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NUMERICAL DIFFERENTIATION VIA HERMITE WAVELETS

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Abstract: In this research, numerical technique based on Hermite wavelets is established to find the numerical differentiation. Proposed technique is based on the expansion of unknown function into a series of basis of Hermite wavelets. Some numerical experiments have been performed to illustrate the accuracy of the proposed technique.

AMS Subject Classification: 65N99
Key Words: Hermite wavelet; numerical differentiation; numerical examples

1. Introduction

Wavelets theory is a relatively new and emerging area in mathematical research and is being extensively used as a powerful tool in various science and engineering disciplines. Wavelets are mathematical functions which have been widely used in digital signal processing for waveform representation and segmentations, image compression, time-frequency analysis, quick algorithms for easy implementations and many other fields of pure and applied mathematics. In the recent years, the different types of wavelet methods have found their way for the numerical solution of different kinds of integral equations arising in mathematical physics models and many other scientific and engineering problems. Several numerical techniques based on Haar wavelets have been established for solving ordinary differential equations, partial differential equations and inte-

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gral equations and are discussed in [1], [2], [3], [4], [5], [6], [7], [13] and [15]. Hermite wavelet based numerical method has been developed for solving boundary value problems in [8]. Hermite collocation method has been established for solving fractional differential equations in [9]. In [10], Hermite wavelet method has been developed for accurate solving fractional Jaulent-Miodek equation associated with energy dependent Schrodinger potential. Numerical solution of two-dimensional hyperbolic telegraph equation has been established with the aid of Hermite wavelets in [11]. Bratu's problem has been solved with the help of Hermite wavelet approach in [12]. Numerical solution of nonlinear singular initial value problems by using operational matrices of integration of Hermite wavelets has been discussed in [14].

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. Graphical methods give results to a low degree of accuracy whereas numerical methods give high accuracy in comparison to graphical methods. With the revolution of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers. Numerical differentiation is the process of calculating the derivatives of a function at some particular value of the independent variable by means of a set of given values of that function. Many numerical techniques have been developed to find the numerical differentiation such as Newton's forward and backward interpolation formula, central formula, Stirling's formula, etc. The main objective of this research is to find the numerical differentiation with the help of Hermite wavelets.

2. Hermite wavelet and its properties

Wavelets constitute a family of functions from dilation and translation of a single function known as mother wavelet. The continuous variation of dilation parameter α and translation parameter β, form a family of continuous wavelets as:

$$\psi_{\alpha,\beta}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi\Big(\frac{x-\beta}{\alpha}\Big), \quad \alpha, \beta \in R, \quad \alpha \neq 0, \tag{1}$$

If the dilation and translation parameters are restricted to discrete values by setting $\alpha = {\alpha_0}^{-k}$, $\beta = n\beta_0{\alpha_0}^{-k}$, $\alpha_0 > 1$, $\beta_0 > 0$, we obtain the following family of discrete wavelets:

$$\psi_{k,n}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi({\alpha_0}^k x - n\beta_0), \quad \alpha, \beta \in R, \quad \alpha \neq 0, \tag{2}$$

where $\psi_{k,n}$, form a wavelet basis for $L^2(R)$. For special case, if $\alpha_0 = 2$ and $\beta_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as:

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_m(2^k m - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $m = 0, 1, 2, 3, ..., M - 1$ and $n = 1, 2, 3, ..., 2^{k-1}$ and k is assumed any positive integer. Also, H_m are Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1 - x^2}$ on the real line R and satisfies the following recurrence relation

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m + 1)H_m(x), \tag{4}$$

where $m = 0, 1, 2, ...,$, $H_0(x) = 1$ and $H_1(x) = 2x$.

