SOME FIXED POINT RESULTS ON EXTENDED b -METRIC SPACES BY UTILIZING C-CLASS FUNCTIONS

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Abstract: In this paper, we employ the concept of C-class functions to prove some fixed point results in the setting of an extended b -metric space. Our result extend and generalize many existing results in the literature. Moreover, we introduce an example to show the validity of our results.

AMS Subject Classification: 47H10, 54H25

Key Words: fixed point; extended b -metric space; altering distance function; C-class functions

1. Introduction

Fixed point theory is one of the paramount tools that used to prove existence and uniqueness theorems in different branches of analysis. The Banach principle is the first fixed point theorem introduced by Banach [3] in 1922. Then after, many researchers paid attention towards fixed point theory, for example see [1, 4, 5, 8, 11, 13, 17]. The notion of metric space has been generalized by many authors. One such generalizations is b -metric spaces in the sense of Bakhtin

Received: April 25, 2020

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[2] and Czerwik [6], for some works in b -metric space see [10, 12, 14, 16]. In 2017, Kamran et al. [7] introduced the notion of extended b -metric spaces as a generalization of b -metric spaces, for some work in this space see [9, 15]. In this paper, we utilize the notion of C-class functions which introduced by Ansari [1] to formulate and prove many fixed point results.

2. Preliminaries

Definition 1. ([7]) Let X be a nonempty set and $\theta : X \times X \longrightarrow [1, \infty)$. A function $d_\theta : X \times X \longrightarrow [0, \infty)$ is called an extended b -metric space if, for all $x, y, z \in X$, it satisfies the following properties:

- (i) $d_\theta(x, y) = 0$ iff $x = y$;
- (ii) $d_\theta(x, y) = d_\theta(y, x)$;
- (iii) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b -metric space.

Remark 2. If we take $\theta(x, y) = s$ for $s \geq 1$, then we obtain the definition of a b -metric space.

Example 3. Let $X = \{2, 3, 4\}$. Define $\theta : X \times X \longrightarrow [1, \infty)$ and $d_\theta : X \times X \longrightarrow [0, \infty)$ as follows:

$$\theta(x, y) = 2 + x + y,$$

$$d_\theta(2, 2) = d_\theta(3, 3) = d_\theta(4, 4) = 0,$$

$$d_\theta(2, 3) = d_\theta(3, 2) = 30, d_\theta(2, 4) = d_\theta(4, 2) = 200,$$

$$d_\theta(3, 4) = d_\theta(4, 3) = 2000.$$

Then (X, d_θ) is an extended b -metric space.

The notions of a Cauchy sequence and a convergent sequence in extended b -metric spaces are defined as follows:

Definition 4. ([7]) Let (X, d_θ) be an extended b -metric space and (x_n) be a sequence in X .

1. A sequence (x_n) in X is said to converge to $x \in X$ if, for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$ for all $n \geq N_\epsilon$. In this case, we write $\lim_{n \rightarrow \infty} x_n = x$.
2. A sequence (x_n) in X is said to be Cauchy if, for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d_\theta(x_n, x_m) < \epsilon$ for all $n, m \geq N_\epsilon$.

An extended b -metric space (X, d_θ) is said to be complete if every Cauchy sequence in X is convergent.

Example 5. [7] Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions define on $[a, b]$. X is a complete extended b -metric space by considering

$$d_\theta(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2,$$

with $\theta(x, y) = |x(t)| + |y(t)| + 2$, where $\theta : X \times X \rightarrow [1, \infty)$.

Definition 6. ([18]) Let A and B be two nonempty subsets of a space X . A mapping $T : A \cup B \rightarrow A \cup B$ is called cyclic if $T(A) \subset B$ and $T(B) \subset A$.

Definition 7. [1] A continuous function $F : [0, \infty)^2 \rightarrow \mathbb{R}$ is called an C-class function if it satisfies:

1. $F(s, t) \leq s$,
2. $F(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

We denote the set of C-class by $\mathcal{C}$.

Example 8. ([1]) For $s, t \in [0, \infty)$, define the functions $F : [0, \infty)^2 \rightarrow \mathbb{R}$ by

- (a) $F(s, t) = s - t$.
- (b) $F(s, t) = \alpha s$ for some $\alpha \in (0, 1)$.
- (c) $F(s, t) = s/(1 + t)^r$ for some $r \in (0, \infty)$.
- (d) $F(s, t) = (s - t)/(1 + t)$.

