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ABSTRACT FRACTIONAL CALCULUS FOR m-ACCRETIVE OPERATORS

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Abstract: In this paper we aim to construct an abstract model of a differential operator with a fractional integro-differential operator composition in final terms, where modeling is understood as an interpretation of concrete differential operators in terms of the infinitesimal generator of a corresponding semigroup. We study such operators as a Kipriyanov operator, Riesz potential, difference operator.

Along with this, we consider transforms of m-accretive operators as a generalization, introduce a special operator class and provide a description of its spectral properties.

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1. Introduction

To write this paper, we were firstly motivated by the boundary value problems of the Sturm-Liouville type for fractional differential equations. Many authors devoted their attention to the topic, nevertheless this kind of problems are relevant for today. First of all, it is connected with the fact that they model various physical - chemical processes: filtration of liquid and gas in highly porous fractal medium; heat exchange processes in medium with fractal structure and memory; casual walks of a point particle that starts moving from the origin by self-similar fractal set; oscillator motion under the action of elastic forces which is characteristic for viscoelastic media, etc. In particular, we would like to study the eigenvalue problem for a differential operator with a fractional derivative in final terms, in this connection such operators as a Kipriyanov fractional differential operator, Riesz potential, difference operator are involved.

In the case corresponding to a selfadjoint senior term we can partially solve the problem having applied the results of the perturbation theory, within the framework of which the following papers are well-known [13], [18], [24], [25],[23], [34]. Generally, to apply the last paper results for a concrete operator L we must be able to represent it by a sum $L = T + A$, where the senior term T must be either a selfadjoint or normal operator. In other cases we can use methods of the papers [21],[20], which are relevant if we deal with non-selfadjoint operators and allow us to study spectral properties of operators whether we have the mentioned above representation or not. These methods are applicable to some type of sectorial operators as opposed to similar results of [23], which can be applied to study a non-selfadjoint operator (see a detailed remark in [34]) if its numerical range of values is a subset of a parabolic domain of the complex plain.

In many papers [3]-[5], [28] the eigenvalue problem was studied by methods of a theory of functions and it is remarkable that special properties of the fractional derivative were used in these papers, bellow we present a brief review. The singular number problem for the resolvent of a second order differential operator with the Riemann-Liouville fractional derivative in final terms was considered in the paper [3]. It was proved that the resolvent belongs to the Hilbert-Schmidt class. The problem of completeness of the root functions system was studied in the paper [4], also similar problems were considered in the paper [5].

However, we deal with a more general operator – a differential operator with a fractional integro-differential operator composition in final terms, which covers the operator mentioned above. Note that several types of compositions of fractional integro-differential operators were studied by such mathematicians as Prabhakar [31], Love [22], Erdélyi [9], McBride [26], Kiryakova [16], Nakhushev [29], etc.

The central idea of this paper is to built a model that gives us a representation of a composition of fractional differential operators in terms of the semigroup theory. For instance we can represent a second order differential operator as some kind of a transform of the infinitesimal generator of a shift semigroup. Continuing this line of reasonings we generalize a differential operator with a fractional integro-differential composition in final terms to some transform of the corresponding infinitesimal generator and introduce a class of transforms of m-accretive operators. Further, we use methods obtained in the papers [20],[21] to study spectral properties of non-selfadjoint operators acting in a complex separable Hilbert space, these methods alow us to obtain an asymptotic equivalence between the real component of the resolvent and the resolvent of the real component of an operator. Due to such an approach we obtain relevant results since an asymptotic formula for the operator real component can be established in many cases (see [2], [32]). Thus, a classification in accordance with resolvent belonging to the Schatten-von Neumann class is obtained, a sufficient condition of completeness of the root vectors system is formulated. As the most significant result we obtain an asymptotic formula for the eigenvalues.

2. Preliminaries

Let $C, C_i,\ i \in \mathbb{N}_0$ be real constants. We assume that a value of C is positive and can be different in various formulas but values of C_i are certain. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\tilde{L}$ the closure of an operator L. We establish the following agreement on using symbols $\tilde{L}^i := (\tilde{L})^i$, where i is an arbitrary symbol. Denote by $\mathrm{D}(L)$, $\mathrm{R}(L)$, $\mathrm{N}(L)$ the *domain of definition*, the *range*, and the *kernel* or *null space* of an operator L respectively. The deficiency (codimension) of $\mathrm{R}(L)$, dimension of $\mathrm{N}(L)$ are denoted by $\operatorname{def} T$, $\operatorname{nul} T$ respectively. Assume that L is a closed operator acting on $\mathfrak{H}$, $\mathrm{N}(L) = 0$, let us define a Hilbert space $\mathfrak{H}_L := \big\{ f, g \in \mathrm{D}(L),\ (f,g)_{\mathfrak{H}_L} = (Lf, Lg)_{\mathfrak{H}} \big\}$. Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements and we have a bounded embedding provided by the inequality

$$\|f\|_{\mathfrak{H}} \leq C_0 \|f\|_{\mathfrak{H}_+},\, C_0 > 0,\ f \in \mathfrak{H}_+,$$

moreover any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$. Let L be a closed operator, for any closable operator S such that $\tilde{S} = L$, its domain $\mathrm{D}(S)$ will be called a core of L. Denote by $\mathrm{D}_0(L)$ a core of a closeable operator L. Let $\mathrm{P}(L)$ be the resolvent set of an operator L and $R_L(\zeta)$, $\zeta \in \mathrm{P}(L)$, $[R_L := R_L(0)]$ denotes the resolvent of an operator L. Denote by $\lambda_i(L)$, $i \in \mathbb{N}$ the eigenvalues of an operator L. Suppose L is a compact operator and $N := (L^*L)^{1/2}$, $r(N) := \dim \mathrm{R}(N)$; then the eigenvalues of the operator N are called the *singular numbers* (*s-numbers*) of the operator L and are denoted by $s_i(L)$, $i = 1, 2, ..., r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0$, $i = r(N) + 1, 2, ...$ According to the terminology of the monograph [10] the dimension of the root vectors subspace corresponding to a certain eigenvalue λ_k is called the *algebraic multiplicity* of the eigenvalue λ_k. Let $\nu(L)$ denotes the sum of all algebraic multiplicities of an operator L. Let $\mathfrak{S}_p(\mathfrak{H})$, $0 < p < \infty$ be a Schatten-von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. By definition, put

$$\mathfrak{S}_p(\mathfrak{H}) := \left\{ L : \mathfrak{H} \to \mathfrak{H}, \sum_{i=1}^{\infty} s_i^p(L) < \infty,\ 0 < p < \infty \right\}.$$

Suppose L is an operator with a compact resolvent and $s_n(R_L) \leq C\, n^{-\mu}$, $n \in \mathbb{N}$, $0 \leq \mu < \infty$; then we denote by $\mu(L)$ order of the operator L in accordance with the definition given in the paper [34]. Denote by $\mathfrak{Re} L := (L + L^*)/2$, $\mathfrak{Im} L := (L - L^*)/2i$ the real and imaginary components of an operator L respectively. In accordance with the terminology of the monograph [12] the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\, f \in \mathrm{D}(L),\, \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of an operator L. An operator L is called *sectorial* if its numerical range belongs to a closed sector $\mathfrak{L}_\gamma(\theta) := \{\zeta : |\arg(\zeta - \gamma)| \leq \theta < \pi/2\}$, where γ is the vertex and θ is the semi-angle of the sector $\mathfrak{L}_\gamma(\theta)$. An operator L is called *bounded from below* if the following relation holds $\mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq \gamma_L \|f\|^2_{\mathfrak{H}}$, $f \in \mathrm{D}(L)$, $\gamma_L \in \mathbb{R}$, where γ_L is called a lower bound of L. An operator L is called *accretive* if $\gamma_L = 0$. An operator L is called *strictly accretive* if $\gamma_L > 0$. An operator L is called *m-accretive* if the next relation holds $(A + \zeta)^{-1} \in \mathcal{B}(\mathfrak{H})$, $\|(A + \zeta)^{-1}\| \leq (\mathrm{Re}\zeta)^{-1}$, $\mathrm{Re}\zeta > 0$. An operator L is called *m-sectorial* if L is sectorial and $L + \beta$ is m-accretive for some constant β. An operator L is called *symmetric* if one is densely defined and the following equality holds $(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}}$, $f, g \in \mathrm{D}(L)$.

Consider a sesquilinear form $t[\cdot,\cdot]$ (see [12]) defined on a linear manifold of the Hilbert space $\mathfrak{H}$. Denote by $t[\cdot]$ the quadratic form corresponding to the sesquilinear form $t[\cdot,\cdot]$. Let $\mathfrak{h}=(t+t^*)/2,\ \mathfrak{k}=(t-t^*)/2i$ be a real and imaginary component of the form t respectively, where $t^*[u,v]=\overline{t[v,u]},\ \mathrm{D}(t^*)=\mathrm{D}(t)$. According to these definitions, we have $\mathfrak{h}[\cdot]=\mathrm{Re}\,t[\cdot],\ \mathfrak{k}[\cdot]=\mathrm{Im}\,t[\cdot]$. Denote by $\tilde{t}$ the closure of a form t. The range of a quadratic form $t[f],\ f\in\mathrm{D}(t),\ \|f\|_{\mathfrak{H}}=1$ is called *range* of the sesquilinear form t and is denoted by $\Theta(t)$. A form t is called *sectorial* if its range belongs to a sector having a vertex γ situated at the real axis and a semi-angle $0\leq\theta<\pi/2$. Suppose t is a closed sectorial form; then a linear manifold $\mathrm{D}_0(t)\subset\mathrm{D}(t)$ is called *core* of t, if the restriction of t to $\mathrm{D}_0(t)$ has the closure t (see [12, p.166]). Due to Theorem 2.7, [12, p.323] there exist unique m-sectorial operators $T_t,T_{\mathfrak{h}}$ associated with the closed sectorial forms $t,\mathfrak{h}$ respectively. The operator $T_{\mathfrak{h}}$ is called a *real part* of the operator T_t and is denoted by $Re\,T_t$. Suppose L is a sectorial densely defined operator and $t[u,v]:=(Lu,v)_{\mathfrak{H}},\ \mathrm{D}(t)=\mathrm{D}(L)$; then due to Theorem 1.27, [12, p.318] the corresponding form t is closable, due to Theorem 2.7, [12, p.323] there exists a unique m-sectorial operator $T_{\tilde{t}}$ associated with the form $\tilde{t}$. In accordance with the definition [12, p.325] the operator $T_{\tilde{t}}$ is called a *Friedrichs extension* of the operator L.

Assume that $T_t\,(0\leq t<\infty)$ is a semigroup of bounded linear operators on $\mathfrak{H}$, by definition put

$$Af=-\lim_{t\to+0}\left(\frac{T_t-I}{t}\right)f,$$

where $\mathrm{D}(A)$ is a set of elements for which the last limit exists in the sense of the norm $\mathfrak{H}$. In accordance with definition [30, p.1] the operator $-A$ is called the *infinitesimal generator* of the semigroup T_t.

Let $f_t:I\to\mathfrak{H},\ t\in I:=[a,b],\ -\infty<a<b<\infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$\sum_{i=0}^{n}f_{\xi_i}\Delta t_i\xrightarrow{\mathfrak{H}}\int\limits_I f_t dt,\quad \lambda\to 0, \tag{1}$$

where $(a=t_0<t_1<...<t_n=b)$ is an arbitrary splitting of the segment $I,\ \lambda:=\max\limits_i(t_{i+1}-t_i),\ \xi_i$ is an arbitrary point belonging to $[t_i,t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see[17, p.248]) i.e. $f_t\xrightarrow{\mathfrak{H}}f_{t_0},\ t\to t_0,\ \forall t_0\in I$. The improper integral is understood as a limit

$$\int\limits_a^b f_t dt\xrightarrow{\mathfrak{H}}\int\limits_a^c f_t dt,\quad b\to c,\quad c\in[-\infty,\infty]. \tag{2}$$

Using notations of the paper [14] we assume that Ω is a convex domain of the n - dimensional Euclidean space $\mathbb{E}^n$, P is a fixed point of the boundary $\partial\Omega$, $Q(r,\mathbf{e})$ is an arbitrary point of Ω; we denote by $\mathbf{e}$ a unit vector having a direction from P to Q, denote by $r = |P-Q|$ the Euclidean distance between the points P, Q, and use the shorthand notation $T := P+\mathbf{e}t,\ t\in\mathbb{R}$. We consider the Lebesgue classes $L_p(\Omega),\ 1\leq p<\infty$ of complex valued functions. For the function $f\in L_p(\Omega)$, we have

$$\int\limits_{\Omega}|f(Q)|^p dQ=\int\limits_{\omega}d\chi\int\limits_{0}^{d(\mathbf{e})}|f(Q)|^p r^{n-1}dr<\infty, \tag{3}$$

where $d\chi$ is an element of solid angle of the unit sphere surface (the unit sphere belongs to $\mathbb{E}^n$) and ω is a surface of this sphere, $d := d(\mathbf{e})$ is the length of the segment of the ray going from the point P in the direction $\mathbf{e}$ within the domain Ω. Without lose of generality, we consider only those directions of $\mathbf{e}$ for which the inner integral on the right-hand side of equality (3) exists and is finite. It is the well-known fact that these are almost all directions. We use a shorthand notation $P\cdot Q = P^i Q_i = \sum_{i=1}^n P_i Q_i$ for the inner product of the points $P=(P_1,P_2,...,P_n),\ Q=(Q_1,Q_2,...,Q_n)$ which belong to $\mathbb{E}^n$. Denote by $D_i f$ a weak partial derivative of the function f with respect to a coordinate variable with index $1\leq i\leq n$, in the one-dimensional case we use a unified form of notations, i.e. $D_1 f = df/dx = f^{'}$. We assume that all functions have a zero extension outside of $\bar{\Omega}$. Everywhere further, unless otherwise stated, we use the notations of the papers [10], [12], [14], [15], [33].

