $r = -1 \pm \kappa$, with δ in Case C defined by (27).

Case A	Specifications of r and κ	$\begin{cases} \text{(i)} & r = \kappa,\ \kappa = -\frac{2}{3} \\ \text{(ii)} & r = -\kappa,\ \kappa = \frac{2}{3} \\ \text{(iii)} & r = -1+\kappa,\ \kappa = -\frac{1}{6} \\ \text{(iv)} & r = -1-\kappa,\ \kappa = \frac{1}{6} \end{cases}$
	Admitted infinitesimal generators	$\begin{cases} X_1 = \partial_t, \quad X_2 = e^{-t}\partial_x \\ X_3 = e^{-2\,t/3}\,(x\,\partial_x - \partial_t - u\,\partial_u) \\ X_4 = x\,e^t\,(x\,\partial_x - 3\,u\,\partial_u) \\ X_5 = x\,\partial_x - \partial_t - \frac{3}{2}\,u\,\partial_u \end{cases}$
Case B	Specifications of r and κ	$\begin{cases} \text{(i)} & r = \kappa,\ \kappa = -1 \\ \text{(ii)} & r = -\kappa,\ \kappa = 1 \\ \text{(iii)} & r = -1+\kappa,\ \kappa = -\frac{1}{2} \\ \text{(iv)} & r = -1-\kappa,\ \kappa = \frac{1}{2} \end{cases}$
	Admitted infinitesimal generators	$\begin{cases} X_1 = \partial_t, \quad X_2 = e^{-t}\partial_x \\ X_3 = x\,\partial_x - \partial_t - u\,\partial_u, \\ X_4 = t\,x\,\partial_x - t\,\partial_t - u\left(t + \frac{1}{2}\right)\,\partial_u \end{cases}$
Case C	Specifications of r and κ	$\begin{cases} \text{(i)} & r = \kappa,\ \kappa \notin \left\{-\frac{1}{2}, -\frac{2}{3}, -1\right\} \\ \text{(ii)} & r = -\kappa,\ \kappa \notin \left\{\frac{1}{2}, \frac{2}{3}, 1\right\} \\ \text{(iii)} & r = -1+\kappa,\ \kappa \notin \left\{\pm\frac{1}{2}, -\frac{1}{6}\right\} \\ \text{(iv)} & r = -1-\kappa,\ \kappa \notin \left\{\pm\frac{1}{2}, \frac{1}{6}\right\} \end{cases}$
	Admitted infinitesimal generators	$\begin{cases} X_1 = \partial_t, \quad X_2 = e^{-t}\partial_x \\ X_3 = e^{-2\,(1+\delta)\,t}\,(x\,\partial_x - \partial_t - u\,\partial_u) \\ X_4 = x\,\partial_x - \partial_t + (u/\delta)\,\partial_u. \end{cases}$

Table 2: Commutator table for Case A.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	$-\frac{2}{3}X_3$	X_4	0
X_2	X_2	0	0	$2\,(X_1 + X_5)$	0
X_3	$\frac{2}{3}X_3$	0	0	0	$-\frac{2}{3}X_3$
X_4	$-X_4$	$-2\,(X_1 + X_5)$	0	0	0
X_5	0	0	$\frac{2}{3}X_3$	0	0

Table 3: Commutator table for Case B.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	X_3
X_2	X_2	0	0	0
X_3	0	0	0	$-X_3$
X_4	$-X_3$	0	X_3	0

Table 4: Commutator table for Case C, where $\delta = \kappa, -\kappa, \kappa-\frac{1}{2}, -\kappa-\frac{1}{2}$ in the specifications (i), (ii), (iii) and (iv), respectively.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	$-2(1+\delta)X_3$	0
X_2	X_2	0	0	0
X_3	$2(1+\delta)X_3$	0	0	$-2(1+\delta)X_3$
X_4	0	0	$2(1+\delta)X_3$	0

The commutator tables for the corresponding Lie algebras are given in Tables 2, 3 and 4.