Then these functions are elements of $\mathcal{C}$.

Definition 9. ([8]) Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a continuous, nondecreasing function. Then ψ is called an altering distance function if $\psi(t) = 0 \Leftrightarrow t = 0$.

We denote the set of altering distance functions by Φ_a .

Definition 10. ([1]) Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a continuous mapping. Then φ is called an ultra altering distance function if $\varphi(t) > 0$ for all $t > 0$.

We denote the set of ultra altering distance functions by Φ_u .

3. Main result

In this section, we utilize the notion of C-class functions to introduce some fixed point results for a cyclic mapping.

Theorem 11. Let (X, d_θ) be a complete extended b -metric space. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Let $f : A \cup B \rightarrow A \cup B$ be a cyclic mapping. Suppose that there exist $F \in \mathcal{C}$, $\psi \in \Phi_a$ and $\phi \in \Phi_u$ such that:

$$\psi(d_\theta(fx, fy)) \leq F(\psi(d_\theta(x, y)), \phi(d_\theta(x, y))) \quad \text{for all } x, y \in X. \quad (1)$$

Assume for $x_0 \in X$, we have

$$\lim_{n, m \rightarrow \infty} \theta(x_n, x_m) = 1, \quad (2)$$

where $x_j = f^j x_0$. If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. Then $x_1 = f x_0 \in B$ and $x_2 = f x_1 \in A$. Continuing this process, we obtain a sequence (x_n) in X , such that $f x_n = x_{n+1}$ with $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}$.

First, we want to show that: $\lim_{n \rightarrow \infty} d_\theta(x_n, x_{n+1}) = 0$.

Let $n \in \mathbb{N}$,

$$\begin{aligned} \psi(d_\theta(x_n, x_{n+1})) &= \psi(d_\theta(f x_{n-1}, f x_n)) \\ &\leq F(\psi(d_\theta(x_{n-1}, x_n)), \phi(d_\theta(x_{n-1}, x_n))) \\ &\leq \psi(d_\theta(x_{n-1}, x_n)). \end{aligned} \quad (3)$$

Since $\psi \in \Phi_a$, then we have

$$d_\theta(x_n, x_{n+1}) \leq d_\theta(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}. \quad (4)$$

This shows that $(d_\theta(x_n, x_{n+1}))$ is decreasing. Then, there exists some $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d_\theta(x_n, x_{n+1}) = r. \quad (5)$$

Assume that $r > 0$. Letting $n \rightarrow \infty$ in (3). Since F , ψ and φ are continuous, we get

$$\psi(r) \leq F(\psi(r), \phi(r)). \quad (6)$$

So $\psi(r) = 0$ or $\phi(r) = 0$. Since $\phi(r) > 0$, we have $r = 0$, which is a contradiction. Hence, $r = 0$ and so

$$\lim_{n \rightarrow \infty} d_\theta(x_n, x_{n+1}) = 0. \quad (7)$$

Now, we want to show that (x_n) is a Cauchy sequence. Assume (x_n) is not a Cauchy sequence. Then, there exists $\epsilon > 0$ and subsequences (x_{n_k}) and (x_{m_k}) of (x_n) with $n_k > m_k > k$ such that

$$d_\theta(x_{n_k}, x_{m_k}) \geq \epsilon. \quad (8)$$

Further, corresponding to m_k , we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (8).

So

$$d_\theta(x_{n_k-1}, x_{m_k}) < \epsilon. \quad (9)$$

Then we have

$$\begin{aligned} 0 < \epsilon &\leq d_\theta(x_{n_k}, x_{m_k}) \leq \theta(x_{n_k}, x_{m_k})[d_\theta(x_{n_k}, x_{n_k-1}) + d_\theta(x_{n_k-1}, x_{m_k})] \\ &\leq \theta(x_{n_k}, x_{m_k})[d_\theta(x_{n_k}, x_{n_k-1}) + \epsilon]. \end{aligned} \quad (10)$$

Letting $k \rightarrow \infty$ and using (7) and (2), we get

$$\lim_{k \rightarrow \infty} d_\theta(x_{n_k}, x_{m_k}) = \epsilon. \quad (11)$$

