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A SPECIAL CASE OF MUTUALLY INTERSECTING REGULAR PYRAMIDS

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Abstract: The primary focus of the paper is on the specific features of the form of the intersecting line as a result of a mutual intersection of regular pyramids with a common base. Explored are the possible applications of the theorem concerning the intersection of two second – order surfaces, sharing a common base and passing through a common second – order curve.

AMS Subject Classification: 97G80, 14N99
Key Words: intersection of pyramids; descriptive geometry; engineering graphics

1. Introduction

When designing and constructing products of mechanical engineering, a substantial part of the process is devoted to the intersection of the individual elements and components, and the forms obtained are often diverse, multi – faceted in their character and interesting in terms of the geometry of the spatial or planar intersecting curves.

One of the most prevalent intersecting forms are those of the conial and cylindrical surfaces, which are well – known to have a common genesis in terms of the theory of infinite elements [5], [3]. Other frequently occurring intersecting forms are the rib surfaces, such as pyramidal and prismatic ones. These

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surfaces are prime sources of rotating shapes, which can be viewed as a result of an infinite increase in the number of the polyhedral sides per face. This would explain why, for example, the methods for finding the points of intersection between the rotating shapes can also be equally appropriate ,with some reservations, to the rib shapes as well.

Let us look at one of the well – accepted theorems related to the intersection of spherical (rounded) shapes, which states that: two second-order surfaces that cut through a curve of a second – order, along which they do not touch each other, shall cross in yet another second – order curve. Hence, the form of their cross – section area is a plane curve of the second order. It is often part of an ellipse or even a full ellipse. Besides, if two of the points at which the cross-sections of the spherical shapes meet, and through which a random plane splits the two surfaces, stand on their common second -order curve (the common base of the intersecting shapes), then the rest shall lie on another curve of the second order (the line of their intersection).

A drawing visualizing the theorem is presented in Figure 1, showing the mutual intersection of rotating truncated cone with an inclined circular cylinder, sharing a common base. These shapes shall cut through yet another second – order curve. Points of this curve are obtained using a sheaf of planes, parallel to the common base of the respective shapes. Each of the planes runs through the cone and the cylinder in two circular cross – section areas, which, in turn, meet at points lying on the line of intersection between these shapes. This is how points 2 and 4 are determined. Point 3 splits the contour construction of the intersecting cylinder and cone found in a plane that is parallel to the front projection plane. Since the cross – section is symmetrical to that plane, its first projection then is a line segment $1 \equiv 5 - 3$.

The purpose of the present paper is to examine the changes in the form of the line of intersection between pyramids with a common base, taking the common origin of pyramidal and conical surfaces for granted, in accordance with the statement that the latter can be obtained as a result of an infinite increase in the number of the polyhedral sides.

The achievement of this goal requires the accomplishment of the following tasks:

- Study into the intersection of rib shapes, with a common base and of the same height, placed in a particular position;

- Study into the intersection of rib shapes, with a common base and of different height, placed in a particular position;

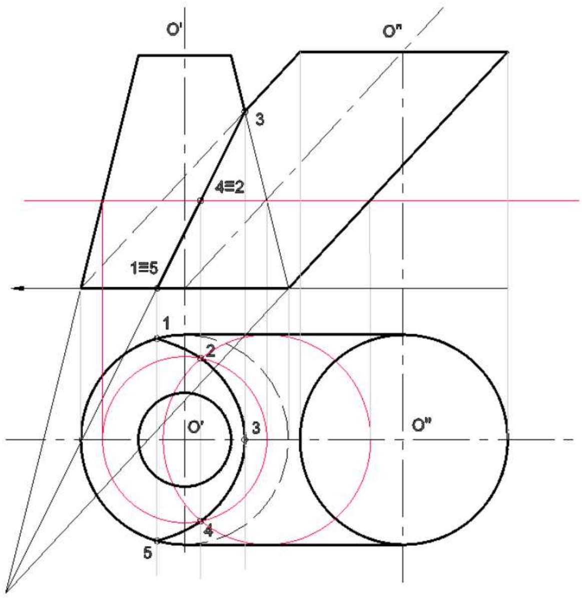


Figure 1: Mutual intersection of rotating truncated cone with a sloping circular cylinder, with a common base

- Making a comparison with intersecting rotating shapes meeting the same requirements.

2. Exposition

Displayed in Figure 2 is a visualization of the intersection of a pair of cones, following by an experiment, performed with a change in the height of one of the mutually intersecting cones. Apparently, the solution does not seem to change and each time it is a curve of a second order. It is, therefore, interesting to know whether this is the case with mutually intersecting pyramids and prisms.