2.1. Auxiliary propositions

In this paragraph we present propositions devoted to properties of accretive operators and related questions. For a reader convenience, we would like to establish well-known facts of the operator theory under a point of view that is necessary for the following reasonings.

Lemma 1. *Assume that A is a closed densely defined operator, the following condition holds*

$$\|(A+t)^{-1}\|_{\mathrm{R}\rightarrow\mathfrak{H}}\leq\frac{1}{t},\ \ t>0, \tag{4}$$

where a notation $\mathrm{R} := \mathrm{R}(A+t)$ is used. Then the operators A, A^* are m-accretive.

Proof. Using (4), consider

$$\|f\|^2_{\mathfrak{H}} \leq \frac{1}{t^2}\|(A+t)f\|^2_{\mathfrak{H}};\ \|f\|^2_{\mathfrak{H}} \leq \frac{1}{t^2}\left\{\|Af\|^2_{\mathfrak{H}} + 2t\mathrm{Re}(Af,f)_{\mathfrak{H}} + t^2\|f\|^2_{\mathfrak{H}}\right\};$$

$$t^{-1}\|Af\|^2_{\mathfrak{H}} + 2\mathrm{Re}(Af,f)_{\mathfrak{H}} \geq 0,\ f \in \mathrm{D}(A).$$

Let t be tended to infinity, then we obtain

$$\mathrm{Re}(Af,f)_{\mathfrak{H}} \geq 0,\ f \in \mathrm{D}(A). \tag{5}$$

It means that the operator A has an accretive property. Due to (5), we have $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\} \subset \Delta(A)$, where $\Delta(A) = \mathbb{C} \setminus \overline{\Theta(A)}$. Applying Theorem 3.2, [12, p.268], we obtain that $A-\lambda$ has a closed range and $\mathrm{nul}(A-\lambda) = 0$, $\mathrm{def}(A-\lambda) = \mathrm{const}$, $\forall \lambda \in \Delta(A)$. Let $\lambda_0 \in \Delta(A)$, $\mathrm{Re}\lambda_0 < 0$. Note that in consequence of inequality (5), we have

$$\mathrm{Re}(f,(A-\lambda)f)_{\mathfrak{H}} \geq -\mathrm{Re}\lambda\|f\|^2_{\mathfrak{H}},\ f \in \mathrm{D}(A). \tag{6}$$

Since the operator $A-\lambda_0$ has a closed range, then

$$\mathfrak{H} = \mathrm{R}(A-\lambda_0) \oplus \mathrm{R}(A-\lambda_0)^{\perp}.$$

We remark that the intersection of the sets $\mathrm{D}(A)$ and $\mathrm{R}(A-\lambda_0)^{\perp}$ is zero, because if we assume the contrary, then applying inequality (6), for arbitrary element $f \in \mathrm{D}(A) \cap \mathrm{R}(A-\lambda_0)^{\perp}$ we get

$$-\mathrm{Re}\lambda_0\|f\|^2_{\mathfrak{H}} \leq \mathrm{Re}(f,[A-\lambda_0]f)_{\mathfrak{H}} = 0,$$

hence $f = 0$. It implies that

$$(f,g)_{\mathfrak{H}} = 0,\ \forall f \in \mathrm{R}(A-\lambda_0)^{\perp},\ \forall g \in \mathrm{D}(A).$$

Since $\mathrm{D}(A)$ is a dense set in $\mathfrak{H}$, then $\mathrm{R}(A-\lambda_0)^{\perp} = 0$. It implies that $\mathrm{def}(A-\lambda_0) = 0$ and if we take into account Theorem 3.2, [12, p.268], then we come to the conclusion that $\mathrm{def}(A-\lambda) = 0$, $\forall \lambda \in \Delta(A)$, hence the operator A is m-accretive.

Now assume that the operator A is m-accretive. Since it is proved that $\mathrm{def}(A+\lambda) = 0$, $\lambda > 0$, then $\mathrm{nul}(A+\lambda)^* = 0$, $\lambda > 0$ (see (3.1) [12, p.267]). In accordance with the well-known fact, we have $\left([\lambda+A]^{-1}\right)^* = [(\lambda+A)^*]^{-1}$. Using the obvious relation $\lambda + A^* = (\lambda+A)^*$, we can deduce $(\lambda+A^*)^{-1} = [(\lambda+A)^*]^{-1}$. Also it is obvious that $\left\|(\lambda+A)^{-1}\right\| = \left\|[(\lambda+A)^{-1}]^*\right\|$, since both operators are bounded. Hence

$$\|\,(\lambda+A^*)^{-1}\,f\|_{\mathfrak{H}} = \|[(\lambda+A)^*]^{-1}f\|_{\mathfrak{H}}$$

$$= \|\left([\lambda+A]^{-1}\right)^* f\|_{\mathfrak{H}} \leq \frac{1}{\lambda}\|f\|_{\mathfrak{H}}, f \in \mathrm{R}(\lambda+A^*),\ \lambda > 0.$$

This relation can be rewritten in the following form

$$\| (\lambda + A^*)^{-1} \|_{\mathrm{R}\to\mathfrak{H}} \leq \frac{1}{\lambda}, \;\; \lambda > 0.$$

Using the proved above fact, we conclude that

$$\| (\lambda + A^*)^{-1} \| \leq \frac{1}{\mathrm{Re}\lambda}, \;\; \mathrm{Re}\lambda > 0. \tag{7}$$

The proof is complete. □

In accordance with the definition given in [17] we can define a positive and negative fractional powers of a positive operator A as follows

$$A^{\alpha} := \frac{\sin \alpha\pi}{\pi} \int\limits_0^{\infty} \lambda^{\alpha-1}(\lambda + A)^{-1} A \, d\lambda;$$

$$A^{-\alpha} := \frac{\sin \alpha\pi}{\pi} \int\limits_0^{\infty} \lambda^{-\alpha}(\lambda + A)^{-1} \, d\lambda, \;\; \alpha \in (0,1). \tag{8}$$

This definition can be correctly extended on m-accretive operators, the corresponding reasonings can be found in [12]. Thus, further we define positive and negative fractional powers of m-accretive operators by formula (8).

Lemma 2. *Assume that $\alpha \in (0,1)$, the operator J is m-accretive, J^{-1} is bounded, then*

$$\|J^{-\alpha} f\|_{\mathfrak{H}} \leq C_{1-\alpha} \|f\|_{\mathfrak{H}}, \; f \in \mathfrak{H}, \tag{9}$$

where $C_{1-\alpha} = 2(1-\alpha)^{-1}\|J^{-1}\| + \alpha^{-1}$.

Proof. Consider

$$J^{-\alpha} = \int\limits_0^1 \lambda^{-\alpha}(\lambda + J)^{-1} d\lambda + \int\limits_1^{\infty} \lambda^{-\alpha}(\lambda + J)^{-1} d\lambda = I_1 + I_2.$$

Using the definition of the integral (1),(2) in a Hilbert space and the fact $J(\lambda + J)^{-1} f = (\lambda + J)^{-1} J f, \; f \in \mathrm{D}(J)$, we can easily obtain

$$\|I_1 f\|_{\mathfrak{H}} \leq \|J^{-1}\| \cdot \left\| \int\limits_0^1 \lambda^{-\alpha} J (\lambda + J)^{-1} f d\lambda \right\|_{\mathfrak{H}}$$

$$\leq \|J^{-1}\|_{\mathrm{R}\to\mathfrak{H}} \cdot \left\{ \|f\|_{\mathfrak{H}} \int\limits_0^1 \lambda^{-\alpha} f d\lambda + \left\| \int\limits_0^1 \lambda^{1-\alpha}(\lambda + J)^{-1} f d\lambda \right\|_{\mathfrak{H}} \right\}$$

$$\leq 2\|J^{-1}\| \cdot \|f\|_{\mathfrak{H}} \int\limits_{0}^{1} \lambda^{-\alpha} d\lambda,\ f \in \mathrm{D}(J);$$

$$\|I_2 f\|_{\mathfrak{H}} = \left\| \int\limits_{1}^{\infty} \lambda^{-\alpha}(\lambda + J)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq \|f\|_{\mathfrak{H}} \int\limits_{1}^{\infty} \lambda^{-\alpha} \|(\lambda + J)^{-1}\| d\lambda$$

$$\leq \|f\|_{\mathfrak{H}} \int\limits_{1}^{\infty} \lambda^{-\alpha-1} d\lambda.$$

Hence $J^{-\alpha}$ is bounded on $\mathrm{D}(J)$. Since $\mathrm{D}(J)$ is dense in $\mathfrak{H}$, then $J^{-\alpha}$ is bounded on $\mathfrak{H}$. Calculating the right-hand sides of the above estimates, we obtain (9). □

3. Main results

In this section we explore a special operator class for which a number of spectral theory theorems can be applied. Further we construct an abstract model of a differential operator in terms of m-accretive operators and call it an m-accretive operator transform, we find such conditions that being imposed guaranty that the transform belongs to the class. As an application of the obtained abstract results we study a differential operator with a fractional integro-differential operator composition in final terms on a bounded domain of the n-dimensional Euclidean space as well as on real axis. One of the central points is a relation connecting fractional powers of m-accretive operators and fractional derivative in the most general sense. By virtue of such an approach we express fractional derivatives in terms of infinitesimal generators, in this regard such operators as a Kipriyanov operator, Riesz potential, difference operator are considered.

3.1. Spectral theorems

Bellow, we give a slight generalization of the results presented in [20].

Theorem 3. *Assume that L is a non-sefadjoint operator acting in $\mathfrak{H}$, the following conditions hold:*

(H1) *There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator L is defined on $\mathfrak{M}$.*

(H2) $|(Lf, g)_{\mathfrak{H}}| \leq C_1 \|f\|_{\mathfrak{H}_+} \|g\|_{\mathfrak{H}_+},\ \mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq C_2 \|f\|^2_{\mathfrak{H}_+},\ f, g \in \mathfrak{M},$ *where C_1, C_2 are positive constants.*

Let W be a restriction of the operator L on the set $\mathfrak{M}$. Then the following propositions are true:

(**A**) *We have the following classification*

$$R_{\tilde{W}} \in \mathfrak{S}_p,\ p = \begin{cases} l,\ l > 2/\mu,\ \mu \leq 1 \\ \qquad\quad 1,\ \mu > 1 \end{cases},$$

where μ is order of $H := Re\,\tilde{W}$. Moreover, under the assumptions $\lambda_n(R_H) \geq C\,n^{-\mu},\ n \in \mathbb{N}$, we have

$$R_{\tilde{W}} \in \mathfrak{S}_p \ \Rightarrow\ \mu p > 1,\ 1 \leq p < \infty.$$

(**B**) *The following relation holds*

$$\sum_{i=1}^{n} |\lambda_i(R_{\tilde{W}})|^p \leq C \sum_{i=1}^{n} \lambda_i^p(R_H),\ 1 \leq p < \infty\ \ (n = 1, 2, ..., \nu(R_{\tilde{W}})), \qquad (10)$$

moreover if $\nu(R_{\tilde{W}}) = \infty$ and $\mu \neq 0$, then the following asymptotic formula holds

$$|\lambda_i(R_{\tilde{W}})| = o\left(i^{-\mu+\varepsilon}\right),\ \ i \to \infty,\ \forall\varepsilon > 0.$$

(**C**) *Assume that $\theta < \pi\mu/2$, where θ is the semi-angle of the sector $\mathfrak{L}_0(\theta) \supset \Theta(\tilde{W})$. Then the system of root vectors of $R_{\tilde{W}}$ is complete in $\mathfrak{H}$.*

Proof. Note that due to the first condition H2, by virtue of Theorem 3.4, [12, p.268] the operator W is closable. Let us show that $\tilde{W}$ is sectorial. By virtue of condition H2, we get

$$\mathrm{Re}(\tilde{W}f, f)_{\mathfrak{H}} \geq C_2\|f\|^2_{\mathfrak{H}_+} \geq C_2\varepsilon\|f\|^2_{\mathfrak{H}_+} + \frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}};$$

$$\mathrm{Re}(\tilde{W}f, f)_{\mathfrak{H}} - k|\mathrm{Im}(\tilde{W}f, f)_{\mathfrak{H}}| \geq (C_2\varepsilon - kC_1)\|f\|^2_{\mathfrak{H}_+}$$

$$+\frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}} = \frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}},$$

where $k = \varepsilon C_2/C_1$. Hence $\Theta(\tilde{W}) \subset \mathfrak{L}_\gamma(\theta),\ \gamma = C_2(1-\varepsilon)/C_0$. Thus, the claim of Lemma 1 [20] is true regarding the operator $\tilde{W}$. Using this fact, we conclude that the claim of Lemma 2, [20] is true regarding the operator $\tilde{W}$ i.e. $\tilde{W}$ is m-accretive.

Using the first representation theorem (Theorem 2.1, [12, p.322]) we have a one-to-one correspondence between m-sectorial operators and closed sectorial sesquilinear forms, i.e. $\tilde{W} = T_t$ by symbol, where t is a sesquilinear form corresponding to the operator $\tilde{W}$. Hence $H := Re\,\tilde{W}$ is defined (see [12, p.337]). In accordance with Theorem 2.6, [12, p.323] the operator H is selfadjoint, strictly accretive.