Also

$$\begin{aligned} d_\theta(x_{n_k}, x_{m_k}) &\leq \theta(x_{n_k}, x_{m_k})d_\theta(x_{n_k}, x_{n_k-1}) + \theta(x_{n_k}, x_{m_k})\theta(x_{n_k-1}, x_{m_k}) \\ &\quad [d_\theta(x_{n_k-1}, x_{m_k-1}) + d_\theta(x_{m_k-1}, x_{m_k})] \\ &\leq \theta(x_{n_k}, x_{m_k})d_\theta(x_{n_k}, x_{n_k-1}) + \theta(x_{n_k}, x_{m_k})\theta(x_{n_k-1}, x_{m_k}) \\ &\quad \theta(x_{n_k-1}, x_{m_k-1})d_\theta(x_{n_k-1}, x_{n_k}) + \theta(x_{n_k}, x_{m_k}) \end{aligned}$$

$$\begin{aligned}
& \theta(x_{n_k-1}, x_{m_k})\theta(x_{n_k-1}, x_{m_k-1})\theta(x_{n_k}, x_{m_k-1})d_\theta(x_{n_k}, x_{m_k}) \\
& + \theta(x_{n_k}, x_{m_k})\theta(x_{n_k-1}, x_{m_k})\theta(x_{n_k-1}, x_{m_k-1}) \\
& \theta(x_{n_k}, x_{m_k-1})d_\theta(x_{m_k}, x_{m_k-1}) + \theta(x_{n_k}, x_{m_k})\theta(x_{n_k-1}, x_{m_k}) \\
& d_\theta(x_{m_k-1}, x_{m_k}).
\end{aligned} \tag{12}$$

As $k \rightarrow \infty$ in the above inequalities and using (2), (7) and (11), we get:

$$\lim_{k \rightarrow \infty} d_\theta(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{13}$$

Then by (1), we have

$$\begin{aligned}
\psi(d_\theta(x_{n_k}, x_{m_k})) & \leq F(\psi(d_\theta(x_{n_k-1}, x_{m_k-1})), \phi(d_\theta(x_{n_k-1}, x_{m_k-1}))) \\
& \leq \psi(d_\theta(x_{n_k-1}, x_{m_k-1})).
\end{aligned} \tag{14}$$

Letting $k \rightarrow \infty$, we obtain

$$\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon). \tag{15}$$

So $\psi(\epsilon) = 0$ or $\phi(\epsilon) = 0$. This implies that $\epsilon = 0$, which is a contradiction.

Hence, (x_n) is a Cauchy sequence. So there exists $u \in X$ such that $\lim_{n \rightarrow \infty} x_n = u$.

Since f is continuous, we have: $\lim_{n \rightarrow \infty} f x_n = f u$. On the other hand: $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} x_{n+1} = u$, then by uniqueness of the limit, we have $f u = u$.

Since $(x_{2n}) \in A$ and A is closed, we have $u \in A$. Also, since $(x_{2n+1}) \in B$ and B is closed, we have $u \in B$. Hence, u is a fixed point of f in $A \cap B$.

To prove the uniqueness of u , we assume there exists $v \in X$ such that $f v = v$. Then by (1), we have

$$\begin{aligned}
\psi(d_\theta(u, v)) & = \psi(d_\theta(f u, f v)) \leq F(\psi(d_\theta(u, v)), \phi(d_\theta(u, v))) \\
& \leq \psi(d_\theta(u, v)).
\end{aligned} \tag{16}$$

So, $\psi(d_\theta(u, v)) = 0$ or $\phi(d_\theta(u, v)) = 0$. This implies that $d_\theta(u, v) = 0$. Hence $u = v$. Thus, f has a unique fixed point in $A \cap B$. $\square$

By choosing $A = B = X$ in Theorem 11 we get the following result:

Corollary 12. *Let (X, d_θ) be a complete extended b -metric space, and let $f : X \rightarrow X$ be a mapping. Suppose that there exist $F \in \mathcal{C}$, $\phi \in \Phi_a$ and $\phi \in \Phi_u$ such that:*

$$\psi(d_\theta(f x, f y)) \leq F(\psi(d_\theta(x, y)), \phi(d_\theta(x, y))) \text{ for all } x, y \in X. \tag{17}$$

Assume for $x_0 \in X$, we have

$$\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) = 1, \quad (18)$$

where $x_j = f^j x_0$. If f is continuous, then f has a unique fixed point in X .