An AutoCAD software product was used to conduct the experiment, to create the drawings and to produce the visualizations of the relevant situations. For the purposes of this experiment, use was made of octagonal pyramids of the same height and sharing a common base. The choice of octagonal pyramids was in light of the principle that the number of the angles should be reasonably permissible, so that it is not too large to yield illegible and cluttered drawings,

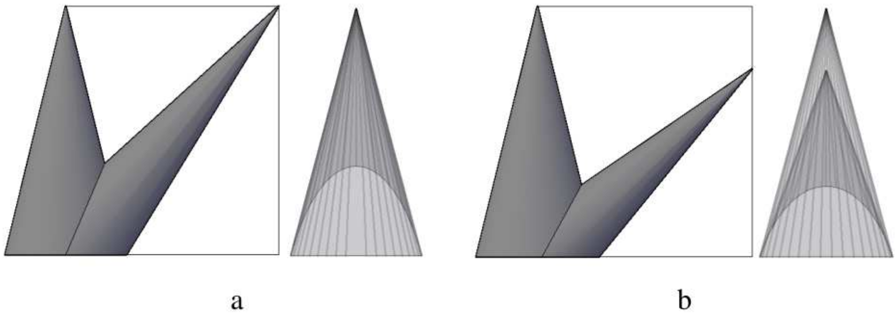


Figure 2: Mutually intersecting circular cones with a common base

and at the same time not too small to compromise the final result.

The following restrictions were observed during the experiment:

- The intersecting pyramids should have a common base;
- The pyramids should have an even number of angles;
- The pyramids should be of the same height ;
- The pyramids should be of different heights.

On the whole, two special cases were found to fulfil these requirements:

- In the first case, the intersection between the two pyramids begins and ends in two diametrically opposite angles of their common base – a polygon. If a line segment is built between these two main edge angels, then it will be perpendicular to the plane defined by the vertices of the intersecting pyramids and the center at the base. As is visible from the drawing and the visualization in Figure 4, these are the angles in which not only the pairs of the surrounding edges VB and $V'B$ intersect, but also that of VF and $V'F$. Thus, the line of intersection begins and ends in the other quadrilateral angles $BAVV'C$ and $FGVV'E$ at the base (Figure 3);
- As for the second case, the crossing starts from the plane (or the line of intersection) upon which lie the pair of the mutually intersecting surrounding sides BVC and $BV'C$ or in other words, two of the main edges of the common base are parallel to the plane defined by the the vertices of the pyramids and the the center of their common base.

2.1. Intersection of regular pyramids with a common base and of the same height placed in a particular position.

2.1.1. First special case

Displayed in Figure 3 and Figure 4 is the first of the special cases related to the intersection of two pyramids of the same height. Indeed, if we look back at the theorem about the intersection of two second – order surfaces with a common base (governing circle), circumscribed around the octagonal base of the pyramids, it follows that their common conic cross-section area which is projected in the circle will also be tangential to the contour in which the two pyramids cut across. This is immediately obvious from the drawing in Figure 4 and the visualization in Figure 3b. Therefore, with such an arrangement of the pyramids, their intersection starts from the quadrilateral angle $BAVV'C$ at the base of the pyramids, that intersects the pair of the surrounding edges VB and $V'B$ lying in the same plane, which would be tangential to the conical surface, if such a surface was drawn around the pyramids, and in which case the edges VB and $V'B$ would also form the frame of that conical surface. The intersection between the shapes will continue between the other pairs of edges at points KMN and will end in the other pair of surrounding edges cutting through the common base of the pyramids at point F, i.e. in the other quadrilateral angle $FGVV'E$. The broken line $BKMNF$, is projected in the common base of the horizontal projection, as a tangent to a second – order curve, and as a straight line in the frontal projection plane – from $B_1 \equiv F_1$ to M_1 through $K_1 \equiv N_1$. This is a guarantee that the line of intersection between the shapes lies in the same plane.

Furthermore, it can be said that the triangles AVC and $AV'C$ stand in one and the same plane. If straight lines are drawn through the edges AV and $V'C$ they will intersect at a point, marked in the drawing as T (Figure 4a). This particular point and the vertices of the intersecting pyramids construct a plane in which lie the triangles AVC and $AV'C$, as well as their common intersection point K.

Solving the problem displayed in Figure 4b, reached is another conclusive proof of the statement under discussion. In this case, use was made of a plane γ passing through the two shapes. The cross–sections are two identical octagons homothetic to the octagonal common base. The place where these polygons intersect are the points that lie at the intersection line of two shapes.

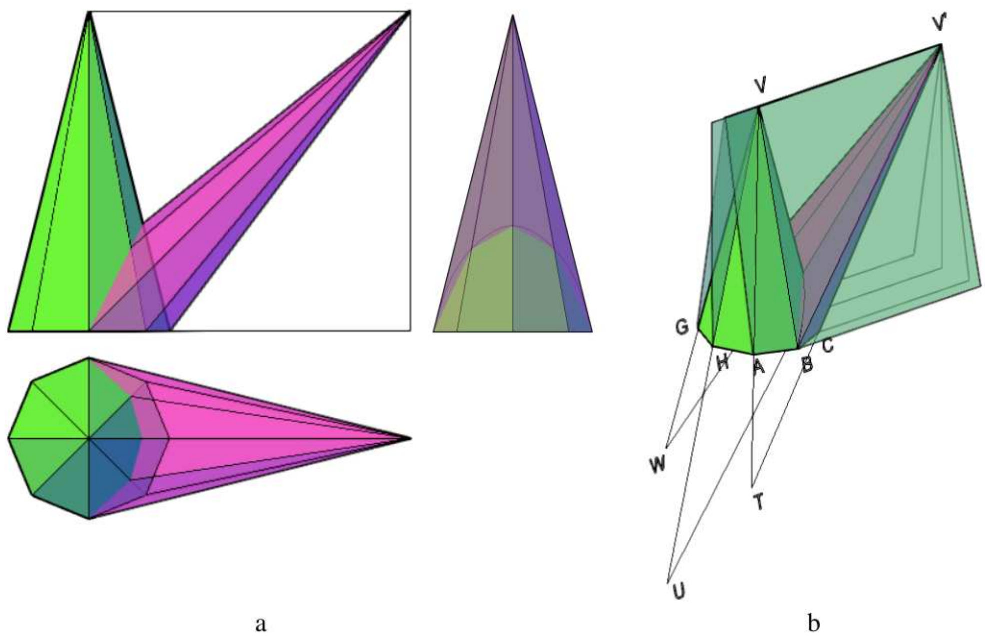


Figure 3: Visualisations of mutually intersecting regular pyramids, with a common base and of the same height – first special case