A compact embedding provided by $\mathfrak{h}[f] \geq C_2\|f\|_{\mathfrak{H}_+} \geq C_2/C_0\|f\|_{\mathfrak{H}}$, $f \in \mathrm{D}(h)$ proves that R_H is compact (see proof of Theorem 4, [20]) and as a result of the application of Theorem 3.3, [12, p.337], we get $R_{\tilde{W}}$ is compact. Thus the claim of Theorem 4, [20] remains true regarding the operators R_H, $R_{\tilde{W}}$.

In accordance with Theorem 2.5, [12, p.323], we get $W^* = T_{t^*}$ (since $W^* = \tilde{W}^*$). Now if we denote $t_1 := t^*$, then it is easy to calculate $\mathfrak{k} = -\mathfrak{k}_1$. Since t is sectorial, than $|\mathfrak{k}_1| \leq \tan\theta \cdot \mathfrak{h}$. Hence, in accordance with Lemma 3.1, [12, p.336], we get $\mathfrak{k}[u,v] = (BH^{1/2}u, H^{1/2}v)$, $\mathfrak{k}_1[u,v] = -(BH^{1/2}u, H^{1/2}v)$, $u,v \in \mathrm{D}(H^{1/2})$, where $B \in \mathcal{B}(\mathfrak{H})$ is a symmetric operator. Let us prove that B is selfadjoint. Note that in accordance with Lemma 3.1, [12, p.336] $\mathrm{D}(B) = \mathrm{R}(H^{1/2})$, in accordance with Theorem 2.1, [12, p.322], we have $(Hf,f)_{\mathfrak{H}} \geq C_2/C_0\|f\|^2_{\mathfrak{H}}$, $f \in \mathrm{D}(H)$, using the reasonings of Theorem 5, [20], we conclude that $\mathrm{R}(H^{1/2}) = \mathfrak{H}$, i.e. $\mathrm{D}(B) = \mathfrak{H}$. Hence B is selfadjoint. Using Lemma 3.2, [12, p.337], we obtain a representation $\tilde{W} = H^{1/2}(I + iB)H^{1/2}$, $W^* = H^{1/2}(I - iB)H^{1/2}$. Noting the fact $\mathrm{D}(B) = \mathfrak{H}$, we can easily obtain $(I \pm iB)^* = I \mp iB$. Since B is selfadjoint, then $\mathrm{Re}([I \pm iB]f,f)_{\mathfrak{H}} = \|f\|^2_{\mathfrak{H}}$. Using this fact and applying Theorem 3.2, [12, p.268], we conclude that $\mathrm{R}(I \pm iB)$ is a closed set. Since $\mathrm{N}(I \pm iB) = 0$, then $\mathrm{R}(I \mp iB) = \mathfrak{H}$ (see (3.2), [12, p.267]). Thus, we obtain $(I \pm iB)^{-1} \in \mathcal{B}(\mathfrak{H})$. Taking into account the above facts, we get $R_{\tilde{W}} = H^{-1/2}(I + iB)^{-1}H^{-1/2}$, $R_{W^*} = H^{-1/2}(I - iB)^{-1}H^{-1/2}$. In accordance with the well-known theorem (see Theorem 5, [35, p.557]), we have $R^*_{\tilde{W}} = R_{W^*}$. Note that the relations $(I \pm iB) \in \mathcal{B}(\mathfrak{H})$, $(I \pm iB)^{-1} \in \mathcal{B}(\mathfrak{H})$, $H^{-1/2} \in \mathcal{B}(\mathfrak{H})$ allow as to obtain the following formula by direct calculations

$$\mathfrak{Re} R_{\tilde{W}} = \frac{1}{2}H^{-1/2}(I + B^2)^{-1}H^{-1/2}.$$

This formula is a crucial point of the matter, we can repeat the rest part of the proof of Theorem 5, [20] in terms $H := Re\,\tilde{W}$. By virtue of these facts, Theorems 7-9, [20], can be reformulated in terms $H := Re\,\tilde{W}$, since they are based on Lemmas 1, 3, Theorems 4, 5, [20].

□

Remark 4. Consider a condition $\mathfrak{M} \subset \mathrm{D}(W^*)$, in this case the operator $\mathcal{H} := \mathfrak{Re}\,W$ is defined on $\mathfrak{M}$, the fact is that $\tilde{\mathcal{H}}$ is selfadjoint, bounded from bellow (see Lemma 3, [20]). Hence a corresponding sesquilinear form (denote this form by h) is symmetric and bounded from bellow also (see Theorem 2.6, [12, p.323]). It can be easily shown that $h \subset \mathfrak{h}$, but using this fact we cannot claim in general that $\tilde{\mathcal{H}} \subset H$ (see [12, p.330]). We just have an inclusion $\tilde{\mathcal{H}}^{1/2} \subset H^{1/2}$ (see [12, p.332]). Note that the fact $\tilde{\mathcal{H}} \subset H$ follows from a condition $\mathrm{D}_0(\mathfrak{h}) \subset \mathrm{D}(h)$ (see Corollary 2.4, [12, p.323]). However, it is proved (see proof of Theorem 4, [20]) that relation H2 guaranties that $\tilde{\mathcal{H}} = H$. Note that the last relation is very useful in applications, since in most concrete cases we can find a concrete form of the operator $\mathcal{H}$.

3.2. Transform

Consider a transform of an m-accretive operator J acting in $\mathfrak{H}$

$$Z^{\alpha}_{G,F}(J) := J^{*}GJ + FJ^{\alpha},\ \alpha \in [0,1), \tag{11}$$

where symbols G, F denote operators acting in $\mathfrak{H}$. Further, using a relation $L = Z^{\alpha}_{G,F}(J)$ we mean that there exists an appropriate representation for the operator L.

The following theorem gives us a tool to describe spectral properties of transform (11), as it will be shown further it has an important application in fractional calculus since allows to represent fractional differential operators as a transform of the infinitesimal generator of a semigroup.

Theorem 5. *Assume that the operator J is m-accretive, J^{-1} is compact, G is bounded, strictly accretive, with a lower bound $\gamma_G > C_{\alpha}\|J^{-1}\| \cdot \|F\|$, $\mathrm{D}(G) \supset \mathrm{R}(J)$, $F \in \mathcal{B}(\mathfrak{H})$, where C_{α} is a constant* (9). *Then $Z^{\alpha}_{G,F}(J)$ satisfies conditions H1 - H2.*

Proof. Since J is m-accretive, then it is closed, densely defined (see [12, p.279], using the fact that $(J+\lambda)^{-1}$, $(\lambda > 0)$ is a closed operator, we conclude that J is closed also). Firstly, we want to check fulfilment of condition H1. Let us choose a space $\mathfrak{H}_J$ as a space $\mathfrak{H}_+$. Since J^{-1} is compact, then we conclude that the following relation holds $\|f\|_{\mathfrak{H}} \leq \|J^{-1}\| \cdot \|Jf\|_{\mathfrak{H}}$, $f \in \mathrm{D}(J)$ and the embedding provided by this inequality is compact. Thus condition H1 is satisfied.

Let us prove that $\mathrm{D}(J^{*}GJ)$ is a core of J. Consider a space $\mathfrak{H}_J$ and a sesquilinear form

$$l_G(u,v) := (GJu, Jv)_{\mathfrak{H}},\ u,v \in \mathrm{D}(J).$$

Observe that this form is a bounded functional on $\mathfrak{H}_J$, since we have

$$|(GJu, Jv)_{\mathfrak{H}}| \leq \|G\| \cdot \|Ju\|_{\mathfrak{H}} \|Jv\|_{\mathfrak{H}}.$$

Hence using the Riesz representation theorem, we have

$$\forall z \in \mathrm{D}(J),\, \exists f \in \mathrm{D}(J):\, (GJz, Jv)_{\mathfrak{H}} = (Jf, Jv)_{\mathfrak{H}}.$$

On the other hand, due to the properties of the operator G, it is clear that the conditions of the Lax-Milgram theorem are satisfied, i.e. $|(GJu, Jv)_{\mathfrak{H}}| \leq \|G\| \cdot \|Ju\|_{\mathfrak{H}} \|Jv\|_{\mathfrak{H}},\; |(GJu, Ju)_{\mathfrak{H}}| \geq \gamma_G \|Ju\|^2_{\mathfrak{H}}$. Note that, in accordance with Theorem 3.24, [12, p.275] the set $\mathrm{D}(J^*J)$ is a core of J, i.e.

$$\forall f \in \mathrm{D}(J),\, \exists \{f_n\}_1^\infty \subset \mathrm{D}(J^*J):\, f_n \underset{J}{\rightarrow} f.$$

Using the Lax-Milgram theorem, in the previously used terms, we get

$$\forall f_n,\, n \in \mathbb{N},\, \exists z_n \in \mathrm{D}(J):\, (GJz_n, Jv)_{\mathfrak{H}} = (Jf_n, Jv)_{\mathfrak{H}}.$$

Combining the above relations, we obtain

$$(GJ\xi_n, Jv)_{\mathfrak{H}} = (J\psi_n, Jv)_{\mathfrak{H}},$$

where $\xi_n := z - z_n,\, \psi_n := f - f_n$. Using the strictly accretive property of the operator G, we have

$$\|J\xi_n\|^2_{\mathfrak{H}} \gamma_G \leq |(GJ\xi_n, J\xi_n)_{\mathfrak{H}}| = |(J\psi_n, J\xi_n)_{\mathfrak{H}}| \leq \|J\psi_n\|_{\mathfrak{H}} \|J\xi_n\|_{\mathfrak{H}}.$$

Taking into account that J^{-1} is bounded, we obtain

$$K_1 \|\xi_n\|_{\mathfrak{H}} \leq \|J\xi_n\|_{\mathfrak{H}} \leq K_2 \|J\psi_n\|_{\mathfrak{H}},\; K_1, K_2 > 0,$$

from what follows that

$$Jz_n \xrightarrow{\mathfrak{H}} Jz.$$

On the other hand, we have

$$(GJz_n, Jv)_{\mathfrak{H}} = (Jf_n, Jv)_{\mathfrak{H}} = (J^*Jf_n, v)_{\mathfrak{H}},\, v \in \mathrm{D}(J).$$

Hence $\{z_n\}_1^\infty \subset \mathrm{D}(J^*GJ)$. Taking into account the above reasonings, we conclude that $\mathrm{D}(J^*GJ)$ is a core of J. Thus, we have obtained the desired result.

Note that $\mathrm{D}_0(J)$ is dense in $\mathfrak{H}$, since J is densely defined. We have proved above

$$\mathrm{Re}\,(J^*GJf, f)_{\mathfrak{H}} = \mathrm{Re}\,(GJf, Jf)_{\mathfrak{H}} \geq \gamma_G \|f\|^2_{\mathfrak{H}_J},$$

$$\left|(J^*GJf, g)_{\mathfrak{H}}\right| = \left|(GJf, Jg)_{\mathfrak{H}}\right| \leq \|G\| \cdot \|Jf\|_{\mathfrak{H}} \|Jg\|_{\mathfrak{H}},\, f, g \in \mathrm{D}_0(J).$$

Similarly, we get

$$\begin{aligned} |(FJ^\alpha f, g)_{\mathfrak{H}}| &\leq \|FJ^\alpha f\|_{\mathfrak{H}} \|g\|_{\mathfrak{H}} \\ &\leq \|J^{-1}\| \cdot \|F\| \cdot \|J^\alpha f\|_{\mathfrak{H}} \|Jg\|_{\mathfrak{H}},\;\; f, g \in \mathrm{D}_0(J). \end{aligned} \tag{12}$$

In accordance with (8), we have $J^{\alpha-1}J \subset J^\alpha$. Therefore, using Lemma 2, we obtain

$$\|J^{\alpha}f\|_{\mathfrak{H}}=\|J^{\alpha-1}Jf\|_{\mathfrak{H}}\leq C_{\alpha}\|Jf\|_{\mathfrak{H}},\,f\in\mathrm{D}_0(J). \tag{13}$$

Combining this fact with (12), we obtain

$$|(FJ^{\alpha}f,g)_{\mathfrak{H}}|\leq C_{\alpha}\|J^{-1}\|\cdot\|F\|\cdot\|f\|_{\mathfrak{H}_J}\|g\|_{\mathfrak{H}_J},\ \ f,g\in\mathrm{D}_0(J),$$

(the case corresponding to $\alpha=0$ is trivial, since the operator J^{-1} is bounded). It follows that

$$\mathrm{Re}(FJ^{\alpha}f,f)\geq-C_{\alpha}\|J^{-1}\|\cdot\|F\|\cdot\|f\|^2_{\mathfrak{H}_J},\ \ f\in\mathrm{D}_0(J).$$

Combining the above facts, we obtain fulfillment of condition H2. □

Definition 6. Define an operator class $\mathfrak{G}_{\alpha}:=\{W:\,W=Z^{\alpha}_{G,F}(J)\}$, where G,F,J satisfy the conditions of Theorem 5.

3.3. The model

In this section we consider various operators acting in a complex separable Hilbert space for which Theorem 3 can be applied, the given bellow results also cover a case $\alpha=0$ after minor changes which are omitted due to simplicity. In accordance with Remark 4, we will stress cases when the relation $\tilde{\mathcal{H}}=H$ can be obtained.

