NUMERICAL SOLUTION OF A PARABOLIC PROBLEM WITH INVOLUTION AND NONLOCAL CONDITIONS

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Abstract: In this paper, the well-posedness theorem on the parabolic differential equation with involution and nonlocal conditions is considered. The stable difference scheme for the approximate solution of this problem is presented. Numerical results are given.

AMS Subject Classification: 35N25, 65J22, 39A14

Key Words: well-posedness; positivity; coercive stability; nonlocal problem

1. Introduction

There always has been a major interest for the numerical solution of partial differential equations due to their widespread applications in physics and technology. For this aim, the well-posedness of various nonlocal boundary value problems for partial differential and difference equations has also been studied extensively by many researchers (for instance, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] and the references given therein). In the last decades, partial differential equations with involutions were also investigated by several authors (for references, see [13], [14], [15], [16]). In [14], Ashyralyev and Sarsenbi established the stability and coercive stability estimates for the mixed problem (with involution with respect to t and the Dirichlet condition in x) of a parabolic equation with involution in Hölder norms. The mixed problem for a parabolic

equation with involution was investigated by the operator tool of monograph [14, Eq. (29)]. The stability estimates in $C([0, T], L_2[-l, l])$ norm and coercive stability estimates in $C_\alpha([0, T], L_2[-l, l])$ and $C_{\alpha_0}([0, T], L_2[-l, l])$ norms for the solution of the problem were established. However, numerical solutions to nonlocal parabolic equations with involution have not been well-investigated.

The present paper is devoted to the study of parabolic equation with involution and nonlocal conditions. The well-posedness theorem on the differential equation of the parabolic problem is proved. A stable difference scheme for the approximate solution of this problem is constructed. Furthermore, stability estimates for the difference scheme of the parabolic problem are established. Numerical results are provided.

2. Well-posedness of the Differential Problem with Involution

In this paper we consider the problem

$$\begin{cases} u_t(t, x) - (a(x)u_x(t, x))_x - \beta(a(-x)u_x(t, -x))_x \\ + \sigma u(t, x) = f(t, x), & -l < x < l, \quad 0 < t < T, \\ u(t, -l) = u(t, l), \quad u_x(t, -l) = u_x(t, l), & 0 \leq t \leq T, \\ u(0, x) = \varphi(x), & -l \leq x \leq l, \end{cases} \quad (1)$$

for the one dimensional parabolic differential equation with involution. Problem (1) has a unique solution $u(t, x)$ under compatibility conditions for the smooth functions $f(t, x)$ ($(t, x) \in (0, T) \times (-l, l)$), $a \geq a(x) = a(-x) \geq \delta > 0$, $\delta - a|\beta| \geq 0$ ($x \in (-l, l)$), $\varphi(x)$, $x \in [-l, l]$, and $\sigma > 0$.

Throughout this paper, let the Sobolev space $W_2^2[-\ell, \ell]$ be defined as the set of all functions $v(x)$ defined on $[-\ell, \ell]$ such that $v(x)$ and the second order derivative function $v''(x)$ are both locally integrable in $L_2[-\ell, \ell]$, equipped with the norm

$$\|v(x)\|_{W_2^2[-\ell, \ell]} = \left(\int_{-\ell}^{\ell} |v(x)|^2 dx \right)^{1/2} + \left(\int_{-\ell}^{\ell} |v''(x)|^2 dx \right)^{1/2}.$$

Theorem 1. Suppose that $\varphi \in W_2^2[-\ell, \ell]$. Let the function $f(t, x) \in C_0^\alpha([0, T], L_2[-\ell, \ell])$. Then the solution of the initial boundary value problem (1) satisfies the coercive stability estimate

$$\|u_t\|_{C_0^\alpha([0, T], L_2[-l, l])} + \|u\|_{C_0^\alpha([0, T], W_2^2[-l, l])}$$

$$\leq M(\delta, \sigma, \beta, l) \left[\|\varphi\|_{W_2^2[-l, l]} + \|f\|_{C_0^\alpha([0, T], L_2[-l, l])} \right],$$

where $M(\alpha, \delta, \sigma, \beta, l)$ does not depend on $f(t, x)$ and $\varphi(x)$. Here $C_0^\alpha([0, T], H)$, $0 < \alpha < 1$, stand for Banach spaces of all abstract continuous functions $\varphi(t)$ defined on $[0, T]$ with values in H satisfying a Hölder condition for which the following norm is finite

$$\|\varphi\|_{C_0^\alpha([0, T], H)} = \|\varphi\|_{C([0, T], H)} + \sup_{0 \leq t < t+\tau \leq T} \frac{(t+\tau)^\alpha \|\varphi(t+\tau) - \varphi(t)\|_H}{\tau^\alpha}.$$