2.1.2. Second special case

In the second special case of two intersecting pyramids, the line of intersection between the shapes does not start from their common base but from the plane where the surrounding sides BVC and $BV'C$ intersect. The intersection of the shapes continues between the other pairs of edges at points MN, up to the plane where the other pair of surrounding sides GVF and $GV'F$ lies. The two pairs of surrounding sides cut across at the intersection line, on which reside the vertices of the two pyramids. The broken straight line $KMNL$, projects into the common base with the horizontal projection, as a tangent to a second–order curve (Figure 7), and as a straight line in the frontal projection plane. This would assure that here again the intersection line between the shapes also lies in the same plane. Figure 5 helps to visualize the way in which the curve of a second order, to which the broken line of the intersection between the shapes is tangential, does not touch upon the edge of the common base of the shapes, but

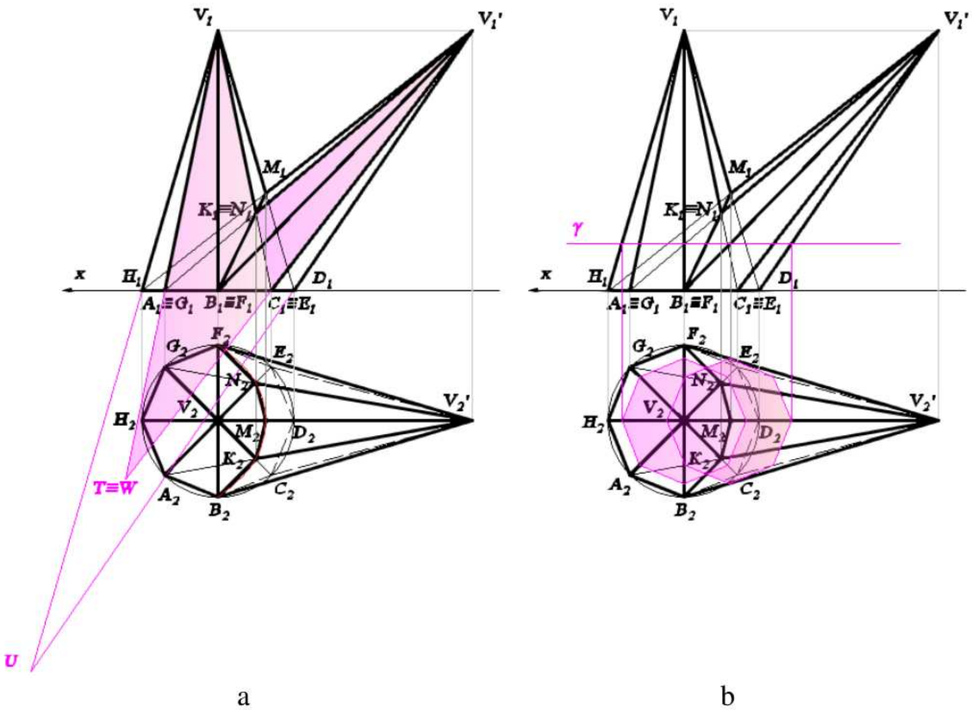


Figure 4: Mutually intersecting regular pyramids with a common base and of the same height – drawing of the first special case

travels across the circle circumscribed around the common base of the pyramids. In this case, the intersection of the shapes starts from point K, passes through points M and N and draws to a close at point L. The solution is depicted in Figure 6 and Figure 7, (for the purposes of which, applied was a sheaf of secant planes meeting in an infinite straight line, where the cross–sections are pairs of identical octagons homothetic to the common octagonal base). This is so because, in this case, there are no two points at which the cross–sections of the shapes are likely to pass through, allowing for a random plane to intersect the two surfaces. The intersection occurs at multiple points (the intersection is in a line segment) as is clearly discernible in Figure 7 when a plane α cuts through the pyramids. Nevertheless, in plane β the intersection does happen at two points only – M and N, which ensures that the other points of the intersection line between these shapes lie on a broken curve, which is also tangential to the

