STABILITY OF BINET-CAUCHY FUNCTIONAL EQUATION ON NON-ARCHIMEDEAN BANACH ALGEBRAS

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Abstract: In this paper, we introduced Binet-Cauchy additive functional equation

$$\left(\sum_{i=1}^n f(x_i y_i) \right) \left(\sum_{j=1}^n f(u_j v_j) \right) = f \left(\sum_{i=1}^n x_i v_i \right) f \left(\sum_{j=1}^n y_j u_j \right) \\ + \sum_{1 \leq i < j \leq n} f(x_i u_j - x_j u_i) f(y_i v_j - y_j v_i)$$

and established related generalized Ulam-Hyers stability results in non-Archimedean Banach modules over a Banach algebra.

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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [27] concerning the stability of group homomorphisms and affirma-

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tively answered for Banach spaces by Hyers [14]. It was further generalized and excellent results obtained by number of authors as [2, 13, 23, 24, 26].

The functional equation

$$f(x+y) = f(x) + f(y) \quad (1)$$

is said to be **additive functional equation** because the additive function $f(x) = ax$ is a solution of the functional equation (1).

Let $\mathbf{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbf{K}$ into $[0, \infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x+y| \leq \max\{|x|, |y|\} \leq |x| + |y|$ for all $x, y \in \mathbf{K}$. The associated field $\mathbf{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is nontrivial, i.e., there is a $z \in \mathbf{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbf{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbf{K}$ with the strong triangle inequality (ultrametric); namely, $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

In any such a space a sequence $\{x_n\}_{n \in \mathbf{N}}$ is Cauchy if and only if $\{x_{n+1} - x_n\}_{n \in \mathbf{N}}$ converges to zero. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{A}$.

In this paper, we investigated the general solution in vector spaces and discussed the generalized Ulam-Hyers stability of Binet-Cauchy functional equation

$$\begin{aligned} \left(\sum_{i=1}^n f(x_i y_i) \right) \left(\sum_{j=1}^n f(u_j v_j) \right) &= f \left(\sum_{i=1}^n x_i v_i \right) f \left(\sum_{j=1}^n y_j u_j \right) + \\ &\quad \sum_{1 \leq i < j \leq n} f(x_i u_j - x_j u_i) f(y_i v_j - y_j v_i) \end{aligned} \quad (2)$$

in non-Archimedean Banach modules over a Banach algebra.

2. General Solution of the Functional Equation (2)

In this section, we investigate the general solution of the functional equation (2). Throughout this section let us consider X and Y be real vector spaces.

Theorem 1. *Let X and Y be real vector spaces. If the mapping $f : X \rightarrow Y$ satisfies the functional equation (2) for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in X$ then $f : X \rightarrow Y$ satisfying the functional equation (1) for all $x, y \in X$.*

Proof. Setting $(x_3, y_3, u_3, v_3, \dots, x_n, y_n, u_n, v_n)$ by $(0, 0, \dots, 0, 0)$ in 2, we get

$$\begin{aligned} & (f(x_1y_1) + f(x_2y_2))(f(u_1v_1) + f(u_2v_2)) \\ &= f(x_1v_1 + x_2v_2)f(y_1u_1 + y_2v_2) + f(x_1u_2 - x_2u_1)f(y_1v_2 - y_2v_1) \end{aligned} \quad (3)$$

for all $x_1, y_1, u_1, v_1, x_2, y_2, u_2, v_2 \in X$.

Replacing $(x_1, y_1, u_1, v_1, x_2, y_2, u_2, v_2)$ by $(x^{\frac{1}{4}}, x^{\frac{1}{4}}, \dots, x^{\frac{1}{4}}, x^{\frac{1}{4}})$ in (3), we obtain

$$n^2 (f(\sqrt{x}))^2 = (f(n\sqrt{x}))^2 \quad (4)$$

for all $x \in X$. Setting $u_1 = 0$ and $v_1 = 0$ in (3), we get

$$(f(x))^2 = f(x^2) \quad (5)$$

for all $x \in X$. Replacing x by $\sqrt{x}$ in (5), we arrive

$$(f(\sqrt{x}))^2 = f(x) \quad (6)$$

for all $x \in X$. Using (6) in (4), we reach

$$n^2 (f(x)) = (f(n^2x)) \quad (7)$$

for all $x \in X$. Put $n^2 = k$ in above equation (7), we get

$$k(f(x)) = (f(kx)) \quad (8)$$

for all $x \in X$. Taking $x_1 = v_1 = 0$ and $x_2 = y_1 = y_2 = u_1 = u_2 = v_2 = \sqrt{x}$ in (4) and using (8), we obtain

$$f(-x) = -f(x) \quad (9)$$

for all $x \in X$. Therefore f is an odd function. Letting $x_1 = y_1 = u_2 = v_2 = \sqrt{x}$; $x_2 = y_2 = -\sqrt{y}$; $u_1 = \sqrt{y}$ and $v_1 = 0$ in (4), we derive (1). Hence completes the proof. $\square$