Kipriyanov operator

Here, we study a case $\alpha\in(0,1)$. Assume that $\Omega\subset\mathbb{E}^n$ is a convex domain, with a sufficient smooth boundary (C^3 class) of the n-dimensional Euclidian space. For the sake of the simplicity we consider that Ω is bounded, but the results can be extended to some type of unbounded domains. In accordance with the definition given in the paper [19], we consider the directional fractional integrals. By definition, put

$$(\mathfrak{I}^{\alpha}_{0+}f)(Q):=\frac{1}{\Gamma(\alpha)}\int\limits_0^r\frac{f(P+t\mathbf{e})}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt,$$

$$(\mathfrak{I}^{\alpha}_{d-}f)(Q):=\frac{1}{\Gamma(\alpha)}\int\limits_r^d\frac{f(P+t\mathbf{e})}{(t-r)^{1-\alpha}}\,dt,\ \ f\in L_p(\Omega),\ 1\leq p\leq\infty.$$

Also, we consider auxiliary operators, the so-called truncated directional fractional derivatives (see [19]). By definition, put

$$(\mathfrak{D}^{\alpha}_{0+,\,\varepsilon}f)(Q)=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{r-\varepsilon}\frac{f(Q)r^{n-1}-f(P+\mathbf{e}t)t^{n-1}}{(r-t)^{\alpha+1}r^{n-1}}dt$$

$$+\frac{f(Q)}{\Gamma(1-\alpha)}r^{-\alpha},\ \varepsilon\leq r\leq d,\ \ (\mathfrak{D}^{\alpha}_{0+,\,\varepsilon}f)(Q)=\frac{f(Q)}{\varepsilon^{\alpha}},\ \ 0\leq r<\varepsilon;$$

$$(\mathfrak{D}^{\alpha}_{d-,\,\varepsilon}f)(Q)=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{r+\varepsilon}^{d}\frac{f(Q)-f(P+\mathbf{e}t)}{(t-r)^{\alpha+1}}dt$$

$$+\frac{f(Q)}{\Gamma(1-\alpha)}(d-r)^{-\alpha},\ 0\leq r\leq d-\varepsilon,$$

$$(\mathfrak{D}^{\alpha}_{d-,\,\varepsilon}f)(Q)=\frac{f(Q)}{\alpha}\left(\frac{1}{\varepsilon^{\alpha}}-\frac{1}{(d-r)^{\alpha}}\right),\ \ d-\varepsilon<r\leq d.$$

Now, we can define the directional fractional derivatives as follows

$$\mathfrak{D}^{\alpha}_{0+}f=\lim_{\substack{\varepsilon\to 0\\(L_p)}}\mathfrak{D}^{\alpha}_{0+,\varepsilon}f,\ \mathfrak{D}^{\alpha}_{d-}f=\lim_{\substack{\varepsilon\to 0\\(L_p)}}\mathfrak{D}^{\alpha}_{d-,\varepsilon}f,\ \ 1\leq p\leq\infty.$$

The properties of these operators are described in detail in the paper [19]. Similarly to the monograph [33] we consider left-side and right-side cases. For instance, $\mathfrak{I}^{\alpha}_{0+}$ is called a left-side directional fractional integral and $\mathfrak{D}^{\alpha}_{d-}$ is called a right-side directional fractional derivative. We suppose $\mathfrak{I}^{0}_{0+}=I$. Nevertheless, this fact can be easily proved dy virtue of the reasonings corresponding to the one-dimensional case and given in [33]. We also consider integral operators with a weighted factor (see [33, p.175]) defined by the following formal construction

$$\left(\mathfrak{I}^{\alpha}_{0+}\mu f\right)(Q):=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{r}\frac{(\mu f)(P+t\mathbf{e})}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt,$$

where μ is a real-valued function.

Consider a linear combination of an uniformly elliptic operator, which is written in the divergence form, and a composition of a fractional integro-differential operator, where the fractional differential operator is understood as the adjoint operator regarding the Kipriyanov operator (see [14],[15],[21])

$$L:=-\mathcal{T}+\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-},\ \sigma\in[0,1),$$

$$\mathrm{D}(L)=H^{2}(\Omega)\cap H^{1}_{0}(\Omega),$$

where $\mathcal{T} := D_j(a^{ij}D_i\cdot)$, $i,j = 1,2,...,n$, under the following assumptions regarding coefficients

$$a^{ij}(Q) \in C^2(\bar{\Omega}),\ \mathrm{Re}a^{ij}\xi_i\xi_j \geq \gamma_a|\xi|^2,\ \ \gamma_a > 0,$$

$$\mathrm{Im}a^{ij} = 0\ (n \geq 2),\ \ \rho \in L_\infty(\Omega). \tag{14}$$

Note that in the one-dimensional case the operator $\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}$ is reduced to a weighted fractional integro-differential operator composition, which was studied properly by many researchers (see introduction, [33, p.175]). Consider a shift semigroup in a direction acting on $L_2(\Omega)$ and defined as follows $T_t f(Q) := f(P+\mathbf{e}[r+t]) = f(Q+\mathbf{e}t)$. We can formulate the following proposition.

Lemma 7. *The semigroup T_t is a C_0 semigroup of contractions.*

Proof. By virtue of the continuous in average property, we conclude that T_t is a strongly continuous semigroup. It can be easily established due to the following reasonings, using the Minkowski inequality, we have

$$\left\{\int\limits_{\Omega}|f(Q+\mathbf{e}t)-f(Q)|^2dQ\right\}^{\frac{1}{2}} \leq \left\{\int\limits_{\Omega}|f(Q+\mathbf{e}t)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}$$

$$+\left\{\int\limits_{\Omega}|f(Q)-f_m(Q)|^2dQ\right\}^{\frac{1}{2}}+\left\{\int\limits_{\Omega}|f_m(Q)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}$$

$$= I_1 + I_2 + I_3 < \varepsilon,$$

where $f \in L_2(\Omega)$, $\{f_n\}_1^\infty \subset C_0^\infty(\Omega)$; m is chosen so that $I_1, I_2 < \varepsilon/3$ and t is chosen so that $I_3 < \varepsilon/3$. Thus, there exists such a positive number t_0 that

$$\|T_t f - f\|_{L_2} < \varepsilon,\ t < t_0,$$

for arbitrary small $\varepsilon > 0$. Using the assumption that all functions have the zero extension outside $\bar{\Omega}$, we have $\|T_t\| \leq 1$. Hence we conclude that T_t is a C_0 semigroup of contractions (see [30]). □

Lemma 8. *Suppose $\rho \in \mathrm{Lip}\lambda$, $\lambda > \alpha$, $0 < \alpha < 1$; then*

$$\rho \cdot \mathfrak{I}^{\alpha}_{0+}(L_2) = \mathfrak{I}^{\alpha}_{d-}(L_2);\ \rho \cdot \mathfrak{I}^{\alpha}_{d-}(L_2) = \mathfrak{I}^{\alpha}_{d-}(L_2).$$

Proof. Consider an operator

$$(\psi_{\varepsilon}^{+} f)(Q) = \begin{cases} \int\limits_{0}^{r-\varepsilon} \frac{f(Q)r^{n-1} - f(T)t^{n-1}}{(r-t)^{\alpha+1} r^{n-1}} dt, & \varepsilon \leq r \leq d, \\ \frac{f(Q)}{\alpha} \left(\frac{1}{\varepsilon^{\alpha}} - \frac{1}{r^{\alpha}} \right), & 0 \leq r < \varepsilon, \end{cases} \tag{15}$$

where $T = P + \mathbf{e}t$. We should prove that there exists a limit

$$\psi_{\varepsilon}^{+} \rho f \xrightarrow{L_2} \psi f,\ f \in \mathfrak{I}_{0+}^{\alpha}(L_2),$$

where ψf is some function corresponding to f. We have

$$(\psi_{\varepsilon}^{+} \rho f)(Q) = \int\limits_{0}^{r-\varepsilon} \frac{\rho(Q) f(Q) r^{n-1} - \rho(T) f(T) t^{n-1}}{(r-t)^{\alpha+1} r^{n-1}} dt$$

$$= \rho(Q) \int\limits_{0}^{r-\varepsilon} \frac{f(Q)r^{n-1} - f(T)t^{n-1}}{(r-t)^{\alpha+1} r^{n-1}} dt$$

$$+ \int\limits_{0}^{r-\varepsilon} \frac{f(T)[\rho(Q) - \rho(T)]}{(r-t)^{\alpha+1}} \left(\frac{t}{r} \right)^{n-1} dt = A_{\varepsilon}(Q) + B_{\varepsilon}(Q),\ \varepsilon \leq r \leq d;$$

$$(\psi_{\varepsilon}^{+} \rho f)(Q) = \rho(Q) f(Q) \frac{1}{\alpha} \left(\frac{1}{\varepsilon^{\alpha}} - \frac{1}{r^{\alpha}} \right),\ 0 \leq r < \varepsilon.$$

Hence, we get

$$\|\psi_{\varepsilon_{n+1}}^{+} \rho f - \psi_{\varepsilon_n}^{+} \rho f\|_{L_2(\Omega)} \leq \|\psi_{\varepsilon_{n+1}}^{+} \rho f - \psi_{\varepsilon_n}^{+} \rho f\|_{L_2(\Omega')}$$

$$+ \|\psi_{\varepsilon_{n+1}}^{+} \rho f - \psi_{\varepsilon_n}^{+} \rho f\|_{L_2(\Omega_n)},$$

where $\{\varepsilon_n\}_1^{\infty} \subset \mathbb{R}_+$ is a strictly decreasing sequence that is chosen in an arbitrary way, $\Omega_n := \omega \times \{0 < r < \varepsilon_n\}$, $\Omega' := \Omega \setminus \Omega_n$. It is clear that

$$\|A_{\varepsilon_{n+1}} - A_{\varepsilon_n}\|_{L_2(\Omega')} \leq \|\rho\|_{L_\infty(\Omega)} \|\psi_{\varepsilon_{n+1}}^{+} f - \psi_{\varepsilon_n}^{+} f\|_{L_2(\Omega')}.$$

Since in accordance with Theorem 2.3, [19] the sequence $\psi_{\varepsilon_n}^{+} f$, $(n = 1, 2, ...)$ is fundamental for the defined function f, with respect to the $L_2(\Omega)$ norm, then the sequence A_{ε_n} is also fundamental with respect to the $L_2(\Omega')$ norm. Having used the Hölder properties of ρ, we have

$$\|B_{\varepsilon_{n+1}} - B_{\varepsilon_n}\|_{L_2(\Omega')}$$

$$\leq M\left\{\int\limits_{\Omega'}\left(\int\limits_{r-\varepsilon_n}^{r-\varepsilon_{n+1}}\frac{|f(T)|}{(r-t)^{\alpha+1-\lambda}}\left(\frac{t}{r}\right)^{n-1}dt\right)^2dQ\right\}^{\frac{1}{2}}.$$

Note that applying Theorem 2.3, [19], we have

$$\left\{\int\limits_{\Omega}\left(\int\limits_{0}^{r}\frac{|f(T)|}{(r-t)^{\alpha+1-\lambda}}\left(\frac{t}{r}\right)^{n-1}dt\right)^2dQ\right\}^{\frac{1}{2}}\leq C\|f\|_{L_2}.$$

Hence the sequence $\{B_{\varepsilon_n}\}_1^\infty$ is fundamental with respect to the $L_2(\Omega')$ norm. Therefore,

$$\|\psi^+_{\varepsilon_{n+1}}\rho f-\psi^+_{\varepsilon_n}\rho f\|_{L_2(\Omega')}\to 0,\;\; n\to\infty.$$

Consider

$$\|\psi^+_{\varepsilon_{n+1}}\rho f-\psi^+_{\varepsilon_n}\rho f\|_{L_2(\Omega_n)}\leq\|\psi^+_{\varepsilon_{n+1}}\rho f-\psi^+_{\varepsilon_n}\rho f\|_{L_2(\Omega_{n+1})}$$

$$+\left\{\int\limits_{\omega}d\chi\int\limits_{\varepsilon_{n+1}}^{\varepsilon_n}|A_{\varepsilon_{n+1}}(Q)+B_{\varepsilon_{n+1}}(Q)|^2rdr\right\}^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\left\{\int\limits_{\omega}d\chi\int\limits_{\varepsilon_{n+1}}^{\varepsilon_n}\left|\rho(Q)f(Q)\left(\frac{1}{\varepsilon_n^\alpha}-\frac{1}{r^\alpha}\right)\right|^2rdr\right\}^{\frac{1}{2}}=I_1+I_2+I_3.$$

We have

$$I_1\leq\frac{1}{\alpha}\left(\frac{1}{\varepsilon_n^\alpha}-\frac{1}{\varepsilon_{n+1}^\alpha}\right)\|\rho\|_{L_\infty}\int\limits_{\omega}d\chi\int\limits_{0}^{\varepsilon_{n+1}}f(Q)rdr$$

$$\leq\frac{1}{\alpha}\left(\frac{1}{\varepsilon_n^\alpha}-\frac{1}{\varepsilon_{n+1}^\alpha}\right)\|\rho\|_{L_\infty}\int\limits_{\omega}\left\{\int\limits_{0}^{\varepsilon_{n+1}}|f(Q)|^2rdr\right\}^{\frac{1}{2}}\left\{\int\limits_{0}^{\varepsilon_{n+1}}rdr\right\}^{\frac{1}{2}}d\chi$$

$$\leq\frac{1}{\sqrt{2}\alpha}\left(\frac{1}{\varepsilon_n^\alpha}-\frac{1}{\varepsilon_{n+1}^\alpha}\right)\varepsilon_{n+1}\|\rho\|_{L_\infty}\|f\|_{L_2}.$$

Hence $I_1\to 0$, $n\to\infty$. Using the estimates used above, it is not hard to prove that $I_2,I_3\to 0$, $n\to\infty$. The proof is left to a reader. Therefore,

$$\|\psi^+_{\varepsilon_{n+1}}\rho f-\psi^+_{\varepsilon_n}\rho f\|_{L_2(\Omega_n)}\to 0,\;\; n\to\infty.$$

Combining the obtained results, we have

$$\|\psi^+_{\varepsilon_{n+1}}\rho f-\psi^+_{\varepsilon_n}\rho f\|_{L_2(\Omega)}\to 0,\;\; n\to\infty.$$

Using Theorem 2.2, [19], we obtain the desired result for the case corresponding to the class $\mathfrak{I}^{\alpha}_{0+}(L_2)$. The proof corresponding to the class $\mathfrak{I}^{\alpha}_{d-}(L_2)$ is absolutely analogous. □

The following theorem is formulated in terms of the infinitesimal generator $-A$ of the semigroup T_t.