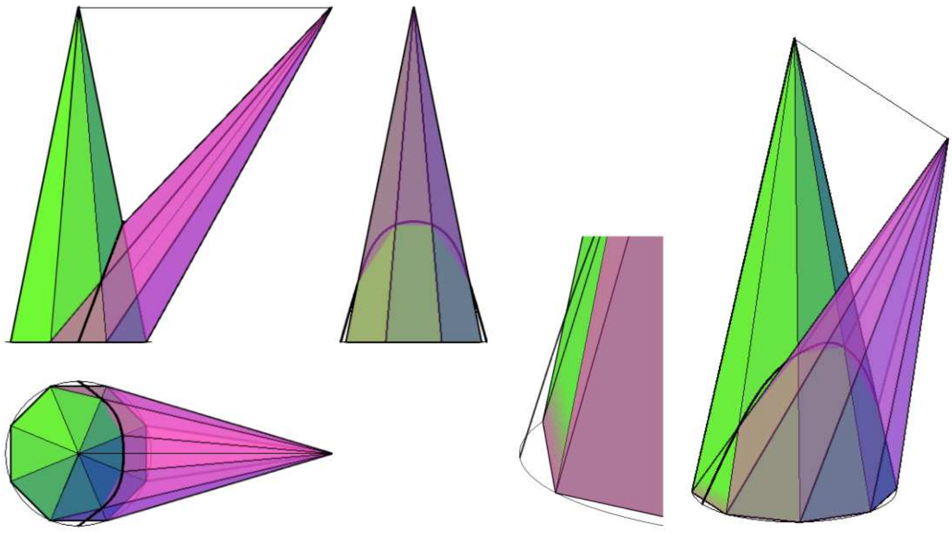
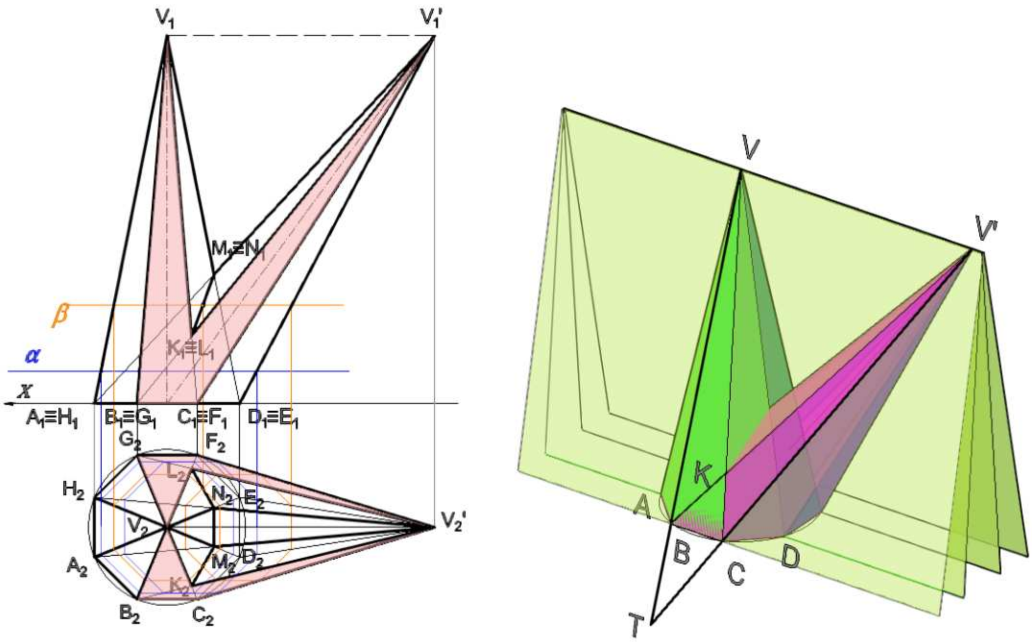


Figure 5: Visualization ofmutually intersecting regular pyramids, with a common base and of the same height – second special casee

curve of the second order starting from point K, passing through the points M and N and terminating at point L.

Further evidence for the statement mentioned above is yet another solution through the use of a sheaf of secant planes cutting across the outermost straight line running through the vertices of the two pyramids. Triangles AVC and $AV'C$ remain in the same plane. If straight lines are drawn through the edges AV and $V'C$, then they will meet at the point indicated in the drawing as T (Figure 6 b). This point and the vertices of the mutually intersecting pyramids built up a plane, found in which are not only the triangles AVC and $AV'C$ but their common intersection point K as well. Moreover, as displayed in the computer visualization, the triangle locked between points BKC defines itself as part of the two surrounding surface areas of mutually intersecting pyramids AVC and $AV'C$, which, in turn, is another shred of evidence that the pair of the surrounding sides lie in the same plane, [2].

It is obvious from the figures that in two cases the broken intersection line between the mutually intersecting pyramids lies in a plane which appears tangent to a curve of the second – order.



a b

Figure 6: Mutually intersecting regular pyramids with a common base and of the same height – drawing of the second special case

It can therefore be concluded that under the conditions stipulated above, the theorem referring to the intersection of two spherical shapes, is also applicable to rib shapes, of the same height.

2.2. Intersection of regular pyramids with a common base and different heights placed in a particular position.

2.2.1. First case

Presented in Figure 8 and Figure 9 is the first case of intersection of two pyramids with a common base but of different heights. In the experiment depicted in figure1 where two cones intersect, it appears that the intersection line between the shapes is a curve of the second order, even if the height of the two cones is different, but is this the case for mutually intersecting pyramids?