3. Stability Results

In this section, we present the generalized Ulam - Hyers stability of the functional equation (2).

Throughout this section, let $\mathcal{B}$ be a Banach-algebra with norm $\| \cdot \|_A$ and ${}_{\mathcal{B}}X$ be left Banach $\mathcal{B}$ -module with norm $\| \cdot \|$ and ${}_{\mathcal{B}}Y$ be non-Archimedean left Banach $\mathcal{B}$ -module with norm $\| \cdot \|$. Define a mapping $F : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}Y$ by

$$\begin{aligned} F(x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n) &= \left(\sum_{i=1}^n f(x_i y_i) \right) \left(\sum_{j=1}^n f(u_j v_j) \right) \\ &\quad - f \left(\sum_{i=1}^n x_i v_i \right) f \left(\sum_{j=1}^n y_j u_j \right) \\ &\quad - \sum_{1 \leq i < j \leq n} f(x_i u_j - x_j u_i) f(y_i v_j - y_j v_i) \end{aligned}$$

for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$.

Theorem 2. Let $j \in \{-1, 1\}$. Let $\alpha : {}_{\mathcal{B}}X^{4n} \rightarrow [0, \infty)$ be a function such that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x_1, n^{kj}y_1, \dots, n^{kj}u_n, n^{kj}v_n)}{n^{kj}}, \text{ converges to } \Re \text{ and} \\ \lim_{k \rightarrow \infty} \frac{\alpha(n^{kj}x_1, n^{kj}y_1, \dots, n^{kj}u_n, n^{kj}v_n)}{n^{kj}} < \infty \end{aligned} \quad (10)$$

for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$ and let $f : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}Y$ be a function satisfies the inequality

$$\|F(x_1, y_1, \dots, u_n, v_n)\| \leq \alpha(x_1, y_1, \dots, u_n, v_n) \quad (11)$$

for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$ and

$$\begin{aligned} \|f(x_1 y_1 \cdots u_n v_n) - f(x_1) f(y_1) \cdots f(u_n) f(v_n)\| \\ \leq \alpha(x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n) \end{aligned} \quad (12)$$

for all $x, y \in {}_{\mathcal{B}}X$. Then there exists a unique additive homomorphism $\mathcal{H} : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}Y$ such that

$$\|f(x) - \mathcal{H}(x)\|$$

$$\leq \frac{1}{|k|} \max \left\{ \frac{1}{|k|^i} \alpha \left((k^i x)^{\frac{1}{4}}, \dots, (k^i x)^{\frac{1}{4}} \right) : 0 \leq j < n \right\} \quad (13)$$

for all $x \in_{\mathcal{B}} X$. The mapping $\mathcal{H}(x)$ is defined by

$$\mathcal{H}(x) = \lim_{m \rightarrow \infty} \frac{f(k^{mj}x)}{|k|^{mj}} \quad (14)$$

for all $x \in_{\mathcal{B}} X$.

Proof. Assume $j = 1$. Replacing $(x_1, y_1, \dots, x_n, y_n, u_n, v_n)$ by $(x^{\frac{1}{4}}, x^{\frac{1}{4}}, x^{\frac{1}{4}}, x^{\frac{1}{4}}, \dots, x^{\frac{1}{4}}, x^{\frac{1}{4}})$ and using (5) in (11), we get

$$\|n^2 f(x) - f(n^2 x)\| \leq \alpha \left(x^{\frac{1}{4}}, \dots, x^{\frac{1}{4}} \right) \quad (15)$$

for all $x \in_{\mathcal{B}} X$. Above equation (15) can be rewritten as

$$\left\| f(x) - \frac{f(kx)}{k} \right\| \leq \frac{1}{|k|} \alpha \left(x^{\frac{1}{4}}, \dots, x^{\frac{1}{4}} \right), \quad (16)$$