Proof. Following the techniques from [14], we introduce a differential operator A^x defined by the formula

$$A^x u(x) = -(a(x)u_x(x))_x - \beta(a(-x)u_x(-x))_x + \sigma u(x) \quad (2)$$

with the domain

$$D(A^x) = \{u, u_{xx} \in L_2[-l, l] : u(-l) = u(l), u_x(-l) = u_x(l)\}.$$

Then, we can rewrite the problem

$$u_t(t) + Au(t) = f(t), \quad 0 < t < T, \quad u(0) = \varphi \quad (3)$$

in a Hilbert space $H = L_2[-l, l]$ with the unbounded operator $A = A^x$ defined by formula (2). Here, $f(t) = f(t, x)$ and $u(t) = u(t, x)$ are known and unknown abstract functions defined on $[0, T]$ with values in $H = L_2[-l, l]$, respectively, and $\varphi = \varphi(x)$ and $a = a(x)$ are given smooth elements of $H = L_2[-l, l]$. Therefore, the proof of Theorem 1 is based on the theorem on coercive stability of abstract problem (3) (see [14]), the self-adjointness and positive definiteness of the space operator A^x defined by formula (2). $\square$

3. Well-posedness of the Difference Scheme

Now, we study the stable difference scheme for the approximate solution of problem (1). The discretization of this problem is carried out in two stages. In the first stage, we define the grid space

$$[-l, l]_h = \{x_n : x_n = nh, \quad -M \leq n \leq M, \quad Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([-l, l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_r\}_{-M}^M$ defined on $[-l, l]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [-l, l]_h} |\varphi^h(x)|^2 h \right)^{1/2}.$$

To the differential operator A^x given in (2), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\overline{x}}(x))_{x,r} - \beta(a(-x)\varphi_{\overline{x}}(-x))_{x,r} + \sigma\varphi_r\}_{-M+1}^{M-1}, \quad (4)$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_{-M}^M$ satisfying the conditions $\varphi_{-M} = \varphi_M$ and $\varphi_{-M+1} - \varphi_{-M} = \varphi_M - \varphi_{M-1}$ ($-3\varphi_{-M+2} + 4\varphi_{-M+1} - \varphi_{-M} = -\varphi_M + 4\varphi_{M-1} - 3\varphi_{M-2}$).

It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h} . With the help of A_h^x , we reach the abstract problem

$$\begin{cases} u_t^h(t, x) + A_h^x u^h(t, x) = f^h(t, x), & x \in [-l, l]_h, \quad 0 < t < T, \\ u^h(0, x) = \varphi^h(x), & x \in [-l, l]_h. \end{cases} \quad (5)$$

In the second stage, we replace the problem (5) with first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = f_k^h, & f_k^h(x) = f(t_k, x), \\ t_k = k\tau, \quad 1 \leq k \leq N, \quad N\tau = T, & x \in [-l, l]_h, \\ u_0^h(x) = \varphi^h(x), & x \in [-l, l]_h. \end{cases} \quad (6)$$

We have the following stability and coercive stability results.

Theorem 2. *The solution of the difference problem (6) satisfies stability and coercive the stability estimates*

$$\begin{aligned} & \left\| \left\{ u_k^h \right\}_0^N \right\|_{C_\tau(L_{2h})} \\ & \leq M_3(\delta, \sigma, \beta, l) \left[\|\varphi^h\|_{L_{2h}} + \left\| \left\{ f_k^h \right\}_1^N \right\|_{C_\tau(L_{2h})} \right], \\ & \left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_1^N \right\|_{C_\tau^\alpha(L_{2h})} + \left\| \left\{ u_k^h \right\}_0^N \right\|_{C_\tau^\alpha(W_{2h}^2)} \end{aligned}$$

$$\leq M(\alpha, \delta, \sigma, \beta, l) \left[\|\varphi^h\|_{W_{2h}^2} + \left\| \left\{ f_k^h \right\}_1^N \right\|_{C_\tau^\alpha(L_{2h})} \right],$$

where $M_3(\delta, \sigma, \beta, l)$, $M(\alpha, \delta, \sigma, \beta, l)$ do not depend on τ, h, f_k^h , $1 \leq k \leq N$ and $\varphi^h(x)$. Here, we have the normed space

$$C_\tau^\alpha(H) = C^\alpha([0, T]_\tau, H), \quad 0 < \alpha < 1,$$

of abstract grid functions $\{\varphi_k\}_{k=1}^N$ with values on H and the norm

$$\begin{aligned} \|\{\varphi_k\}_{k=1}^N\|_{C_\tau^\beta(E)} &= \max_{1 \leq k \leq N} \|\varphi_k\|_H \\ &+ \max_{1 \leq k < k+r \leq N} \|\varphi_{k+r} - \varphi_k\|_H \frac{((k+r)\tau)^\alpha}{(r\tau)^\alpha}. \end{aligned}$$