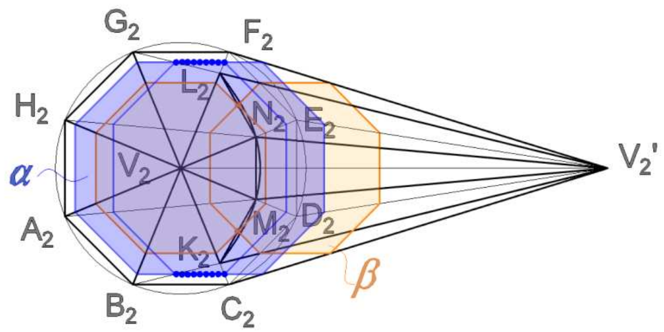


Figure 7: Solution by the method of "sheaf of planes intersecting in an infinite straight line"

The pair of surrounding edges VB and $V'B$ that appears outermost for the intersection (Figure 8b) lie in the same plane, which is also true for the other pair of edges VF and $V'F$, and explains why the line of the mutual intersection between these shapes begins at the place where they intersect, just as it is the case with intersecting pyramids of equal height, contrary to the situation with the other pairs of edges.

As depicted in the drawing laid out in Figure 9, when the two pyramids pass through plane β, the resultant homothetic to the common octagonal base cross-sections, are no longer identical. Herein, the polygonal section of the inclined pyramid is smaller than the section of the right pyramid. This is not the case for the pyramids of the previous groups, where the cross – sections with a given plane are of the same size. In this particular case, the intersection line of the shapes, although projected into the common base of the horizontal projection as a broken curve $B_2K_2K_2'M_2N_2N_2'F_2$ (Figure 9), does not lie any longer in the same plane. In the frontal projection, the points do not stand on the same straight line. The evidence that the intersecting line between the shapes does not lie in the same plane is by no means conclusive.

Shown further in Figure 8b is that the plane in which the triangle ACV, lies and the plane of the triangle ACV' are not incidental as is the case with the intersecting pyramids of one and the same height (Figure 4b and Figure 6b), despite the fact that V and V' reside in the same plane. In view of the foregoing, it follows that if the two triangles are not incidental, the only place where they can intersect is their common base. Consequently, the straight line on which the edge CV' lies, shall not cross with the straight line on which the

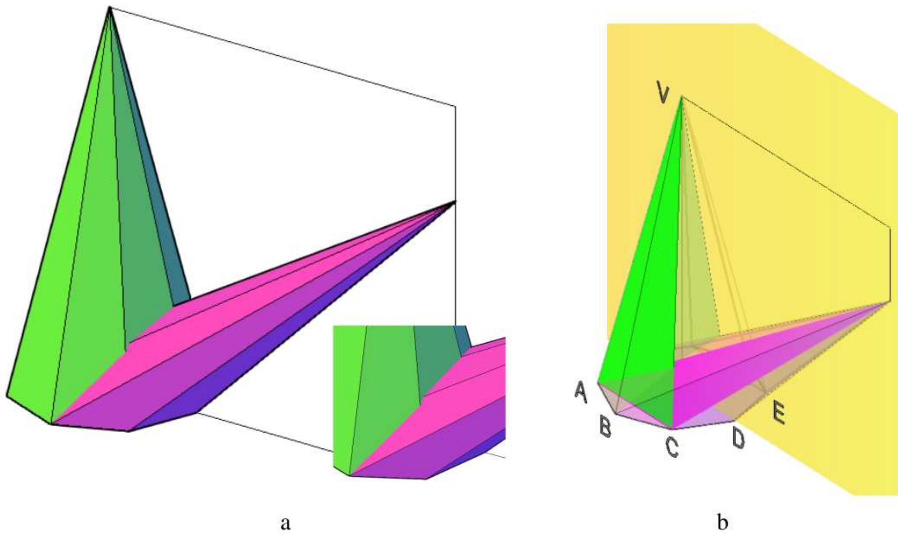


Figure 8: Visualization of mutually intersecting regular pyramids, with a common base but of different heights – first special case

edge AV stands.

Another piece of evidence is further provided in the drawings in Figure 10, setting out a comparison of the two situations that have been discussed so far.

- If a plane (perpendicular to the frontal projection plane) is constructed across two intersecting pyramids of the same height, through the points of intersection of the straight lines lying on the pairs of the intersecting edges (i.e. through the points TUW), then this is the precise plane where the line of intersection between the shapes should also lie. With this in view, it can be said that all the points - $TUWBKMNF$ abide in one and the same second – order curve that is tangential to the line of intersection between these shapes. This is clearly displayed in Figure 10 a, and visually illustrated in Figure13 b.

- If we follow these guidelines to draw up two intersecting pyramids of different heights as depicted in Figure 10 b, by constructing a plane (perpendicular to the frontal projection plane) through points TUW, it becomes obvious that this is not the plane in which the points or even a single point of the intersection line between these shapes are likely to reside. This is