where $k = n^2$, for all $x \in_{\mathcal{B}} X$. Replacing x by kx in (16) and divided by k , we get

$$\left\| \frac{f(kx)}{k} - \frac{f(k^2 x)}{k^2} \right\| \leq \frac{1}{|k|^2} \alpha \left((kx)^{\frac{1}{4}}, \dots, (kx)^{\frac{1}{4}} \right) \quad (17)$$

for all $x \in_{\mathcal{B}} X$. Combining (16) and (17), we obtain

$$\begin{aligned} & \left\| f(x) - \frac{f(k^m x)}{k^m} \right\| \\ & \leq \frac{1}{|k|} \max \left\{ \alpha \left(x^{\frac{1}{4}}, \dots, x^{\frac{1}{4}} \right), \alpha \left((kx)^{\frac{1}{4}}, \dots, (kx)^{\frac{1}{4}} \right) \right\} \end{aligned} \quad (18)$$

for all $x \in_{\mathcal{B}} X$. Using induction on a positive integer m , we obtain that

$$\begin{aligned} & \left\| f(x) - \frac{f(k^m x)}{k^m} \right\| \\ & \leq \frac{1}{|k|} \max \left\{ \frac{1}{|k|^i} \alpha \left((k^i x)^{\frac{1}{4}}, \dots, (k^i x)^{\frac{1}{4}} \right) : 0 \leq i < m \right\} \end{aligned} \quad (19)$$

for all $x \in_{\mathcal{B}} X$. In order to prove the convergence of the sequence $\left\{ \frac{f(k^m x)}{k^m} \right\}$, replace x by $k^l x$ and divided by k^l in (19), for any $l, m > 0$, we arrive to

$$\left\| \frac{f(k^l x)}{k^l} - \frac{f(k^{l+m} x)}{k^{l+m}} \right\|$$

$$\leq \frac{1}{|k|} \max \left\{ \frac{1}{|k|^{i+l}} \alpha \left(\left(k^{i+l} x \right)^{\frac{1}{4}}, \dots, \left(k^{i+l} x \right)^{\frac{1}{4}} \right) : 0 \leq i < m \right\} \quad (20)$$

for all $x \in {}_{\mathcal{B}}X$. Since the right hand side of the inequality (20) tends to 0 as $m \rightarrow \infty$, the sequence $\left\{ \frac{f(k^m x)}{k^m} \right\}$ is a Cauchy sequence. Since ${}_{\mathcal{B}}Y$ is complete, there exists a mapping $\mathcal{H} : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}Y$ such that

$$\mathcal{H}(x) = \lim_{m \rightarrow \infty} \frac{f(k^m x)}{k^m}, \quad \forall x \in {}_{\mathcal{B}}X.$$

Letting $m \rightarrow \infty$ in (19), we see that (13) holds for all $x \in {}_{\mathcal{B}}X$. Now we need to prove $\mathcal{H}$ satisfies (2), replacing $(x_1, y_1, \dots, u_n, v_n)$ by $(k^m x_1, k^m y_1, \dots, k^m u_n, k^m v_n)$ and divided by k^m in (11), we arrive

$$\begin{aligned} & \frac{1}{|k|^m} \|F(k^m x_1, k^m y_1, k^m u_1, k^m v_1, \dots, k^m x_n, k^m y_n, k^m u_n, k^m v_n)\| \\ & \leq \frac{1}{|k|^m} \alpha(k^m x_1, k^m y_1, k^m u_1, k^m v_1, \dots, k^m x_n, k^m y_n, k^m u_n, k^m v_n) \end{aligned}$$

for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$. Letting $m \rightarrow \infty$ in the above inequality, we arrive

$$\|\mathcal{H}(k^m x_1, k^m y_1, k^m u_1, k^m v_1, \dots, k^m x_n, k^m y_n, k^m u_n, k^m v_n)\| = 0.$$