Proof. Difference scheme (6) can be written as the following abstract difference scheme

$$\frac{u_k^h - u_{k-1}^h}{\tau} + A_h^x u_k^h = f_k^h, \quad 1 \leq k \leq N, \quad u_0^h = \varphi^h,$$

in a Hilbert space L_{2h} with operator $A_h = A_h^x$ defined by formula (4). Here and in future, $f_k^h = f_k^h(x)$ is given abstract mesh function, $u_k^h = u_k^h(x)$ is unknown mesh function. The proof of Theorem 2 is based on the stability and coercive stability of the difference scheme (6) and on the self-adjointness and positive definiteness of the space difference operator A_h^x in L_{2h} (see [14]). $\square$

4. Numerical Experiment

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature. In this section for the approximate solution of a problem, we use the first and second order of accuracy difference schemes and the error analysis is given.

We consider the problem

$$\begin{cases} u_t(t, x) - u_{xx}(t, x) - \frac{1}{2}u_{x,x}(t, -x) + u(t, x) \\ = \cos t \sin x + \frac{3}{2} \sin t \sin x, \quad x \in (-\pi, \pi), t \in (0, \pi), \\ u(0, x) = 0, x \in [-\pi, \pi], \\ u(t, -\pi) = u(t, \pi), \quad u_x(t, -\pi) = u_x(t, \pi), \quad t \in [0, \pi] \end{cases} \quad (7)$$

for parabolic equation with involution and nonlocal condition. The exact solution of this problem is

$$u(t, x) = \sin t \sin x, \quad -\pi \leq x \leq \pi, \quad 0 \leq t \leq \pi.$$

Here and in future, we denote the set $[0, \pi]_\tau \times [-\pi, \pi]_h$ of all grid points

$$[0, \pi]_\tau \times [-\pi, \pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \leq k \leq N, \\ N\tau = \pi, x_n = nh, -M \leq n \leq M, Mh = \pi\}.$$

For the numerical solution of problem (7), we present the first order of accuracy difference scheme

$$\begin{cases} \tau^{-1} (u_n^k - u_n^{k-1}) - h^{-2} (u_{n+1}^k - 2u_n^k + u_{n-1}^k) \\ - \frac{1}{2} h^{-2} (u_{-n+1}^k - 2u_{-n}^k + u_{-n-1}^k) + u_n^k \\ = \cos t_k \sin x_n + \frac{3}{2} \sin t_k \sin x_n, \\ 1 \leq k \leq N, -M+1 \leq n \leq M-1, \\ u_n^0 = 0, -M \leq n \leq M, \\ u_{-M}^k = u_M^k, u_{-M+1}^k - u_{-M}^k = u_M^k - u_{M-1}^k, 0 \leq k \leq N. \end{cases} \quad (8)$$

For obtaining the solution of difference scheme (8), we rewrite it in the matrix form

$$A U^k + B U^{k-1} = R \varphi^k, \quad 1 \leq k \leq N, \quad U^0 = \varphi, \quad (9)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & -1 \\ a & b & a & 0 & . & . & . & . & . & 0 & c & -a & c \\ 0 & a & b & a & . & . & . & . & . & c & -a & c & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a+c & b-a & a+c & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & c & -a & c & . & . & . & . & . & a & b & a & 0 \\ c & -a & c & 0 & . & . & . & . & . & 0 & a & b & a \\ -1 & 1 & 0 & 0 & . & . & . & . & . & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & d & 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & d & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & d & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & d & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}$$

and

$$a = -\frac{1}{h^2}, \quad b = \frac{1}{\tau} + \frac{2}{h^2} + 1, \quad c = -\frac{1}{2h^2}, \quad d = \frac{1}{\tau},$$

$$U^s = \begin{bmatrix} U_{-M}^s \\ \vdots \\ U_M^s \end{bmatrix} \quad \text{for } s = k, k-1, \quad \varphi = \begin{bmatrix} \varphi(x_{-M}) \\ \vdots \\ \varphi(x_M) \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix}, \quad \varphi^k = \begin{bmatrix} 0 \\ \varphi_{-M+1}^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}.$$