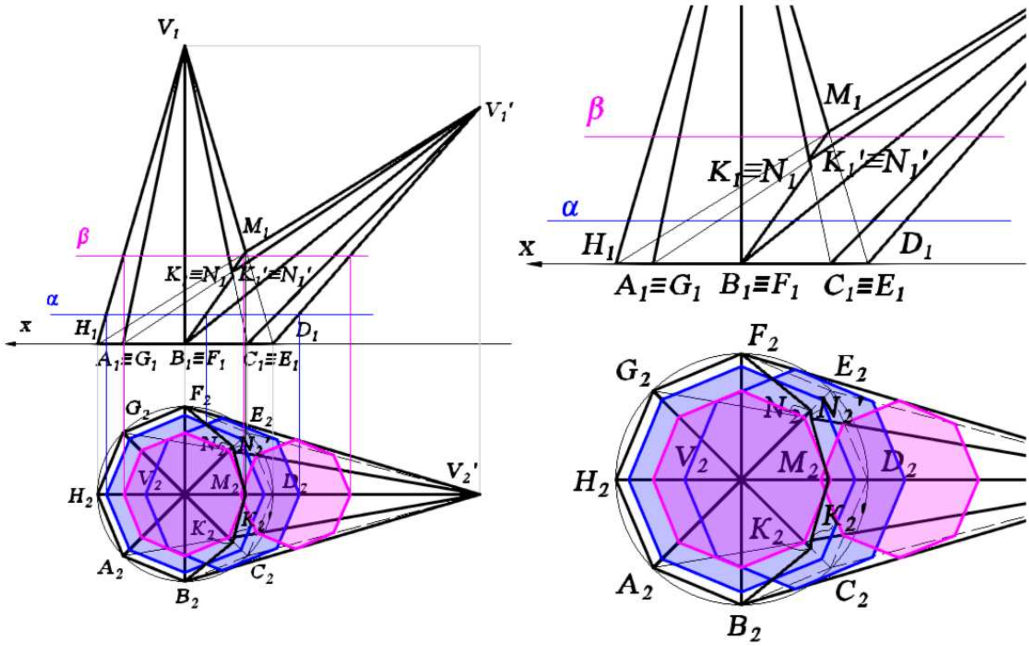


Figure 9: Visualization of mutually intersecting regular pyramids, with a common base but of different heights – drawing of the first special case

due to the fact that the points TUW are not the place where the straight lines, upon which the edges of the pyramids lie, intersect, as depicted in Figure 9, in which it can also be clearly seen that the homothetic cross – sections of the pyramids with a plane are not of the same size. Hence, the straight lines whereupon the edges of the pyramids lie are viewed as crossed lines rather than being lines of intersection.

In a nutshell, taking everything into consideration, we can infer that pairs of intersecting edges are likely to form only in the pairs of the surrounding edges that are outermost for the intersection itself – i.e. VB and $V'B$, as well as in the other pair of edges VF and $V'F$, the place where the line of the mutual intersection between the shapes starts and ends, and also in the edges lying in the plane where the heights (and the vertices) of two intersecting pyramids belong to.

It can also be concluded that intersection only in the pairs of the surrounding edges that are outermost for the intersection itself is likely to be observed in pyramids with an even number of angles, consistent with the arithmetic progression $a_n = a_{n-1} + 4$, such as $a_1 = 6$.

Intersection where there will be a single pair of intersecting edges in the plane in which lie the heights of two intersecting shapes and two more pairs of intersecting edges VB and $V'B$ that are outermost for the intersection itself, and a pair of edges VF and $V'F$, shall be represented through the arithmetical progression $a_n = a_{n-1} + 4$, such as $a_1 = 4$.

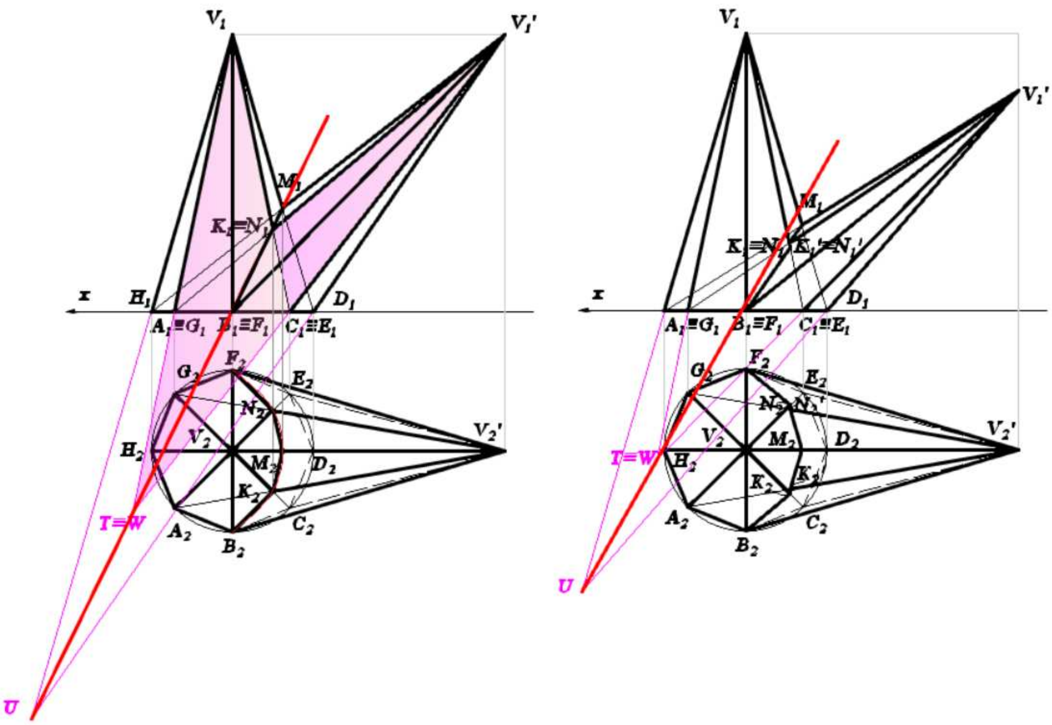


Figure 10: Mutually intersecting regular pyramids, with a common base – comparison