3. Function approximation

Consider any square integrable function $u(x)$ can be expanded in terms of infinite series of Hermite basis functions as:

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x), \tag{5}$$

where $C_{n,m}$ are constants of this infinite series, known as Hermite wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T \Psi(x), \tag{6}$$

where C and Ψ are $2^{k-1}M \times 1$ matrices and are given by

$$C^T = [C_{1,0}, ..., C_{1,M-1}, ..., C_{2^{k-1},0}, ..., C_{2^{k-1},M-1}] \tag{7}$$

and

$$\Psi = [\psi_{1,0}, ..., \psi_{1,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}]^T. \tag{8}$$

4. Proposed scheme for numerical differentiation

Suppose we are given the following set of values of $y = f(x)$ for a set of values of x: i.e.

$$y(x_0) = y_0,\ y(x_1) = y_1,\ y(x_2) = y_2,\ \cdots,\ y(x_{n-1}) = y_{n-1}. \tag{9}$$

x	x_0	x_1	$\cdots$	x_{n-1}
$y = f(x)$	y_0	y_1	$\cdots$	y_{n-1}

To find the first, second and third derivatives of y at any value of x, consider the approximation

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x). \tag{10}$$

Using the conditions given in (9), we obtain

$$y_0 = y(x_0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_0), \tag{11}$$

$$y_1 = y(x_1) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_1), \tag{12}$$

$$\vdots$$

$$y_{n-1} = y(x_{n-1}) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_{n-1}). \tag{13}$$

Solving the above system of n algebraic equations, we obtain the wavelet coefficients. Differentiating (10), three times w.r.t x, we obtain

$$y'(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi'_{n,m}(x), \tag{14}$$

$$y''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi''_{n,m}(x), \tag{15}$$

$$y'''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi'''_{n,m}(x), \tag{16}$$

$$\vdots$$

Substituting the values of wavelet coefficients into (14), (15) and (16), we obtain the first, second and third derivatives of y.

5. Numerical experiments

In this section, some numerical examples have been performed to illustrate the accuracy of the proposed technique.

Example 1: Consider the data in Table 1:

x	0	0.2	0.4	0.6	0.8
$y = f(x)$	1	1.008	1.064	1.216	1.512

Table 1: Data for Example 1.

Using the Newton forward or backward interpolation formula, we obtain the polynomial $y(x) = x^3 + 1$. Taking $k = 1, M = 5$ and the approximation is:

$$y(x) = \sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x), \tag{17}$$

i.e.

$$y(x) = C_{1,0}\psi_{1,0}(x) + C_{1,1}\psi_{1,1}(x) + \cdots + C_{1,4}\psi_{1,4}(x), \tag{18}$$

Substituting $x = 0, 0.2, 0.4, 0.6, 0.8$ respectively, in (18), we obtain

$$y(0) = C_{1,0}\psi_{1,0}(0) + C_{1,1}\psi_{1,1}(0) + \cdots + C_{1,4}\psi_{1,4}(0), \tag{19}$$

$$y(0.2) = C_{1,0}\psi_{1,0}(0.2) + C_{1,1}\psi_{1,1}(0.2) + \cdots + C_{1,4}\psi_{1,4}(0.2), \tag{20}$$

$$\vdots$$

$$y(0.8) = C_{1,0}\psi_{1,0}(0.8) + C_{1,1}\psi_{1,1}(0.8) + \cdots + C_{1,4}\psi_{1,4}(0.8). \tag{21}$$

From (19) - (21), we obtain system of 5 algebraic equations. Solving such a system of algebraic equations, we obtain wavelet coefficients:

$$c_{1,0} = 1.163172839656745e + 000, \quad c_{1,1} = 2.077094356529901e - 001,$$

$$c_{1,2} = 8.308377426119595e - 002, \quad c_{1,3} = 1.384729571019938e - 002,$$

$$c_{1,4} = 1.665334536937735e - 016.$$

Substituting these wavelet coefficients into (18), we obtain

$$y(x) = 1.163172839656745.\psi_{1,0}(x) + 0.207709435652990.\psi_{1,1}(x) + \cdots + 1.665334536937735e - 016.\psi_{1,4}(x). \quad (22)$$