Theorem 9. *We claim that* $L = Z^{\alpha}_{G,F}(A)$. *Moreover if* γ_a *is sufficiently large in comparison with* $\|\rho\|_{L_\infty}$, *then* L *satisfies conditions H1-H2, where we put* $\mathfrak{M} := C_0^\infty(\Omega)$, *if we additionally assume that* $\rho \in \mathrm{Lip}\lambda$, $\lambda > \alpha$, *then* $\tilde{\mathcal{H}} = H$.

Proof. By virtue of Corollary 3.6, [30, p.11], we have

$$\|(\lambda + A)^{-1}\| \leq \frac{1}{\mathrm{Re}\lambda}, \quad \mathrm{Re}\lambda > 0. \tag{16}$$

Inequality (16) implies that A is m-accretive. Using formula (8), we can define positive fractional powers $\alpha \in (0,1)$ of the operator A. Applying the Balakrishnan formula, we obtain

$$A^{\alpha} f := \frac{\sin \alpha\pi}{\pi} \int\limits_0^\infty \lambda^{\alpha-1}(\lambda + A)^{-1} A f \, d\lambda$$

$$= \frac{1}{\Gamma(-\alpha)} \int\limits_0^\infty \frac{T_t - I}{t^{\alpha+1}} f dt, \quad f \in \mathrm{D}(A). \tag{17}$$

Hence, in the concrete form of writing we have

$$A^{\alpha} f(Q) = \frac{1}{\Gamma(-\alpha)} \int\limits_0^\infty \frac{f(Q + \mathbf{e}t) - f(Q)}{t^{\alpha+1}} dt$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_r^{d(\mathbf{e})} \frac{f(Q) - f(P + \mathbf{e}t)}{(t - r)^{\alpha+1}} dt + \frac{f(Q)}{\Gamma(1-\alpha)} \{d(\mathbf{e}) - r\}^{-\alpha}$$

$$= \mathfrak{D}^{\alpha}_{d-} f(Q), \quad f \in \mathrm{D}(A), \tag{18}$$

where $d(\mathbf{e})$ is the distance from the point P to the edge of Ω along the direction $\mathbf{e}$. Note that a relation between positive fractional powers of the operator A and the Riemann-Liouville fractional derivative was demonstrated in the one-dimensional case in the paper [6].

Consider a restriction $A_0 \subset A$, $\mathrm{D}(A_0) = C_0^\infty(\Omega)$ of the operator A. Note that, since the infinitesimal generator $-A$ is a closed operator (see [30]), then

A_0 is closeable. It is not hard to prove that $\tilde{A}_0$ is an m-accretive operator. For this purpose, note that since the operator A is m-accretive, then by virtue of (5), we get

$$\mathrm{Re}(\tilde{A}_0 f, f)_{\mathfrak{H}} \geq 0, \;\; f \in \mathrm{D}(\tilde{A}_0).$$

This gives us an opportunity to conclude that

$$\|f\|^2_{\mathfrak{H}} \leq \frac{1}{t^2}\left\{\|\tilde{A}_0 f\|^2_{\mathfrak{H}} + 2t\mathrm{Re}(\tilde{A}_0 f, f)_{\mathfrak{H}} + t^2\|f\|^2_{\mathfrak{H}}\right\};$$

$$\|f\|^2_{\mathfrak{H}} \leq \frac{1}{t^2}\|(\tilde{A}_0 + t)f\|^2_{\mathfrak{H}}, \;\; t > 0.$$

Therefore

$$\|(\tilde{A}_0 + t)^{-1}\|_{\mathrm{R}\to\mathfrak{H}} \leq \frac{1}{t}, \;\; t > 0,$$

where $\mathrm{R} := \mathrm{R}(\tilde{A}_0 + t)$. Hence, in accordance with Lemma 1, we obtain that the operator $\tilde{A}_0$ is m-accretive. Since there does not exist an accretive extension of an m-accretive operator (see [12, p.279]) and $\tilde{A}_0 \subset A$, then $\tilde{A}_0 = A$. It is easy to prove that

$$\|Af\|_{L_2} \leq C\|f\|_{H^1_0}, \; f \in H^1_0(\Omega), \tag{19}$$

for this purpose we should establish a representation $Af(Q) = -(\nabla f, \mathbf{e})_{\mathbb{E}^n}$, $f \in C^\infty_0(\Omega)$ the rest of the proof is left to a reader. Thus, we get $H^1_0(\Omega) \subset \mathrm{D}(A)$, and as a result $A^\alpha f = \mathfrak{D}^\alpha_{d-} f, \; f \in H^1_0(\Omega)$. Let us find a representation for the operator G. Consider an operator

$$Bf(Q) = \int_0^r f(P + \mathbf{e}[r - t])dt, \; f \in L_2(\Omega).$$

It is not hard to prove that $B \in \mathcal{B}(L_2)$, applying the generalized Minkowski inequality, we get

$$\|Bf\|_{L_2} \leq \int\limits_0^{\mathrm{diam}\,\Omega} dt \left(\int\limits_\Omega |f(P + \mathbf{e}[r - t])|dQ\right)^{1/2} \leq C\|f\|_{L_2}.$$

The fact $A^{-1}_0 \subset B$ follows from properties of the one-dimensional integral defined on smooth functions. It is a well-known fact (see Theorem 2 [35, p.555]) that since A_0 is closeable and there exists a bounded operator A^{-1}_0, then there exists a bounded operator $A^{-1} = \tilde{A}^{-1}_0 = \widetilde{A^{-1}_0}$. Using this relation we conclude that $A^{-1} \subset B$. It is obvious that

$$\int\limits_\Omega A\,(B\mathcal{T} f \cdot g)\, dQ = \int\limits_\Omega AB\mathcal{T} f \cdot g\, dQ$$

$$+\int\limits_{\Omega} B\mathcal{T}f\cdot Ag\,dQ,\ \ f\in C^{2}(\bar{\Omega}),\ \ g\in C_0^{\infty}(\Omega). \tag{20}$$

Using the divergence theorem, we get

$$\int\limits_{\Omega} A\,(B\mathcal{T}f\cdot g)\ dQ=\int\limits_{S}(\mathbf{e},\mathbf{n})_{\mathbb{E}^n}(B\mathcal{T}f\cdot g)(\sigma)d\sigma, \tag{21}$$

where S is the surface of Ω. Taking into account that $g(S)=0$ and combining (20),(21), we get

$$-\int\limits_{\Omega} AB\mathcal{T}f\cdot\bar{g}\,dQ=\int\limits_{\Omega} B\mathcal{T}f\cdot\overline{Ag}\,dQ,\ f\in C^{2}(\bar{\Omega}),g\in C_0^{\infty}(\Omega). \tag{22}$$

Suppose that $f\in H^2(\Omega)$, then there exists a sequence $\{f_n\}_1^{\infty}\subset C^2(\bar{\Omega})$ such that $f_n\xrightarrow{H^2}f$ (see [35, p.346]). Using this fact, it is not hard to prove that $\mathcal{T}f_n\xrightarrow{L_2}\mathcal{T}f$. Therefore $AB\mathcal{T}f_n\xrightarrow{L_2}\mathcal{T}f$, since $AB\mathcal{T}f_n=\mathcal{T}f_n$. It is also clear that $B\mathcal{T}f_n\xrightarrow{L_2}B\mathcal{T}f$, since B is continuous. Using these facts, we can extend relation (22) to the following

$$-\int\limits_{\Omega}\mathcal{T}f\cdot\bar{g}\,dQ=\int\limits_{\Omega}B\mathcal{T}f\,\overline{Ag}\,dQ,\ \ f\in \mathrm{D}(L),\ g\in C_0^{\infty}(\Omega). \tag{23}$$

It was previously proved that $H_0^1(\Omega)\subset \mathrm{D}(A),\ A^{-1}\subset B$. Hence $GAf=B\mathcal{T}f,\ f\in \mathrm{D}(L)$, where $G:=B\mathcal{T}B$. Using this fact we can rewrite relation (23) in a form

$$-\int\limits_{\Omega}\mathcal{T}f\cdot\bar{g}\,dQ=\int\limits_{\Omega}GAf\,\overline{Ag}\,dQ,\ f\in \mathrm{D}(L),\ g\in C_0^{\infty}(\Omega). \tag{24}$$

Note that in accordance with the fact $A=\tilde{A}_0$, we have

$$\forall g\in \mathrm{D}(A),\ \exists\{g_n\}_1^{\infty}\subset C_0^{\infty}(\Omega),\ \ g_n\underset{A}{\longrightarrow}g.$$

Therefore, we can extend relation (24) to the following

$$-\int\limits_{\Omega}\mathcal{T}f\cdot\bar{g}\,dQ=\int\limits_{\Omega}GAf\,\overline{Ag}\,dQ,\ \ f\in \mathrm{D}(L),\ g\in \mathrm{D}(A). \tag{25}$$

Relation (25) indicates that $GAf\in \mathrm{D}(A^*)$ and it is clear that $-\mathcal{T}\subset A^*GA$. On the other hand in accordance with Chapter VI, Theorem 1.2, [7], we have that $-\mathcal{T}$ is a closed operator, hence in accordance with Lemma 1 the operator $-\mathcal{T}$ is m-accretive. Note that there does not exist an accretive extension of an m-accretive operator, but it can be proved easily that A^*GA is accretive. Therefore $-\mathcal{T}=A^*GA$. Further, applying Theorem 2.1, [19], we get $(\mathfrak{I}^{\sigma}_{0+}\rho\,\cdot)\in$

$\mathcal{B}(L_2)$. Let us recall the previously proved fact $\mathfrak{D}^{\alpha}_{d-}f = A^{\alpha}f,\ f \in H^1_0(\Omega)$. Thus, the representation $L = Z^{\alpha}_{GF}(A)$, where $G := B\mathcal{T}B$, $F := (\mathfrak{I}^{\sigma}_{0+}\rho\,\cdot)$ has been established.

Let us prove that the operator L satisfy conditions H1–H2. Choose the space $L_2(\Omega)$ as a space $\mathfrak{H}$, the set $C_0^{\infty}(\Omega)$ as a linear manifold $\mathfrak{M}$, and the space $H^1_0(\Omega)$ as a space $\mathfrak{H}_+$. By virtue of the Rellich-Kondrashov theorem, we have $H^1_0(\Omega) \subset\subset L_2(\Omega)$. Thus, condition H1 is fulfilled. Using simple reasonings, we come to the following inequality

$$\left|\int\limits_{\Omega} \mathcal{T}f \cdot \bar{g}\, dQ\right| \le C\|f\|_{H^1_0}\|g\|_{H^1_0},\quad f,g \in C_0^{\infty}(\Omega). \tag{26}$$

Let us prove that

$$|\left(\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}f, g\right)_{L_2}| \le K\|f\|_{H^1_0}\|g\|_{L_2},\ f,g \in C_0^{\infty}(\Omega), \tag{27}$$

where $K = C\|\rho\|_{L_\infty}$. Using a fact that the operator $(\mathfrak{I}^{\sigma}_{0+}\rho\,\cdot)$ is bounded, we obtain

$$\|\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}f\|_{L_2} \le C\|\rho\|_{L_\infty}\|\mathfrak{D}^{\alpha}_{d-}f\|_{L_2},\ f \in C_0^{\infty}(\Omega). \tag{28}$$

Taking into account that A^{-1} is bounded, A is m-accretive, applying Lemma 2 analogously to (13), we conclude that $\|A^{\alpha}f\|_{L_2} \le C\|Af\|_{L_2},\ f \in \mathrm{D}(A)$. Using (18),(19), we get $\|\mathfrak{D}^{\alpha}_{d-}f\|_{L_2} \le C\|f\|_{H^1_0},\ f \in C_0^{\infty}(\Omega)$. Combining this relation with (28), we obtain

$$\|\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}f\|_{L_2} \le K\|f\|_{H^1_0},\ f \in C_0^{\infty}(\Omega).$$

Using this inequality, we can easily obtain (27), from what follows that

$$\mathrm{Re}(\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}f, f)_{L_2} \ge -K\|f\|^2_{H^1_0},\quad f \in C_0^{\infty}(\Omega).$$

On the other hand, using a uniformly elliptic property of the operator $\mathcal{T}$ it is not hard to prove that

$$-\mathrm{Re}(\mathcal{T}f, f) \ge \gamma_a\|f\|_{H^1_0},\quad f \in C_0^{\infty}(\Omega), \tag{29}$$

the proof of this fact is obvious and left to a reader (see [19]). Now, if we assume that $\gamma_a > K$, then we obtain the fulfillment of condition H2.