2.2.2. Second case

In the second special case of two intersecting pyramids of different heights, the line of intersection between these shapes starts at their common base, which is in contrast to the corresponding group of pyramids of the same height. The pairs of the surrounding sides BVC and $BV'C$ as well as GVF and $GV'F$ no longer lie in the same plane (Figure 11). The intersection commences at point C and terminates at point F (Figure 12). The intersecting line between these shapes turns out to be a spatial broken line which does not lie in the same plane. Accordingly, the intersection of the shapes with a random plane, parallel to the base, does not result in the occurrence of two identical and homothetic to the base polygons, but in a polygon (in the right pyramid) that is larger than the other – in the inclined pyramid.

Additionally, as presented in Figure 11 and Figure 12, the triangles BVC and $BV'C$ are again not incidental, and the place where the intersecting line of these shapes begins is point C. The other pair of triangles GVF and $GV'F$, where the intersection terminates are also not incidental, with the intersection between these shapes ending at point F.

Undoubtedly, within such a context, it should be noted there are no pairs of edges lying in the same plane at all. The intersection of the edges of one of the pyramids with the other one occurs in the surrounding sides.

It might be inferred that such an intersection will also be observed in pyramids with an even number of angles, in accordance with the arithmetic progression $a_n = a_{n-1} + 4$, such as $a_1 = 4$. Intersection of edges is likely to take place in only two of the pairs of surrounding edges that are outermost to the intersection itself, like VC and $V'C$, and a pair of edges VF and $V'F$ (Figure12).

Intersection at which there will be a pair of intersecting edges in the plane in which lie the heights of the two intersecting shapes and two more pairs of surrounding edges that are outermost for the intersection itself, such as VC and $V'C$, and a pair of edges VF and $V'F$ (see Figure 12) shall follow the arithmetic progression $a_n = a_{n-1} + 4$, where $a_1 = 6$.

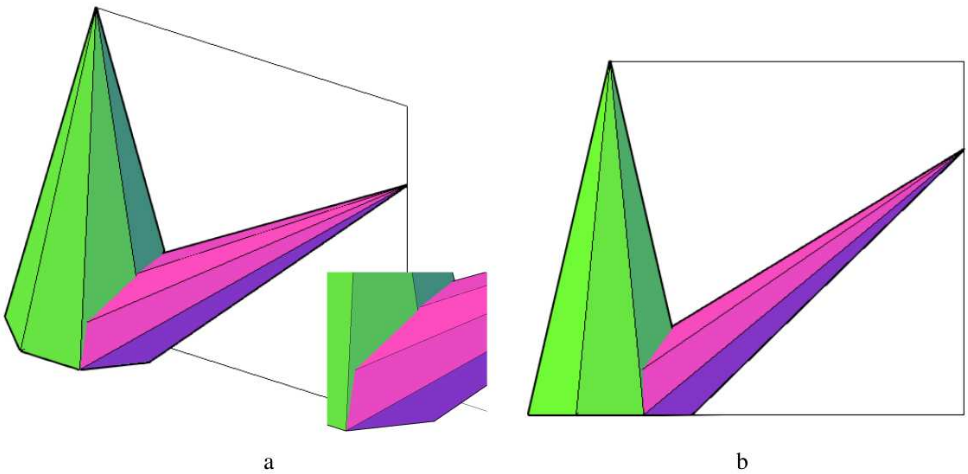


Figure 11: Visualization of mutually intersecting regular pyramids, with a common base but of different heights – second special case

2.3. Comparison of the intersection of rotating shapes, with a common base (of the same and different heights), and pyramids that meet the same requirements, placed in a particular position.

The theorem for two surfaces of second order that intersect in a second – order curve along which they do not touch, proves appropriate only to the first case as reviewed above and only if the intersecting pyramids are of the same height. In this case, the form of the mutual intersection of the pyramids is a curve which is tangential to a curve of the second order. Still more, if two of the points, where the cross – sections of the pyramids meet so as a random plane can intersect the two surfaces, lie on a broken curve (the common base), which appears to be tangential to a second – order curve, then the remaining edges are considered tangential to yet another curve of the second order (the line of their intersection). All that has been discussed above can be visually compressed in Figure 13.

It should be noted, however, with certain reservations, that the proposed theorem applies even to the second case of the first group at issue, but not applicable at all when the pyramids are of different heights.

Considering all the cases that have been discussed so far in this paper, the following theorem can be deduced:

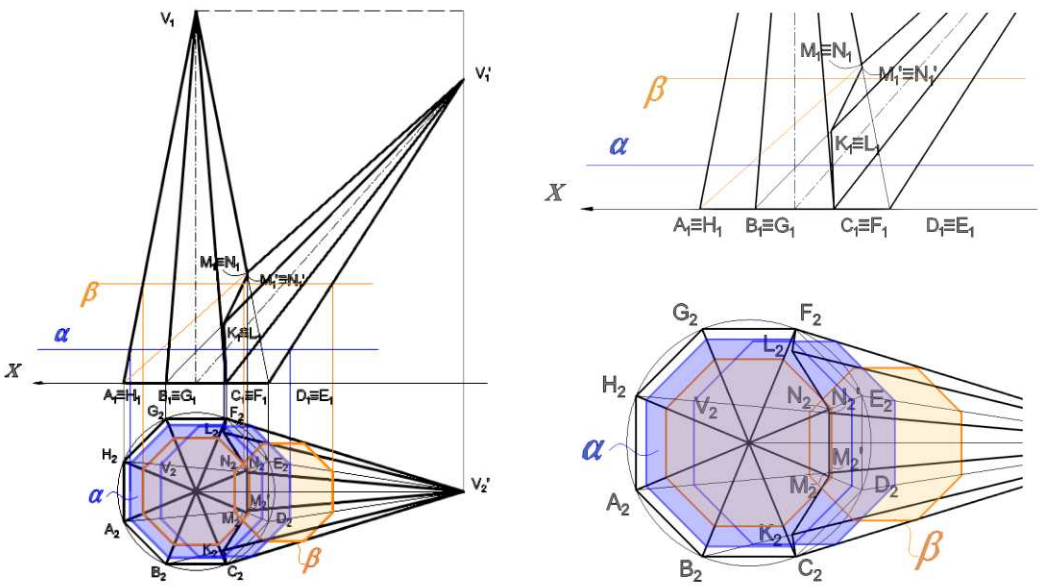


Figure 12: Drawing of mutually intersecting regular pyramids, with a common base but of different heights – second special case

Theorem 1. *The mutual intersection of two pyramids of the same height, with an even number of sides and a common base of a regular polygon, starts and terminates in two diammetrically opposite angles of their common base – a polygon, tangential to a curve of a second order. The pyramids intersect in* $\frac{n}{2}+1$ *pairs of sides (n - the number of the angles of the common base). Their mutual cross – section is a planar broken line that has* $\frac{n}{2}$ *sides, which when viewed from above is projected as a tangent to a curve of the second order.*

3. Conclusion

From the in–depth analysis of the study hereto described, drawn might be the following conclusions:

1. The possible applications of the theorem referring to two surfaces of sec-

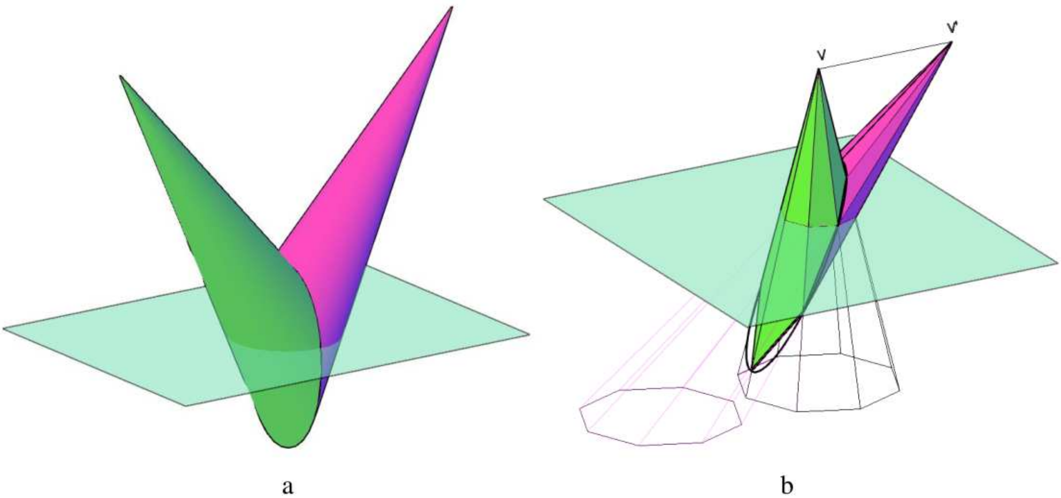


Figure 13: Intersection of rib and rotating shapes

ond order that intersect in a second – order curve is practically feasible only when the two pyramids are of the same height. In such a case the line of intersection between the shapes is a broken line, tangential to a curve of second order, while their cross – section is projected into a circle inscribed in which is their common base – a regular polygon with an even number of sides.

2. In a projection plane parallel to the plane defined by the heights of the mutually intersecting pyramids, the cross – section is a straight line starting from the internal for the two intersecting shapes side or edge and terminating at the centre of their common base. This is a firm guarantee that the line of intersection between the pyramids is a broken plane line.

3. Advanced is a novel theorem which states that "The mutual intersection of two pyramids of the same height, with an even number of sides and a common base of a regular polygon, starts and terminates in two diametrically opposite angles of their common base – a polygon, tangential to a curve of a second order. The pyramids intersect in $\frac{n}{2}+1$ pairs of sides (n – the number of the angles of the common base). Their mutual cross-section is a planar broken line that has $\frac{n}{2}$ sides, which when viewed from above is projected as a tangent to a curve of the second order."

4. With reference to the successful implementation of the proposed theorem, identified has been the following limitations:
   - the pyramids should have an even number of sides and a common base of a regular polygon;
   - the pyramids should be of the same height;
   - the intersection line of the shapes should start and end in two diametrically opposite angles of their common base - a polygon.
5. As for the mutually intersecting pyramids with a common base but of different heights, the theorem as regards two surfaces of second order that intersect in a second – order curve proves inapplicable and irrelevant. In such cases, the broken line of intersection appears to be a spatial broken line.

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