Differentiating (22), twice w.r.t x, we obtain

$$y'(x) = 1.163172839656745.\psi'_{1,0}(x) + 0.207709435652990.\psi'_{1,1}(x) + \cdots + 1.665334536937735e - 016.\psi'_{1,4}(x), \quad (23)$$

$$y''(x) = 1.163172839656745.\psi''_{1,0}(x) + 0.207709435652990.\psi''_{1,1}(x) + \cdots + 1.665334536937735e - 016.\psi''_{1,4}(x), \quad (24)$$

x	0	0.2	0.4	0.6	0.8
$y'(x)$(Hermite)	0	0.12	0.48	1.08	1.92
$y'(x)$(Exact)	0	0.12	0.48	1.08	1.92
$y''(x)$(Hermite)	0	1.20	2.40	3.60	4.80
$y''(x)$(Exact)	0	1.20	2.40	3.60	4.80

Table 2: Comparison of Exact and Hermite wavelets solutions of Example 1.

Results are exactly same as the results obtained with forward or backward interpolation formulae. Table 2 shows the comparison of exact and Hermite wavelet solutions of Example 1.

Example 2: Consider the data in Table 3:

x	0	0.15	0.30	0.45	0.60	0.75	0.90
$y = f(x)$	4	4.4618	4.9499	5.4683	6.0221	6.6170	7.2596

Table 3: Data for Example 2.

Using the Newton forward or backward interpolation formula, we obtain the polynomial $y(x) = e^x + 2x + 3$. Taking $k = 1, M = 7$, the wavelet coefficients are as:

$c_{1,0} = 5.098825671912350,\ c_{1,1} = 0.819973029031273,$
$c_{1,2} = 0.046640807118610,\ c_{1,3} = 0.003894802545886,$
$c_{1,4} = 0.000262763041592,\ c_{1,5} = 0.000015829098890,$
$c_{1,6} = 0.000021105465188.$

x	0	0.15	0.30	0.45	0.60	0.75	0.90
$y'(x)$ (Her.)	2.9965	3.1628	3.3497	3.5680	3.8223	4.1169	4.4605
$y'(x)$ (Exa.)	3.0000	3.1618	3.3499	3.5683	3.8221	4.1170	4.4596
$y''(x)$ (Her.)	1.0815	1.1604	1.3430	1.5715	1.8237	2.1130	2.4882
$y''(x)$ (Exa.)	1.0000	1.1618	1.3499	1.5683	1.8221	2.1170	2.4596

Table 4: Comparison of Exact and Hermite wavelets solutions of Example 2.

Results are nearly same as the results obtained with forward or backward interpolation formulae. Table 4 shows the comparison of exact and Hermite wavelet solutions of first and second derivatives of Example 2.

Example 3: Consider the data given in Table 5:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y = f(x)$	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Table 5: Data for Example 3

Taking $k = 1, M = 7$. The wavelet coefficients are as:
$c_{1,0} = 31.54582822860697,\ c_{1,1} = 2.920821623552001,$
$c_{1,2} = 1.618840222204653,\ c_{1,3} = 1.793657522458886,$
$c_{1,4} = 0.914738891965072,\ c_{1,5} = 0.328152059797574,$
$c_{1,6} = 0.087146609286698.$

Table 6 shows the first and second derivatives of y at different values of x using Hermite wavelets.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y'(x)$ (Her.)	13.6	14.5	10.1	5.3	0.8	−3.6	−7.2
$y''(x)$ (Her.)	68.4	−30.5	−49.8	−45.5	−44.7	−45.3	−16.3

Table 6: Hermite wavelets solutions of Example 3.

6. Conclusion

From the above numerical data, it is concluded that Hermite wavelet is a powerful mathematical tool for computing numerical differentiation. This technique is also applicable to find higher order derivatives.

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