Assume additionally that $\rho \in \mathrm{Lip}\lambda,\ \lambda > \alpha$, let us prove that $C_0^{\infty}(\Omega) \subset \mathrm{D}(L^*)$. Note that

$$\int\limits_{\Omega} D_j(a^{ij}D_i f)\, g dQ = \int\limits_{\Omega} f\, \overline{D_j(a^{ji}D_i g)} dQ,\ f \in \mathrm{D}(L),\ g \in C_0^{\infty}(\Omega).$$

Using this equality, we conclude that $(-\mathcal{T})^*$ is defined on $C_0^{\infty}(\Omega)$. Applying the Fubini theorem, Lemma 8, Lemma 2.6, [19], we get

$$\left(\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}f, g\right)_{L_2} = \left(\mathfrak{D}^{\alpha}_{d-}f, \rho\,\mathfrak{I}^{\sigma}_{d-}g\right)_{L_2} = \left(f, \mathfrak{D}^{\alpha}_{0+}\rho\,\mathfrak{I}^{\sigma}_{d-}g\right)_{L_2},$$

$$f \in \mathrm{D}(L),\ g \in C_0^\infty(\Omega).$$

Therefore the operator $\left(\mathfrak{I}^\sigma_{0+}\rho\,\mathfrak{D}^\alpha_{d-}\right)^*$ is defined on $C_0^\infty(\Omega)$. Taking into account the above reasonings, we conclude that $C_0^\infty(\Omega) \subset \mathrm{D}(L^*)$. Combining this fact with relation H2, we obtain $\tilde{\mathcal{H}} = H$ (see Remark 4). □

Corollary 10. *Consider a one-dimensional case, we claim that $L \in \mathfrak{G}_\alpha$.*

Proof. It is not hard to prove that $\|A_0 f\|_{L_2} = \|f\|_{H_0^1},\ f \in C_0^\infty(\Omega)$. This relation can be extended to the following

$$\|Af\|_{L_2} = \|f\|_{H_0^1},\ f \in H_0^1(\Omega), \tag{30}$$

whence $\mathrm{D}(A) = H_0^1(\Omega)$. Taking into account the Rellich-Kondrashov theorem, we conclude that A^{-1} is compact. Thus, to show that conditions of Theorem 5 are fulfilled we need prove that the operator $G := B\mathcal{T}B$ is bounded and $\mathrm{R}(A) \subset \mathrm{D}(G)$. We can establish the following relation by direct calculations $GA_0 f = B\mathcal{T}f = a^{11}A_0 f,\ f \in C_0^\infty(\Omega)$, where $a^{11} = a^{ij},\ i,j = 1$. Using this equality, we can easily prove that $\|GAf\|_{L_2} \leq C\|Af\|_{L_2},\ f \in \mathrm{D}(A)$. Thus, we obtain the desired result. □

Riesz potential

Consider a space $L_2(\Omega),\ \Omega := (-\infty, \infty)$. We denote by $H_0^{2,\lambda}(\Omega)$ the completion of the set $C_0^\infty(\Omega)$ with the norm

$$\|f\|_{H_0^{2,\lambda}} = \left\{\|f\|^2_{L_2(\Omega)} + \|f''\|^2_{L_2(\Omega,\omega^\lambda)}\right\}^{1/2},\ \ \lambda \in \mathbb{R},$$

where $\omega(x) := (1 + |x|)$. Let us notice the following fact (see Theorem 1 [1]), if $\lambda > 4$, then $H_0^{2,\lambda}(\Omega) \subset\subset L_2(\Omega)$. Consider a Riesz potential

$$I^\alpha f(x) = B_\alpha \int\limits_{-\infty}^{\infty} f(s)|s - x|^{\alpha-1} ds,\ B_\alpha = \frac{1}{2\Gamma(\alpha)\cos\alpha\pi/2},\ \alpha \in (0,1),$$

where f is in $L_p(-\infty,\infty),\ 1 \leq p < 1/\alpha$. It is obvious that

$$I^\alpha f = B_\alpha\Gamma(\alpha)(I^\alpha_+ f + I^\alpha_- f), \quad \text{where} \quad I^\alpha_\pm f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_0^\infty f(s \pm x)s^{\alpha-1} ds,$$

these latter operators are known as fractional integrals on a whole real axis (see [33, p.94]). Assume that the following condition holds $\sigma/2 + 3/4 < \alpha < 1$, where σ is a non-negative constant. Following the idea of the monograph [33, p.176] consider a sum of a differential operator and a composition of fractional integro-differential operators

$$L := \tilde{\mathcal{T}} + I^{\sigma}_{+}\,\rho\, I^{2(1-\alpha)}\frac{d^2}{dx^2}\,,$$

where

$$\mathcal{T} := \frac{d^2}{dx^2}\left(a\frac{d^2}{dx^2}\cdot\right),\ \mathrm{D}(\mathcal{T}) = C_0^{\infty}(\Omega),$$

$$\rho(x)\in L_{\infty}(\Omega),\ a(x)\in L_{\infty}(\Omega)\cap C^2(\Omega), \mathrm{Re}\, a(x) > \gamma_a(1+|x|)^5, \gamma_a > 0.$$

Consider a family of operators

$$T_t f(x) = (2\pi t)^{-1/2}\int\limits_{-\infty}^{\infty} e^{-(x-\tau)^2/2t} f(\tau)d\tau, \quad t>0,$$

$$T_t f(x) = f(x), \quad t=0,\ f\in L_2(\Omega).$$

Lemma 11. *T_t is a C_0 semigroup of contractions.*

Proof. Let us establish the semigroup property. By definition, we have $T_0 = I$. Consider the following formula, note that the interchange of the integration order can be easily substantiated

$$T_t T_{t'} f(x) = \frac{1}{\sqrt{2\pi t}\sqrt{2\pi t'}}\int\limits_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{2t}}du\int\limits_{-\infty}^{\infty} e^{-\frac{(u-\tau)^2}{2t}} f(\tau)d\tau$$

$$= \frac{1}{\sqrt{2\pi t}\sqrt{2\pi t'}}\int\limits_{-\infty}^{\infty} f(\tau)d\tau\int\limits_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{2t}} e^{-\frac{(u-\tau)^2}{2t}}du$$

$$= \frac{1}{\sqrt{2\pi t}\sqrt{2\pi t'}}\int\limits_{-\infty}^{\infty} f(\tau)d\tau\int\limits_{-\infty}^{\infty} e^{-\frac{(x-v-\tau)^2}{2t}} e^{-\frac{v^2}{2t}}dv.$$

On the other hand, in accordance with the formula [36, p.325], we have

$$\frac{1}{\sqrt{2\pi(t+t')}}e^{-\frac{(x-\tau)^2}{2t}} = \frac{1}{\sqrt{2\pi t}\sqrt{2\pi t'}}\int\limits_{-\infty}^{\infty} e^{-\frac{(x-\tau-v)^2}{2t}} e^{-\frac{v^2}{2t}}dv.$$

Hence

$$\frac{1}{\sqrt{2\pi(t+t')}}\int\limits_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2t}} f(\tau)d\tau$$

$$= \frac{1}{\sqrt{2\pi t}\sqrt{2\pi t'}} \int_{-\infty}^{\infty} f(\tau)d\tau \int_{-\infty}^{\infty} e^{-\frac{(x-v-\tau)^2}{2t}} e^{-\frac{v^2}{2t}} dv,$$

from what immediately follows the fact $T_t T_{t'} f = T_{t+t'} f$. Let us show that T_t is a C_0 semigroup of contractions. Observe that

$$(2\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\tau^2/2t} d\tau = 1.$$

Therefore, using the generalized Minkowski inequality (see (1.33) [33, p.9]), we get

$$\|T_t f\|_{L_2} = \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x+s) N_t(s) ds \right|^2 dx \right)^{1/2}$$

$$\leq \int_{-\infty}^{\infty} N_t(s) ds \left(\int_{-\infty}^{\infty} |f(x+s)|^2 dx \right)^{1/2} = \|f\|_{L_2},\ f \in C_0^\infty(\Omega),$$

where $N_t(x) := (2\pi t)^{-1/2} e^{-x^2/2t}$. It is clear that the last inequality can be extended to $L_2(\Omega)$, since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$. Thus, we conclude that T_t is a semigroup of contractions.

Let us establish a strongly continuous property. Assuming that $z = (x - \tau)/\sqrt{t}$, we get in an obvious way

$$\|T_t f - f\|_{L_2} = \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} N_1(z) \left[f(x - \sqrt{t}z) - f(x) \right] dz \right|^2 dx \right)^{1/2}$$

$$\leq \int_{-\infty}^{\infty} N_1(z) \left(\int_{-\infty}^{\infty} \left[f(x - \sqrt{t}z) - f(x) \right]^2 dx \right)^{1/2} dz,\ \ f \in L_2(\Omega),$$

where $N_1 = N_t|_{t=1}$. Observe that, for arbitrary fixed t, z we have

$$N_1(z) \left(\int_{-\infty}^{\infty} \left[f(x - \sqrt{t}z) - f(x) \right]^2 dx \right)^{1/2}$$

$$\leq N_1(z) \left(\int_{-\infty}^{\infty} \left[f(x - \sqrt{t}z) \right]^2 dx \right)^{1/2} + N_1(z)\|f\|_{L_2} \leq 2N_1(z)\|f\|_{L_2}.$$

Applying the Fatou–Lebesgue theorem, we get

$$\overline{\lim_{t\to 0}}\int\limits_{\infty}^{\infty} N_1(z)\left(\int\limits_{\infty}^{\infty}\left[f(x-\sqrt{t}z)-f(x)\right]^2 dx\right)^{1/2} dz$$

$$\leq \int\limits_{\infty}^{\infty} N_1(z)\,\overline{\lim_{t\to 0}}\left(\int\limits_{\infty}^{\infty}\left[f(x-\sqrt{t}z)-f(x)\right]^2 dx\right)^{1/2} = 0,$$

from what follows that $\|T_t f - f\|_{L_2} \to 0,\ t \to 0$. Hence T_t is a C_0 semigroup of contractions. □

The following theorem is formulated in terms of the infinitesimal generator $-A$ of the semigroup T_t.

Theorem 12. *We claim that* $L = Z^{\alpha}_{G,F}(A)$. *Moreover, if* $\min\{\gamma_a, \delta\}$ $(\delta > 0)$ *is sufficiently large in comparison with* $\|\rho\|_{L_\infty}$, *then a perturbation* $L + \delta I$ *satisfies conditions H1-H2, where we put* $\mathfrak{M} := C_0^\infty(\Omega)$.

Proof. Let us prove that

$$Af = -\frac{1}{2}\frac{d^2 f}{dx^2} \ \text{ a.e.,} \ \ f \in \mathrm{D}(A).$$

Consider an operator $J_n = n(nI + A)^{-1}$. It is clear that $AJ_n = n(I - J_n)$. Using the formula

$$(nI + A)^{-1} f = \int\limits_0^\infty e^{-nt} T_t f dt, \ \ n > 0,\ f \in L_2(\Omega),$$

we easily obtain

$$J_n f(x) = \frac{n}{\sqrt{2\pi}}\int\limits_0^\infty e^{-nt} t^{-1/2} dt \int\limits_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2t}} f(\tau) d\tau$$

$$= \frac{n}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(\tau) d\tau \int\limits_0^\infty e^{-nt-\frac{(x-\tau)^2}{2t}} t^{-1/2} dt$$

$$= \sqrt{\frac{2n}{\pi}}\int\limits_{-\infty}^{\infty} f(\tau) d\tau \int\limits_0^\infty e^{-\sigma^2 - n\frac{(x-\tau)^2}{2\sigma^2}} d\sigma, \ \ t = \sigma^2/n.$$

Applying the following formula (see (3), [36, p.336])

$$\int_0^\infty e^{-(\sigma^2+c^2/\sigma^2)}d\sigma=\frac{\sqrt{\pi}}{2}e^{-2|c|}, \tag{31}$$

we obtain

$$J_n f(x)=\sqrt{\frac{n}{2}}\int_{-\infty}^{\infty} f(\tau)e^{-\sqrt{2n}|x-\tau|}d\tau$$

$$=\sqrt{\frac{n}{2}}\int_{-\infty}^{x} f(\tau)e^{-\sqrt{2n}(x-\tau)}d\tau+\sqrt{\frac{n}{2}}\int_{x}^{\infty} f(\tau)e^{-\sqrt{2n}(\tau-x)}d\tau$$

$$=\sqrt{\frac{n}{2}}e^{-\sqrt{2n}x}\int_{-\infty}^{x} f(\tau)e^{\sqrt{2n}\tau}d\tau+\sqrt{\frac{n}{2}}e^{\sqrt{2n}x}\int_{x}^{\infty} f(\tau)e^{-\sqrt{2n}\tau}d\tau,$$

$$f\in L_2(\Omega).$$

Consider

$$I_1(x)=\int_{-\infty}^{x} f(\tau)e^{\sqrt{2n}\tau}d\tau,\ I_2(x)=\int_{x}^{\infty} f(\tau)e^{-\sqrt{2n}\tau}d\tau.$$

Observe that the functions $f(x)e^{\sqrt{2n}x}$, $f(x)e^{-\sqrt{2n}x}$ have the same Lebesgue points, then in accordance with the known fact, we have $I_1'(x)=f(x)e^{\sqrt{2n}x}$, $I_2'(x)=-f(x)e^{-\sqrt{2n}x}$, where x is a Lebesgue point. Using this result, we get

$$(J_n f(x))'=-n\int_{-\infty}^{x} f(\tau)e^{-\sqrt{2n}(x-\tau)}d\tau+n\int_{x}^{\infty} f(\tau)e^{-\sqrt{2n}(\tau-x)}d\tau \ \text{ a.e.}$$

Analogously, we have almost everywhere

$$(J_n f(x))''=n\sqrt{2n}\int_{-\infty}^{x} f(\tau)e^{-\sqrt{2n}(x-\tau)}d\tau$$

$$+n\sqrt{2n}\int_{x}^{\infty} f(\tau)e^{-\sqrt{2n}(\tau-x)}d\tau-n2f(x)$$

$$=2n(J_n-I)f(x)=-2AJ_nf(x),$$

taking into account the fact $\mathrm{R}(J_n) = \mathrm{R}(R_A(n)) = \mathrm{D}(A)$, we obtain the desired result.