4. Invariant solutions of the STM-NFP equation

Each of the infinitesimal symmetry generators admitted by (1) can be used to construct a family of invariant solutions of the equation. A function $u = \Theta(x,t)$ is an invariant solution of (5) arising from X if it is a solution of (5) and satisfies the *invariant surface* condition,

$$X\left(u - \Theta(x,t)\right) = 0 \quad \text{when} \quad u = \Theta(x,t). \tag{28}$$

The construction of invariant solutions proceeds in a very algorithmic fashion. For each infinitesimal symmetry generators X, one determines from solutions of the associated corresponding system,

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{29}$$

two independent invariants $r(x,t,u)$ and $v(x,t,u)$ (with $v_u \neq 0$) of the associated group. The form of the invariant solution arising from X is now obtained from $v = F(r)$ or

$$u = \Theta(x,t) \tag{30}$$

when solved for u. Upon substitution of (30) into (5) we obtain an ODE that defines Θ, the solution of which completes the construction of the invariant solution.

To avoid "duplicating" invariant solutions we determine optimal systems (in the usual way Olver [17]). For each of the Lie algebras represented in Table 1, we construct an adjoint representation of the underlying Lie group via the Lie series

$$\mathrm{Adj}\big(\exp(\varepsilon\, X_i)\big) X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2}\, \varepsilon^2 [X_i, [X_i, X_j]] - \cdots , \tag{31}$$

where $[X_i, X_j]$ is the Lie bracket of X_i and X_j. For the cases identified in Table 1 the adjoint representations are presented in Table 5, Table 6 and Table 7 for Cases A, B and C, respectively, where the (i,j)-th entry indicates $\mathrm{Ad}\,(\exp(\varepsilon X_i)) X_j$.

$$\Psi_{24}(X_1 \ldots, X_5) = \varepsilon^2 X_2 + X_4 - 2\,\varepsilon(X_1 + X_5), \tag{32}$$

$$\Psi_{42}(X_1 \ldots, X_5) = X_2 + \varepsilon^2 X_4 + 2\,\varepsilon(X_1 + X_5). \tag{33}$$

Table 5: Adjoint representations for Case A, where $\Phi_{ij}(\cdot)$ are defined in (32) and (33).

Adj	X_1	X_2	X_3	X_4	X_5
X_1	X_1	$e^{\varepsilon} X_2$	$e^{\frac{2}{3}\varepsilon} X_3$	$e^{-\varepsilon} X_4$	X_5
X_2	$X_1 - \varepsilon X_2$	X_2	X_3	$\Psi_{24}(\cdot)$	X_5
X_3	$X_1 - \frac{2\varepsilon}{3} X_3$	X_2	X_3	X_4	$X_5 + \frac{2\varepsilon}{3} X_3$
X_4	$X_1 + \varepsilon X_4$	$\Psi_{42}(\cdot)$	X_3	X_4	X_5
X_5	X_1	X_2	$e^{-\frac{2}{3}\varepsilon} X_3$	X_4	X_5

Table 6: Adjoint representations for Case B.

Adj	X_1	X_2	X_3	X_4
X_1	X_1	$e^{\varepsilon} X_2$	X_3	$X_4 - \varepsilon X_3$
X_2	$X_1 - \varepsilon X_2$	X_2	X_3	X_4
X_3	X_1	X_2	X_3	$X_4 + \varepsilon X_3$
X_4	$X_1 + (1 - e^{-\varepsilon}) X_3$	X_2	$e^{-\varepsilon} X_3$	X_4

Table 7: Adjoint representations for Case C.

Adj	X_1	X_2	X_3	X_4
X_1	X_1	$e^{\varepsilon} X_2$	$e^{2(1+\delta)\varepsilon} X_3$	X_4
X_2	$X_1 - \varepsilon X_2$	X_2	X_3	X_4
X_3	$X_1 - 2\,(1+\delta)\,\varepsilon X_3$	X_2	X_3	$X_4 + 2\,(1+\delta)\,\varepsilon X_3$
X_4	X_1	X_2	$e^{-2(1+\delta)\varepsilon} X_3$	X_4

Following Olver's approach [17], we use the adjoint representations to identify equivalent infinitesimal symmetry generators. It turns out that every invariant solution of the STM-NFP equation can be obtained from an invariant solution constructed from one of the elements from the optimal systems presented in Table 8.

Table 8: Optimal system of one-dimensional subalgebras of the STM-NFP equation; α, β and γ are arbitrary constants.

Case A	• $\beta X_2 + X_4 + \gamma X_5$ • $\alpha X_1 + \beta X_2 + X_4$ • X_1	Cases A & B	• $\alpha X_1 + X_4$ • $\alpha X_1 + X_3$ • X_1

Invariant solutions of (1) arising from the representative infinitesimal symmetry generators in the optimal systems represented in Table 8 are obtained in the usual way.