Corollary 13. Let (X, d_θ) be a complete extended b -metric space, and let f be a self mapping defined on X satisfying

$$(1 + \varphi(d_\theta(x, y)))\psi(d_\theta(fx, fy)) \leq \psi(d_\theta(x, y)) - \varphi(d_\theta(x, y))$$

for all $x, y \in X$, where $\varphi \in \Phi_u$ and $\psi \in \Phi_a$. Assume for $x_0 \in X$, we have

$$\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) = 1, \quad (19)$$

where $x_j = f^j x_0$. If f is continuous, then f has a unique fixed point in X .

Proof. Define F by $F(s, t) = (s - t)/(1 + t)$, the result follows from Theorem 11. $\square$

Corollary 14. Let (X, d_θ) be a complete extended b -metric space and $f : A \cup B \rightarrow A \cup B$ be a cyclic mapping. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Assume that there exist $\psi \in \Phi_a$ and $\alpha \in [0, 1)$ such that

$$\psi(d_\theta(fx, fy)) \leq \alpha\psi(d_\theta(x, y)) \quad \text{for all } x, y \in X. \quad (20)$$

Assume for $x_0 \in X$, we have

$$\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) = 1, \quad (21)$$

where $x_j = f^j x_0$. If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. Define F by $F(s, t) = \alpha s$, the result follows from Theorem 11. $\square$

Corollary 15. Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a mapping. Suppose that there exist $F \in \mathcal{C}$, $\psi \in \Phi_a$ and $\phi \in \Phi_u$ such that

$$\psi(d(fx, fy)) \leq F(\psi(d(x, y)), \phi(d(x, y))) \quad \text{for all } x, y \in X. \quad (22)$$

If f is continuous, then f has a unique fixed point in X .

Example 16. Let $X = [-1, 1]$. Define $d_\theta : X \times X \rightarrow \mathbb{R}^+$ by $d_\theta(x, y) = |x - y|$ and $\theta : X \times X \rightarrow [1, \infty)$ by $\theta(x, y) = |x| + |y| + 1$. Let $A = [-1, 0]$, $B = [0, 1]$, and define $f : X \rightarrow X$ by $fx = -x/2$. Also, define $F \in \mathcal{C}$ by $F(s, t) = s - t$ and $\phi \in \Phi_u$ by $\phi(t) = t/4$ and $\psi \in \Phi_a$ by $\psi(t) = t$. Then

(a) d_θ is a complete extended b -metric space on X .

(b) A and B are closed subsets of X .

(c) f is continuous and cyclic.

(d) f satisfy the inequality:

$$\psi(d_\theta(fx, fy)) \leq F(\psi(d_\theta(x, y)), \phi(d_\theta(x, y))) \text{ for all } x, y \in X.$$

(e) Let $x_0 \in X$, we have: $\lim_{n, m \rightarrow \infty} \theta(x_n, x_m) = 1$.

To prove (d), let $x, y \in X$. Then

$$\psi(d_\theta(fx, fy)) = d_\theta(-\frac{x}{2}, -\frac{y}{2}) = |-\frac{x}{2} + \frac{y}{2}| = \frac{1}{2}|x - y|,$$

and

$$F(\psi(d_\theta(x, y)), \phi(d_\theta(x, y))) = d_\theta(x, y) - \phi(d_\theta(x, y)) = \frac{3}{4}|x - y|.$$

So,

$$\psi(d_\theta(fx, fy)) \leq F(\psi(d_\theta(x, y)), \phi(d_\theta(x, y))).$$

Now, to prove (e), let $x_0 \in X$, then $x_n = (-1)^n x_0 / 2^n$ and $x_m = (-1)^m x_0 / 2^m$ and

$$\lim_{n, m \rightarrow \infty} \theta(x_n, x_m) = \lim_{n, m \rightarrow \infty} \left(\frac{|x_0|}{2^n} + \frac{|x_0|}{2^m} + 1 \right) = 1.$$

The example satisfies all the hypotheses of Theorem 11. Hence, f has a unique fixed point in $A \cap B = \{0\}$.

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