In accordance with the reasonings of [36, p.336], we have $C(\Omega) \subset \mathrm{D}(A)$. Denote by A_0 a restriction of A on $C_0^\infty(\Omega)$. Using Lemma 1, we conclude that $\tilde{A}_0 = A$, since there does not exist an accretive extension of an m-accretive operator. Now, it is clear that

$$\|Af\|_{L_2} \leq \|f\|_{H_0^{2,5}}, \quad f \in H_0^{2,5}(\Omega), \tag{32}$$

whence $H_0^{2,5}(\Omega) \subset \mathrm{D}(A)$. Let us establish the representation $L = Z^\alpha_{G,F}(J)$. Since the operator A is m-accretive, then using formula (8), we can define positive fractional powers $\alpha \in (0,1)$ of the operator A. Applying the relations obtained above, we can calculate

$$(\lambda I + A)^{-1} A f(x) = -\frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty e^{-\lambda t} t^{-1/2} dt \int\limits_{-\infty}^\infty e^{-\frac{(x-\tau)^2}{2t}} f''(\tau) d\tau$$

$$= -\frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^\infty f''(\tau) d\tau \int\limits_0^\infty e^{-\lambda t - \frac{(x-\tau)^2}{2t}} t^{-1/2} dt, \quad f \in C_0^\infty(\Omega). \tag{33}$$

Here, substantiation of the interchange of the integration order can be easily obtained due to the properties of the function. We have for arbitrary chosen x, λ:

$$\int\limits_{-A}^{A} f''(\tau) d\tau \int\limits_0^\infty e^{-\lambda t - \frac{(x-\tau)^2}{2t}} t^{-1/2} dt$$

$$= \int\limits_{-A-x}^{A-x} f''(x+s) ds \int\limits_0^1 e^{-\lambda t - s^2/2t} t^{-1/2} dt$$

$$+ \int\limits_{-A-x}^{A-x} f''(x+s) ds \int\limits_1^\infty e^{-\lambda t - s^2/2t} t^{-1/2} dt.$$

Observe that the inner integrals converge uniformly with respect to s, it is also clear that the function under the integrals is continuous regarding to s, t, except of the set of points $(s; t_0)$, $t_0 = 0$. Hence applying the well-known theorem of calculus, we obtain (33). Consider

$$\int\limits_{-\infty}^{\infty} f''(x+s)ds \int\limits_{0}^{\infty} e^{-\lambda t - s^2/2t} t^{-1/2} dt$$

$$= 2\lambda^{-1/2} \int\limits_{-\infty}^{\infty} f''(x+s)ds \int\limits_{0}^{\infty} e^{-\sigma^2 - c^2/\sigma^2} d\sigma = I,$$

where $c^2 = s^2\lambda/2$. Using formula (31), we obtain

$$I = \sqrt{\pi}\lambda^{-1/2} \int\limits_{-\infty}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}|s|}ds$$

$$= \sqrt{\pi}\lambda^{-1/2} \int\limits_{0}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}s}ds + \sqrt{\pi}\lambda^{-1/2} \int\limits_{0}^{\infty} f''(x-s)e^{-\sqrt{2\lambda}s}ds.$$

Thus, combining formulas (8),(33), we conclude that

$$A^{\alpha}f(x) = -\frac{2^{-3/2}}{\Gamma(1-\alpha)\Gamma(\alpha)} \int\limits_{0}^{\infty} \lambda^{\alpha-3/2} d\lambda \int\limits_{-\infty}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}|s|}ds,$$

$$f \in C_0^{\infty}(\Omega).$$

We easily prove that

$$\int\limits_{\varepsilon}^{\infty} f''(x+s)ds \int\limits_{0}^{\infty} \lambda^{\alpha-3/2} e^{-\sqrt{2\lambda}s} d\lambda$$

$$= \int\limits_{0}^{\infty} \lambda^{\alpha-3/2} d\lambda \int\limits_{\varepsilon}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}s}ds, \quad f \in C_0^{\infty}(\Omega). \tag{34}$$

Let us show that

$$\int\limits_{0}^{\infty} \lambda^{\alpha-3/2} d\lambda \int\limits_{\varepsilon}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}s}ds \to \int\limits_{0}^{\infty} \lambda^{\alpha-3/2} d\lambda \int\limits_{0}^{\infty} f''(x+s)e^{-\sqrt{2\lambda}s}ds, \tag{35}$$

when $\varepsilon \to 0$. We have

$$\left| \int\limits_{0}^{\infty} \lambda^{\alpha-3/2} d\lambda \int\limits_{0}^{\varepsilon} f''(x+s)e^{-\sqrt{2\lambda}s}ds \right|$$

$$\leq \|f''\|_{L_\infty}\int\limits_0^\infty \lambda^{\alpha-3/2}d\lambda\int\limits_0^\varepsilon e^{-\sqrt{2\lambda}s}ds$$

$$=\frac{1}{\sqrt{2}}\|f''\|_{L_\infty}\int\limits_0^\infty \lambda^{\alpha-2}\left(1-e^{-\sqrt{2\lambda}\varepsilon}\right)d\lambda$$

$$=\varepsilon^{2(1-\alpha)}2^{3/2-\alpha}\|f''\|_{L_\infty}\int\limits_0^\infty t^{2\alpha-3}\left(1-e^{-t}\right)dt\to 0,\ \ \varepsilon\to 0,$$

from what follows the desired result. Using simple calculations, we get

$$\int\limits_0^\varepsilon f''(x+s)ds\int\limits_0^\infty e^{-\sqrt{2\lambda}s}\lambda^{\alpha-3/2}d\lambda$$

$$=2^{3/2-\alpha}\Gamma(2\alpha-1)\int\limits_0^\varepsilon f''(x+s)s^{1-2\alpha}ds$$

$$\leq C\|f''\|_{L_\infty}\varepsilon^{2(1-\alpha)}\to 0,\ \ \varepsilon\to 0. \tag{36}$$

In accordance with (34), we can write

$$\int\limits_0^\infty f''(x+s)ds\int\limits_0^\infty \lambda^{\alpha-3/2}e^{-\sqrt{2\lambda}s}d\lambda=\int\limits_0^\infty \lambda^{\alpha-3/2}d\lambda\int\limits_\varepsilon^\infty f''(x+s)e^{-\sqrt{2\lambda}s}ds$$

$$+\int\limits_0^\varepsilon f''(x+s)ds\int\limits_0^\infty e^{-\sqrt{2\lambda}s}\lambda^{\alpha-3/2}d\lambda.$$

Passing to the limit at the right-hand side, using (35),(36), we obtain

$$\int\limits_0^\infty \lambda^{\alpha-3/2}d\lambda\int\limits_0^\infty f''(x+s)e^{-\sqrt{2\lambda}s}ds=\int\limits_0^\infty f''(x+s)ds\int\limits_0^\infty \lambda^{\alpha-3/2}e^{-\sqrt{2\lambda}s}d\lambda$$

$$=2^{3/2-\alpha}\Gamma(2\alpha-1)\int\limits_0^\infty f''(x+s)s^{1-2\alpha}ds.$$

Taking into account the analogous reasonings, we conclude that

$$A^{\alpha}f(x) = -\frac{\Gamma(2\alpha-1)}{2^{\alpha}\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_{-\infty}^{\infty} f''(x+s)|s|^{1-2\alpha}ds = K_{\alpha}I^{\alpha}f''(x),$$

$$K_{\alpha} = -\frac{\Gamma(2\alpha-1)\cos\alpha\pi/2}{2^{\alpha-1}\Gamma(1-\alpha)}, \ \ f \in C_0^{\infty}(\Omega).$$

Using the Hardy-Littlewood theorem with limiting exponent (see Theorem 5.3, [33, p.103]), we get

$$\|A^{\alpha}f\|_{L_2} \le C\|I_{+}^{2(1-\alpha)}f''\|_{L_2} + C\|I_{-}^{2(1-\alpha)}f''\|_{L_2} \le C\|f''\|_{L_q},$$
$$f \in C_0^{\infty}(\Omega), \tag{37}$$

where $q = 2/(5-4\alpha)$. Applying the Hölder inequality, we obtain

$$\left(\int\limits_{-\infty}^{\infty} |f''(x)|^q(1+|x|)^{5q/2}(1+|x|)^{-5q/2}dx\right)^{1/q}$$

$$\le \left(\int\limits_{-\infty}^{\infty} |f''(x)|^2(1+|x|)^5dx\right)^{1/2}\left(\int\limits_{-\infty}^{\infty}(1+|x|)^{-5q\gamma/2}dx\right)^{1/q\gamma}$$

$$\le C\|f\|_{H_0^{2,5}}, \ \ f \in C_0^{\infty}(\Omega), \tag{38}$$

where $1 < q < 2$, $\gamma = 2/(2-q) > 1$. Combining (37),(38) and passing to the limit, we get

$$\|A^{\alpha}f\|_{L_2} \le C\|f\|_{H_0^{2,5}}, \ f \in H_0^{2,5}(\Omega). \tag{39}$$

Hence $H_0^{2,5}(\Omega) \subset \mathrm{D}(A^{\alpha})$. Using the Hardy-Littlewood theorem with limiting exponent, we obtain

$$\|I_{+}^{\sigma}\rho I^{2(1-\alpha)}f''\|_{L_2} \le C\|\rho I^{2(1-\alpha)}f''\|_{L_{q_1}} \le C_{\rho}\|f''\|_{L_{q_2}},$$
$$f \in C_0^{\infty}(\Omega), \ C_{\rho} = C\|\rho\|_{L_{\infty}},$$

where $q_1 = 2/(1+2\sigma)$, $q_2 = q_1/(1+2q_1[1-\alpha])$. We can rewrite $q_2 = 2/(1+2\sigma+4[1-\alpha])$, thus $1 < q_2 < 2$. Applying formula (38) and passing to the limit, we get

$$\|I_{+}^{\sigma}\rho I^{2(1-\alpha)}f''\|_{L_2} \ \le C_{\rho}\|f\|_{H_0^{2,5}}, \ f \in H_0^{2,5}(\Omega). \tag{40}$$

Note that

$$\int\limits_{\Omega} \mathcal{T} f\, \bar{g} dx = \int\limits_{\Omega} a f''\, \overline{g''} dx, \;\; f, g \in C_0^{\infty}(\Omega). \tag{41}$$

Therefore $\mathcal{T}$ is accretive, applying Lemma 1 we deduce that $\tilde{\mathcal{T}}$ is m-accretive. Using relation (32),(41) we can easily obtain $\|\tilde{\mathcal{T}} f\|_{L_2} \geq \gamma_a \|f\|_{H_0^{2,5}} \geq C\|Af\|_{L_2}$, $f \in \mathrm{D}(\tilde{\mathcal{T}})$, whence $\mathrm{D}(\tilde{\mathcal{T}}) \subset H_0^{2,5}(\Omega) \subset \mathrm{D}(A)$. Using simple reasonings, we can extend relation (41) and rewrite it in the following form

$$\int\limits_{\Omega} \tilde{\mathcal{T}} f\, \bar{g} dx = \frac{1}{4} \int\limits_{\Omega} aAf\, \overline{Ag} dx, \;\;\; f \in \mathrm{D}(\tilde{\mathcal{T}}),\, g \in \mathrm{D}(A),$$

whence $\tilde{\mathcal{T}} \subset A^{*}GA$, where $G := a/4$. Since the operator $\tilde{\mathcal{T}}$ is m-accretive, $A^{*}GA$ is accretive, then $\tilde{\mathcal{T}} = A^{*}GA$. Hence, taking into account the inclusion $\mathrm{D}(\tilde{\mathcal{T}}) \subset H_0^{2,5}(\Omega)$, relation (40), we conclude that $L = A^{*}GA + FA^{\alpha}$, where $F := \rho I$.

Let us prove that the operator L satisfies conditions H1–H2. Choose the space $L_2(\Omega)$ as a space $\mathfrak{H}$, the set $C_0^{\infty}(\Omega)$ as a linear manifold $\mathfrak{M}$, and the space $H_0^{2,5}(\Omega)$ as a space $\mathfrak{H}_{+}$. By virtue of Theorem 1, [1], we have $H_0^{2,5}(\Omega) \subset\subset L_2(\Omega)$. Thus, condition H1 is satisfied.