1. Case A.

- $\beta X_2 + X_4 + \gamma X_5$

$$u(x,t) = e^{\frac{3\,t}{2}} \left(1 + x^2 e^{2\,t}/\beta\right)^{-\frac{3}{2}} y\,(\zeta)\,, \tag{34}$$

$$\zeta = \sqrt{\beta}\,t/\gamma + \tan^{-1}\left(e^t\,x/\sqrt{\beta}\right), \quad \beta\,\gamma \neq 0,$$

where

$$18\,\beta^{\frac{3}{2}}\,y^{\frac{7}{3}}\,y' + 9\,\beta\,\gamma\,y^{\frac{10}{3}} + 2\,\theta\,\gamma\,\Omega\,\left[3\,y\,y'' - 9\,y^2 - 4\,(y')^2\right] = 0. \tag{35}$$

with

$$\theta = \begin{cases} 1 & \text{for Parts (i) and (ii),} \\ 4 & \text{for Parts (iii) and (iv).} \end{cases} \tag{36}$$

- $\alpha X_1 + \beta X_2 + X_4$

$$u(x,t) = \frac{8\,\alpha^3\,\omega^3\,e^{\frac{3(1-\omega)t}{2}}\,y\,(\zeta)}{\left[2\,e^t\,x + \alpha(1+\omega)\right]^3}, \quad \zeta = \frac{\alpha\,(\omega - 1) - 2\,e^t\,x}{\omega\,\left[2\,e^t\,x + \alpha(1+\omega)\right]\,e^{\omega\,t}}, \tag{37}$$

where

$$18\,\omega^5\,\alpha^2\,\zeta\,y^{\frac{7}{3}}\,y' + \rho\,y^{\frac{10}{3}} + 2\,\theta\,\Omega\left[4\,(y')^2 - 3\,y\,y''\right] = 0, \tag{38}$$

with θ as defined in (36), and

$$\omega = \sqrt{1 - 4\,\beta/\alpha^2}, \quad \alpha^2 - 4\,\beta \geq 0 \quad \text{and} \quad \rho = 9\,\alpha^2\omega^4\,(3\,\omega - 1)\,.$$

- X_1

$$u(x,t) = y(\zeta), \quad \zeta = x, \tag{39}$$

where y is a solution of the ODE

$$4\,\Omega\,(y')^2 - 3\,\Omega\,y\,y'' + 9\,\zeta\,y^{\frac{7}{3}}\,y' + 9\,y^{\frac{10}{3}} = 0. \tag{40}$$

2. Case B.

- $\alpha X_1 + X_4$

$$u(x,t) = e^t\,(t-\alpha)^{\alpha+\frac{1}{2}}\,y(\zeta), \quad \zeta = e^t\,(t-\alpha)^{\alpha}\,x, \tag{41}$$

where

$$2\,\theta\,\Omega\left[y\,y'' - 2\,(y')^2\right] + 2\,\alpha\,\zeta\,y^3\,y' + (1+2\,\alpha)\;y^4 = 0. \tag{42}$$

with

$$\theta = \begin{cases} 1 & \text{for Parts (i) and (ii),} \\ 2 & \text{for Parts (iii) and (iv).} \end{cases} \tag{43}$$

- $\alpha X_1 + X_3$

$$u(x,t) = e^{\frac{t}{1-\alpha}}\,y(\zeta), \quad \zeta = e^{\frac{t}{1-\alpha}}\,x, \tag{44}$$

where

$$\theta\,\Omega\,(\alpha-1)\,[2\,(y')^2 - y\,y''] + \alpha(y^4 + \zeta\,y^3\,y') = 0. \tag{45}$$

with θ as defined in (43).

- X_1

$$u(x,t) = y(x), \tag{46}$$

where

$$\theta\,\Omega\left[y\,y'' - 2\,(y')^2\right] - x\,y^3\,y' - y^4 = 0, \tag{47}$$

with θ as defined in (43).