So, we have the first order difference equation with respect to k with matrix coefficients. From (9) it follows that

$$U^k = -A^{-1}BU^{k-1} + A^{-1}R\varphi^k, \quad k = 1, \dots, N. \quad (10)$$

We also present the following second order of accuracy difference scheme for the approximate solutions of the problem (7)

$$\left\{ \begin{array}{l} \tau^{-1} (u_n^k - u_n^{k-1}) \\ -\frac{1}{2h^2} [(u_{n+1}^k - 2u_n^k + u_{n-1}^k) + (u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1})] \\ -\frac{1}{4h^2} (u_{-n+1}^k - 2u_{-n}^k + u_{-n-1}^k) \\ -\frac{1}{4h^2} (u_{-n+1}^{k-1} - 2u_{-n}^{k-1} + u_{-n-1}^{k-1}) \\ +\frac{1}{2}[u_n^k + u_n^{k-1}] \\ = \cos(t_k - \frac{\tau}{2}) \sin x_n + \frac{3}{2} \sin(t_k - \frac{\tau}{2}) \sin x_n, \\ 1 \leq k \leq N, -M+1 \leq n \leq M-1, \\ u_n^0 = 0, -M \leq n \leq M, \\ u_{-M}^k = u_M^k, \\ -3u_{-M+2}^k + 4u_{-M+1}^k - u_{-M}^k = -u_M^k + 4u_{M-1}^k - 3u_{M-2}^k, \\ 0 \leq k \leq N. \end{array} \right. \quad (11)$$

For obtaining the solution of difference scheme (11), we rewrite it in the matrix form

$$AU^k + BU^{k-1} = R\varphi^{k-\frac{\tau}{2}}, \quad 1 \leq k \leq N, \quad U^0 = \varphi,$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & -1 \\ a & b & a & 0 & . & . & . & . & . & 0 & c & -a & c \\ 0 & a & b & a & . & . & . & . & . & c & -a & c & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a+c & b-a & a+c & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & c & -a & c & . & . & . & . & . & a & b & a & 0 \\ c & -a & c & 0 & . & . & . & . & . & 0 & a & b & a \\ 3 & -4 & 1 & 0 & . & . & . & . & . & 0 & -3 & 4 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & 0 \\ a & d & a & 0 & . & . & . & . & . & 0 & c & -a & c \\ 0 & a & d & a & . & . & . & . & . & c & -a & c & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a+c & d-a & a+c & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & c & -a & c & . & . & . & . & . & a & d & a & 0 \\ c & -a & c & 0 & . & . & . & . & . & 0 & a & d & a \\ 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$a = -\frac{1}{2h^2}, \quad c = -\frac{1}{4h^2}, \quad d = -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2}.$$

Similarly, it follows that

$$U^k = -A^{-1}BU^{k-1} + A^{-1}R\varphi^k, \quad k = 1, \dots, N.$$

Now, we give the results of the numerical analysis. We recorded the numerical solutions u_n^k of these difference schemes at (t_k, x_n) for different N and M values. For their comparison, the errors computed by

$$E = \max_{\substack{1 \leq k \leq N \\ -M \leq n \leq M}} |u(t_k, x_n) - u_n^k|.$$

Table 1 gives the error analysis between the exact solution and the solutions derived by difference schemes. It is constructed for $N = M = 20, 40, 80$, and 160 .

Table 1. Error analysis between the exact solution $u(t_k, x_n)$ and numerical solution u_n^k .

Method	N=M=20	N=M=40	N=M=80	N=M=160
1 st order	$4.2764 \cdot 10^{-2}$	$2.1826 \cdot 10^{-2}$	$1.1022 \cdot 10^{-2}$	$5.5390 \cdot 10^{-3}$
2 nd order	$2.9878 \cdot 10^{-2}$	$6.9966 \cdot 10^{-3}$	$1.6915 \cdot 10^{-3}$	$4.1570 \cdot 10^{-4}$

Thus, the second order of accuracy difference schemes converges faster and are more accurate comparing with the first order of accuracy difference scheme.

5. Conclusion

In this paper, theoretical studies are given that prove well-posedness theorem on the differential equation of the nonlocal problem with involution and stability estimates for the difference scheme of the parabolic problem. To support the theoretical results by a numerical experiment, we constructed a stable difference scheme for the approximate solution of the problem. The obtained results, given in Table 1, support the theoretical results.

6. Acknowledgement

The author would like to thank Prof. Allaberen Ashyralyev for his constructive comments on this paper and continuous support.

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