Hence $\mathcal{H}$ satisfies (2) for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$. This shows that $\mathcal{H}$ is additive. Also

$$\begin{aligned} & \|\mathcal{H}(xy) - \mathcal{H}(x)\mathcal{H}(y)\| \\ & = \lim_{m \rightarrow \infty} \frac{1}{|k|^{2m}} \|f(k^m x k^m y) - f(k^m x) f(k^m y)\| \\ & \leq \lim_{m \rightarrow \infty} \frac{1}{|k|^{2m}} \alpha \left((k^m x)^{\frac{1}{2}}, \dots, (k^m y)^{\frac{1}{2}}, 0, \dots, 0 \right) = 0 \end{aligned}$$

for all $x, y \in {}_{\mathcal{B}}X$. Therefore, $\mathcal{H}$ is additive homomorphism. In order to prove $\mathcal{H}$ is unique, let $\mathcal{H}'(x)$ be another additive mapping satisfying (13) and (2). Then

$$\begin{aligned} \|\mathcal{H}(x) - \mathcal{H}'(x)\| &= \frac{1}{|k|^m} \|\mathcal{H}(k^m x) - \mathcal{H}'(k^m x)\| \\ &\leq \frac{1}{|k|^m} \max \{ \|\mathcal{H}(k^m x) - f(k^m x)\|, \|f(k^m x) - \mathcal{H}'(k^m x)\| \} \\ &\leq \frac{1}{|k|} \max \left\{ \frac{1}{|k|^i} \alpha \left((k^i x)^{\frac{1}{4}}, \dots, (k^i x)^{\frac{1}{4}} \right) : 0 \leq i < m \right\} \end{aligned}$$

$$\rightarrow 0 \quad \text{as } m \rightarrow \infty$$

for all $x \in_{\mathcal{B}} X$. Hence $\mathcal{H}$ is unique.

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of Theorem 2 concerning the stability of (2).

Corollary 3. *Let λ and s be nonnegative real numbers. If a function $f :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} Y$ satisfies the inequality*

$$\begin{aligned} & \|F(x_1, y_1, u_1, v_1 \cdots, x_n, y_n, u_n, v_n)\| \\ & \leq \begin{cases} \lambda, \\ \lambda \sum_{i=1}^n \{||x_i||^s + ||y_i||^s + ||u_i||^s + ||v_i||^s\}, s \neq 1; \\ \lambda \left\{ \prod_{i=1}^n ||x_i||^s ||y_i||^s ||u_i||^s ||v_i||^s \right\}, s \neq \frac{1}{4n}; \\ \lambda \left\{ \prod_{i=1}^n ||x_i||^s ||y_i||^s ||u_i||^s ||v_i||^s \right. \\ \left. + \sum_{i=1}^n (||x_i||^{4ns} + ||y_i||^{4ns} + ||u_i||^{4ns} + ||v_i||^{4ns}) \right\}, s \neq \frac{1}{4n}; \end{cases} \end{aligned} \quad (21)$$

for all $x_1, y_1, u_1, v_1 \cdots, x_n, y_n, u_n, v_n \in_{\mathcal{B}} X$ and

$$\begin{aligned} & \|f(x_1 y_1 \cdots u_n v_n) - f(x_1) f(y_1) \cdots f(u_n) f(v_n)\| \\ & \leq \begin{cases} \lambda, \\ \lambda \{||x_i||^s + ||y_i||^s + ||u_i||^s + ||v_i||^s\}, \\ \lambda \left\{ \prod_{i=1}^n ||x_i||^s ||y_i||^s ||u_i||^s ||v_i||^s \right\}, \\ \lambda \left\{ \prod_{i=1}^n ||x_i||^s ||y_i||^s ||u_i||^s ||v_i||^s \right. \\ \left. + (||x_i||^{4ns} + ||y_i||^{4ns} + ||u_i||^{4ns} + ||v_i||^{4ns}) \right\}, \end{cases} \end{aligned} \quad (22)$$

for all $x_1, y_1, u_1, v_1 \cdots, x_n, y_n, u_n, v_n \in_{\mathcal{B}} X$. Then there exists a unique additive

homomorphism $\mathcal{H} :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} Y$ such that

$$\|f(x) - \mathcal{H}(x)\| \leq \begin{cases} \frac{\lambda}{|k-1|}, \\ \frac{4n\lambda||x||^s}{|k-k^s|}, \\ \frac{\lambda||x||^{4ns}}{\lambda||x||^{4ns}}, \\ \frac{|k-k^{4ns}|}{5n\lambda||x||^{4ns}}, \\ \frac{|k-k^{4ns}|}{|k-k^{4ns}|} \end{cases} \quad (23)$$

for all $x \in_{\mathcal{B}} X$, where $k = n^2$.