3. Case C: (i) & (ii) $[\delta = \kappa$ for (i), $\delta = -\kappa$ for (ii)$]$

- $\alpha X_1 + X_4$

$$u(x,t) = e^{\frac{t}{\delta(\alpha-1)}}\,y(\zeta), \quad \zeta = x\,e^{\frac{t}{1-\alpha}}, \tag{48}$$

where

$$y\,\left[\lambda\,y - \delta\,\alpha\,\zeta\,y'\right] - \Omega\,\delta\,\omega\,y^{2\,\delta}\,\left[y\,y'' + 2\,\delta\,(y')^2\right] = 0, \tag{49}$$

with

$$\omega = (\alpha-1)\;(1+2\,\delta)\,, \quad \lambda = 1 + \delta(1-\alpha).$$

- $X_1 + X_3$

$$u(x,t) = \frac{e^t\, y(\zeta)}{\left(e^{2\,(1+\delta)\,t} - 1\right)^{\frac{1}{2(1+\delta)}}}, \quad \zeta = \frac{e^t\, x}{\left(e^{2\,(1+\delta)\,t} - 1\right)^{\frac{1}{2(1+\delta)}}}. \tag{50}$$

where

$$y^2 + \Omega\,(1 + 2\,\delta)\, y^{2\,\delta}\left[y\, y'' + 2\,(y')^2\,\delta\right] + \zeta\, y\, y' = 0. \tag{51}$$

- X_1

$$u(x,t) = y(x), \tag{52}$$

where

$$\Omega\,(1 + 2\,\delta)\; y^{2\,\delta-1}\left[y\, y'' + 2\,\delta\,(y')^2\right] + x\, y' + y = 0. \tag{53}$$

4. Case C: (iii) & (iv) [$\delta = \kappa$ for (iii) & $\delta = -\kappa$ for (iv)]

- $\alpha X_1 + X_4$

$$u(x,t) = \exp\left\{\frac{2\,t}{\nu\,(\alpha - 1)}\right\} y(\zeta), \quad \zeta = x\, e^{\frac{t}{1-\alpha}}, \tag{54}$$

where

$$\Omega\,\nu\, y^{1+2\,\delta}\, y'' + \Omega\,\nu^2\, y^{2\,\delta}\,(y')^2 + \mu\,\zeta\, y^2\, y' + \omega\, y^3 = 0, \tag{55}$$

with

$$\mu = \frac{\alpha}{\alpha - 1}, \quad \omega = 1 + \frac{2}{(1-\alpha)\,\nu}, \quad \nu = 2\,\delta - 1.$$

- $X_1 + X_3$

$$u(x,t) = \frac{e^t\, y(\zeta)}{\left[e^{(1+2\,\delta)t} - 1\right]^{\frac{1}{1+2\,\delta}}}, \quad \zeta = \frac{x\, e^t}{\left[e^{(1+2\,\delta)t} - 1\right]^{\frac{1}{1+2\,\delta}}}, \tag{56}$$

where

$$y^3 + \Omega\,\rho^2\, y^{2\,\delta}\,(y')^2 - \Omega\,\rho\, y^{1+2\,\delta}\, y'' + \zeta\, y^2\, y' = 0, \quad \rho = 1 - 2\,\delta. \tag{57}$$

- X_1

$$u(x,t) = y(\zeta), \quad \zeta = x, \tag{58}$$

where y is any solution of (57).

5. Concluding remarks

In the work reported in this paper we have carried out complete group classification of the STM-NFP equation and thereby identified all "interesting" particular cases of the equation. We have established that when $r = -1/2$ and $\kappa = \pm 1/2$ the STM-NFP equation is reduced to a first-order linear equation and therefore admits an infinite dimensional Lie algebra. For all the other parameter specifications, the equation admits a two-dimensional Lie algebra spanned by the symmetries X_1 and X_2 stated in (2). This is the principal Lie algebra of the equation. Extensions of the principal Lie algebra occur only if $r = \pm\kappa$ or $r = -1 \pm \kappa$. We have identified all the different scenarios under which the admitted Lie algebra extends. The corresponding parameter specifications possibly define particular interesting entropies associated with the STM-NFP equation. In each of these instances the equation is endowed with remarkable physical properties and is amenable to solution via routines of Lie symmetry analysis. As such, we have performed symmetry reductions of the STM-NFP equation in all such cases, limiting our calculations to essentially different infinitesimal symmetry generators, i.e. those not connected by means of a Lie point transformation of the equation. To do this we have constructed adjoint representations of the symmetry Lie algebras of the STM-NFP equation corresponding to all the instances when the equation admits a nontrivial symmetry Lie algebra. Using the adjoint representations, we have determined one-dimensional optimal systems of Lie algebras of the STM-NFP equation and used them to perform symmetry reductions of the equation. We have thereby characterised practically all invariant solutions of STM-NFP equation by second-order ODEs. Our results have revealed new possibilities for analytical and numerical studies of the equation.

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