Using simple reasonings (the proof is omitted), we come to the following inequality

$$\left| \int\limits_{-\infty}^{\infty} (\tilde{\mathcal{T}} + \delta I) f \cdot \bar{g}\, dx \right| \leq C \|f\|_{H_0^{2,5}} \|g\|_{H_0^{2,5}}, \;\;\; f, g \in C_0^{\infty}(\Omega).$$

Applying the Cauchy-Schwarz inequality, relation (40), we obtain

$$\left| \left(I_{+}^{\sigma} \rho I^{2(1-\alpha)} f'', g \right)_{L_2} \right| \leq C_{\rho} \|f\|_{H_0^{2,5}} \|g\|_{H_0^{2,5}}, \;\; f,\, g \in C_0^{\infty}(\Omega). \tag{42}$$

On the other hand, using the conditions imposed on the function $a(x)$, it is not hard to prove that

$$\mathrm{Re}([\tilde{\mathcal{T}} + \delta I] f, f) \geq \min\{\gamma_a, \delta\} \|f\|^2_{H_0^{2,5}},\; f \in C_0^{\infty}(\Omega).$$

Using relation (42), we can easily obtain

$$\mathrm{Re}(I_{+}^{\sigma} \rho I^{2(1-\alpha)} f'', f) \geq -C_{\rho} \|f\|^2_{H_0^{2,5}},\; f \in C_0^{\infty}(\Omega).$$

Combining the above estimates, we conclude that if the condition $\min\{\gamma_a, \delta\} > C_{\rho}$ holds, then $\mathrm{Re}(Lf, f) \geq C\|f\|^2_{H_0^{2,5}}$, $f \in C_0^{\infty}(\Omega)$. Thus, condition H2 is satisfied. □

Difference operator

Consider a space $L_2(\Omega)$, $\Omega := (-\infty, \infty)$, define a family of operators

$$T_t f(x) := e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(x - k\mu), \;\; f \in L_2(\Omega), \; \lambda, \mu > 0, \; t \geq 0,$$

where convergence is understood in the sense of $L_2(\Omega)$ norm. It is not hard to prove that $T_t : L_2 \to L_2$, for this purpose it is sufficient to note that

$$\left\| \sum_{k=n}^{n+p} \frac{(\lambda t)^k}{k!} f(\cdot - k\mu) \right\|_{L_2} \leq \|f\|_{L_2} \sum_{k=n}^{n+p} \frac{(\lambda t)^k}{k!}. \tag{43}$$

Lemma 13. *T_t is a C_0 semigroup of contractions, the corresponding infinitesimal generator and its adjoint operator are defined by the following expressions*

$$Af(x) = \lambda[f(x) - f(x - \mu)], \; A^* f(x) = \lambda[f(x) - f(x + \mu)], \; f \in L_2(\Omega).$$

Proof. Assume that $f \in L_2(\Omega)$. Analogously to (43), we easily prove that $\|T_t f\|_{L_2} \leq \|f\|_{L_2}$. Consider

$$T_s T_t f(x) = e^{-\lambda s} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \left[e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(x - k\mu - n\mu) \right].$$

Since we have

$$\left\| \sum_{k=0}^{m} \frac{(\lambda t)^k}{k!} f(x - k\mu) \right\|_{L_2} \leq \|f\|_{L_2} \sum_{k=0}^{m} \frac{(\lambda t)^k}{k!},$$

then similarly to the case corresponding to $C(\Omega)$ norm (the prove is based upon the properties of the absolutely convergent double series, see Example 3 [36, p.327]), we conclude that

$$\begin{aligned} T_s T_t f(x) &= e^{-\lambda s} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \left[e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(x - k\mu - n\mu) \right] \\ &= e^{-\lambda(s+t)} \sum_{p=0}^{\infty} \frac{1}{p!} \left[p! \sum_{n=0}^{p} \frac{(\lambda s)^n}{n!} \frac{(\lambda t)^{p-n}}{(p-n)!} f(x - p\mu) \right] \\ &= e^{-\lambda(s+t)} \sum_{p=0}^{\infty} \frac{1}{p!} (\lambda s + \lambda t)^p f(x - p\mu) = T_{s+t} f(x), \end{aligned}$$

where equality is understood in the sense of $L_2(\Omega)$ norm. Let us establish the strongly continuous property. For sufficiently small t, we have

$$\|T_t f - f\|_{L_2} \leq e^{-\lambda t}(e^{\lambda t}-1)\|f\|_{L_2} + e^{-\lambda t}\left\|\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}f(\cdot - k\mu)\right\|_{L_2}$$

$$\leq te^{-\lambda t}\|f\|_{L_2}\left\{C+\sum_{k=0}^{\infty}\frac{(\lambda)^{k+1}t^k}{(k+1)!}\right\},$$

from what follows that

$$\|T_t f - f\|_{L_2} \to 0,\ \ t\to 0.$$

Taking into account the above facts, we conclude that T_t is a C_0 semigroup of contractions. Let us show that

$$Af(x) = \lambda[f(x) - f(x-\mu)],$$

we have (the proof is omitted)

$$\frac{(I-T_t)f(x)}{t}$$

$$= \frac{1-e^{-\lambda t}}{t}f(x) - \lambda e^{-\lambda t}f(x-\mu) - te^{-\lambda t}\sum_{k=2}^{\infty}\frac{\lambda^k t^{k-2}}{k!}f(x-k\mu).$$

Hence

$$\frac{(I-T_t)f}{t} \xrightarrow{L_2} \lambda[f - f(\cdot - \mu)],\ \ t\downarrow 0,$$

thus, we have obtained the desired result. Using change of variables in integral it is easy to show that

$$\int\limits_{-\infty}^{\infty} Af(x)g(x)dx = \int\limits_{-\infty}^{\infty} f(x)\lambda[g(x)-g(x+\mu)]dx,\ f,g\in L_2(\Omega),$$

hence $A^*f(x) = \lambda[f(x) - f(x+\mu)],\ f\in L_2(\Omega)$. The proof is complete. $\square$

It is remarkable that there are some difficulties to apply theorems (A)-(C) to a transform $Z^{\alpha}_{aI,bI}(A)$, where a,b are functions, and the main of them can be said as follows "it is not clear how we should build a space $\mathfrak{H}_+$". However we can consider a rather abstract perturbation of the above transform in order to reveal its spectral properties.

Theorem 14. *Assume that Q is a closed operator acting in $L_2(\Omega)$, $Q^{-1} \in \mathcal{K}(L_2)$, the operator N is strictly accretive, bounded, $\mathrm{R}(Q) \subset \mathrm{D}(N)$. Then a perturbation*

$$L := Z^{\alpha}_{aI,bI}(A) + Q^* N Q, \quad a, b \in L_\infty(\Omega),\ \alpha \in (0,1)$$

satisfies conditions H1–H2, if $\gamma_N > \sigma \|Q^{-1}\|^2$, where we put $\mathfrak{M} := \mathrm{D}_0(Q)$,

$$\sigma = 4\lambda \|a\|_{L_\infty} + \|b\|_{L_\infty} \frac{\alpha \lambda^{\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{k!}.$$

Proof. Let us find a representation for fractional powers of the operator A. Using the Balakrishnan formula (5) [36, p.260], we get

$$A^{\alpha} f = \sum_{k=0}^{\infty} C_k f(x - k\mu), \quad f \in C_0^{\infty}(\Omega), \tag{44}$$

$$C_k = -\frac{\alpha \lambda^k}{k!\Gamma(1-\alpha)} \int_0^{\infty} e^{-\lambda t} t^{k-1-\alpha} dt = -\frac{\alpha \Gamma(k-\alpha)}{k!\Gamma(1-\alpha)} \lambda^{\alpha},$$

$$k = 0, 1, 2, ..., .$$

Let us extend relation (44) to $L_2(\Omega)$. We have almost everywhere

$$\sum_{k=0}^{\infty} C_k g(x - k\mu) - \sum_{k=0}^{\infty} C_k f(x - k\mu) = \sum_{k=0}^{\infty} C_k [g(x - k\mu) - f(x - k\mu)],$$

$$g \in C_0^{\infty}(\Omega),\ f \in L_2(\Omega),$$

since the first sum is a partial sum for a fixed $x \in \mathbb{R}$. In accordance with formula (1.66) [33, p.17], we have $|C_k| \le C\, k^{-1-\alpha}$, hence

$$\left\| \sum_{k=0}^{\infty} C_k [g(\cdot - k\mu) - f(\cdot - k\mu)] \right\|_{L_2} \le \|g - f\|_{L_2} \sum_{k=0}^{\infty} |C_k|.$$

Thus, we obtain

$$\forall f \in L_2(\Omega),\ \exists \{f_n\} \in C_0^{\infty}(\Omega) : f_n \xrightarrow{L_2} f,\ A^{\alpha} f_n \xrightarrow{L_2} \sum_{k=0}^{\infty} C_k f(\cdot - k\mu).$$

Since A^{α} is closed, then

$$A^{\alpha} f = \sum_{k=0}^{\infty} C_k f(x - k\mu),\ f \in L_2(\Omega). \tag{45}$$

Moreover, it is clear that $C_0^\infty(\Omega)$ is a core of A^α. On the other hand, applying formula (8), using the notation $\eta(x)=\lambda[f(x)-f(x-\mu)]$, we get

$$A^\alpha f(x)=\frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty \xi^{\alpha-1}(\xi I+A)^{-1}Af(x)d\xi$$

$$=\frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty \xi^{\alpha-1}d\xi\int\limits_0^\infty e^{-\xi t}T_t\eta(x)dt$$

$$=\frac{\sin\alpha\pi}{\pi}\sum_{k=0}^\infty\frac{\lambda^k}{k!}\eta(x-k\mu)\int\limits_0^\infty \xi^{\alpha-1}d\xi\int\limits_0^\infty e^{-t(\xi+\lambda)}t^k dt$$

$$=\frac{\sin\alpha\pi}{\pi}\sum_{k=0}^\infty\frac{\lambda^k}{k!}\eta(x-k\mu)\int\limits_0^\infty \xi^{\alpha-1}(\xi+\lambda)^{-k-1}d\xi\int\limits_0^\infty e^{-t}t^k dt,$$

$$f\in C_0^\infty(\Omega),$$

we can rewrite the previous relation as follows

$$A^\alpha f(x)=\sum_{k=0}^\infty C'_k[f(x-k\mu)-f(x-(k+1)\mu)],\ \ f\in C_0^\infty(\Omega),\tag{46}$$

$$C'_k=\frac{\lambda^{k+1}\sin\alpha\pi}{\pi}\int\limits_0^\infty \xi^{\alpha-1}(\xi+\lambda)^{-k-1}d\xi.$$

Note that analogously to (45) we can extend formula (46) to $L_2(\Omega)$. Comparing formulas (44),(46) we can check the results calculating directly, we get

$$C'_{k+1}-C'_k=-\frac{\lambda^{k+1}\sin\alpha\pi}{\pi}\int\limits_0^\infty \xi^{\alpha}(\xi+\lambda)^{-k-2}d\xi$$

$$=-\frac{\alpha\Gamma(k+1-\alpha)}{(k+1)!\Gamma(1-\alpha)}\lambda^\alpha=C_{k+1},\ \ C'_0=C_0,\ k\in\mathbb{N}_0.$$

Observe that by virtue of the made assumptions regarding Q, we have $\mathfrak{H}_Q\subset\subset L_2(\Omega)$. Choose the space $L_2(\Omega)$ as a space $\mathfrak{H}$ and the space $\mathfrak{H}_Q$ as a space $\mathfrak{H}_+$. Let $S:=Z^\alpha_{aI,bI}(A)$, $T:=Q^*NQ$. Applying the reasonings of Theorem 5, we conclude that there exists a set $\mathfrak{M}:=\mathrm{D}_0(Q)$, which is dense in $\mathfrak{H}_Q$, such that the operators S,T are defined on its elements. Thus, we obtain the fulfilment of condition H1. Since the operator N is bounded, then $|(Tf,g)|_{L_2}\leq\|N\|\cdot\|f\|_{\mathfrak{H}_Q}\|g\|_{\mathfrak{H}_Q}$. Using formula (45), we can easily obtain $|(Sf,g)|_{L_2}\leq$

$\sigma\|f\|_{L_2}\|g\|_{L_2} \leq \sigma\|Q^{-1}\|^2 \cdot \|f\|_{\mathfrak{H}_Q}\|g\|_{\mathfrak{H}_Q}$, $\sigma = 4\lambda\|a\|_{L_\infty} + \|b\|_{L_\infty}\sum_{k=0}^{\infty}|C_k|$. Using the strictly accretive property of the operator N we get $\mathrm{Re}(Tf, f) \geq \gamma_N\|f\|^2_{\mathfrak{H}_Q}$. On the other hand $\mathrm{Re}(Sf, f) \geq -\sigma\|Q^{-1}\|^2 \cdot \|f\|^2_{\mathfrak{H}_Q}$, hence condition H2 is satisfied. The proof is complete. □

4. Conclusions

In this paper, we studied a true mathematical nature of a differential operator with a fractional derivative in final terms. We constructed a model in terms of the infinitesimal generator of a corresponding semigroup and successfully applied spectral theorems. Further, we generalized the obtained results to some class of transforms of m-accretive operators, what can be treated as an introduction to the fractional calculus of m-accretive operators. As a concrete theoretical achievement of the offered approach, we have the following results: an asymptotic equivalence between the real component of a resolvent and the resolvent of the real component was established for the class; a classification, in accordance with resolvent belonging to the Schatten-von Neumann class, was obtained; a sufficient condition of completeness of the root vectors system were formulated; an asymptotic formula for the eigenvalues was obtained. As an application, there were considered cases corresponding to a finite and infinite measure as well as various notions of fractional derivative under the semigroup theory point of view, such operators as a Kipriyanov operator, Riesz potential, difference operator were involved. The eigenvalue problem for a differential operator with a composition of fractional integro-differential operators in final terms was solved.

In addition, note that minor results are also worth noticing such as a generalization of the well-known von Neumann theorem (see the proof of Theorem 5). In Section 3, it might have been possible to consider an unbounded domain Ω with some restriction imposed upon a solid angle containing Ω, due to this natural way we come to a generalization of the Kipriyanov operator. We should add that various conditions, that may be imposed on the operator F, are worth studying separately since there is a number of applications in the theory of fractional differential equations.

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