Theorem 4. Let $j \in \{-1, 1\}$. Let $\alpha :_{\mathcal{B}} X^{4n} \rightarrow [0, \infty)$ be a function such that (10) for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in_{\mathcal{B}} X$ and let $f :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} Y$ be a function satisfies the inequality (11) for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in_{\mathcal{B}} X$ and

$$\begin{aligned} & \|f(x_1 y_1 \cdots u_n v_n) - f(x_1) y_1 \cdots u_n v_n - x_1 f(y_1) \cdots \\ & u_n v_n - \cdots - x_1 y_1 \cdots u_n f(v_n)\| \leq \alpha(x_1, y_1, \dots, u_n, v_n) \end{aligned} \quad (24)$$

for all $x, y \in_{\mathcal{B}} X$. Then there exists a unique additive derivation $\mathcal{D} :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} X$ such that

$$\begin{aligned} & \|f(x) - \mathcal{D}(x)\| \\ & \leq \frac{1}{|k|} \max \left\{ \frac{1}{|k|^i} \alpha \left((k^i x)^{\frac{1}{4}}, \dots, (k^i x)^{\frac{1}{4}} \right) : 0 \leq j < n \right\} \end{aligned} \quad (25)$$

for all $x \in_{\mathcal{B}} X$. The mapping $\mathcal{D}(x)$ is defined by

$$\mathcal{D}(x) = \lim_{m \rightarrow \infty} \frac{f(k^{mj}x)}{|k|^{mj}} \quad (26)$$

for all $x \in_{\mathcal{B}} X$.

Proof. Assume $j = 1$. By the same reasoning as that in the proof of the Theorem 2, there exist a unique additive mapping $\mathcal{D} :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} X$ satisfying (25). The mapping $\mathcal{D} :_{\mathcal{B}} X \rightarrow_{\mathcal{B}} X$ given by $\mathcal{D}(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^k}$. It follows from (11) that

$$\|\mathcal{D}(xy) - [x\mathcal{D}(y)] - [\mathcal{D}(x)y]\|$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \left\| f(n^{2k}xy) - (n^kx) f(n^ky) - f(n^kx)(n^ky) \right\| \\
&\leq \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \alpha(n^kx, n^ky, 0, \dots, 0) = 0
\end{aligned}$$

for all $x, y \in {}_{\mathcal{B}}X$. Therefore $\mathcal{D} : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}X$ is a additive derivation satisfying (25).

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of Theorem 4 concerning the stability of (2).

Corollary 5. *Let λ and s be nonnegative real numbers. If a function $f : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}X$ satisfies the inequality (21) for all $x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$ and*

$$\begin{aligned}
&\|f(x_1y_1 \cdots u_nv_n) - f(x_1)y_1 \cdots \\
&\quad u_nv_n - x_1f(y_1) \cdots u_nv_n - \cdots - x_1y_1 \cdots u_nf(v_n)\| \\
&\leq \begin{cases} \lambda, \\ \lambda \{ \|x_i\|^s + \|y_i\|^s + \|u_i\|^s + \|v_i\|^s \}, \\ \lambda \left\{ \prod_{i=1}^n \|x_i\|^s \|y_i\|^s \|u_i\|^s \|v_i\|^s \right\}, \\ \lambda \left\{ \prod_{i=1}^n \|x_i\|^s \|y_i\|^s \|u_i\|^s \|v_i\|^s \right. \\ \left. + (\|x_i\|^{4ns} + \|y_i\|^{4ns} + \|u_i\|^{4ns} + \|v_i\|^{4ns}) \right\}, \end{cases} \quad (27)
\end{aligned}$$

$x_1, y_1, u_1, v_1, \dots, x_n, y_n, u_n, v_n \in {}_{\mathcal{B}}X$. Then there exists a unique additive derivation $\mathcal{D} : {}_{\mathcal{B}}X \rightarrow {}_{\mathcal{B}}X$ such that

$$\|f(x) - \mathcal{D}(x)\| \leq \begin{cases} \frac{\lambda}{|k-1|}, \\ \frac{4n\lambda\|x\|^s}{|k-k^s|}, \\ \frac{\lambda\|x\|^{4ns}}{|k-k^{4ns}|}, \\ \frac{5n\lambda\|x\|^{4ns}}{|k-k^{4ns}|} \end{cases} \quad (28)$$

for all $x \in {}_{\mathcal{B}}X$, where $k = n^2$.

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